Advances in Dynamical Systems and Applications.
ISSN 0973-5321 Volume 3 Number 1 (2008), pp. 73–88

http://www.ripublication.com/adsa.htm

Some Oscillation Results for Second Order Functional Dynamic Equations[1]

Lynn Erbe and Raegan Higgins
Department of Mathematics, University of Nebraska–Lincoln,
Lincoln, NE 68588-0130, USA
E-mail: lerbe2@math.unl.edu, s-rhiggin4@math.unl.edu

Abstract

In this paper, we consider the second-order nonlinear dynamic equation

$$x^{\Delta\Delta}(t) + f(t, x^{\sigma}(t), x^{\tau}(t), x^{\xi}(t)) = 0,$$

on a time scale $\mathbb{T}$. Our goal is to establish some new oscillation and nonoscillation results for this equation. Here we assume that $\tau(t) \le t \le \xi(t)$ and $\tau,\ \xi : \mathbb{T} \to \mathbb{T}$, and use the notation $x^{\tau}(t) = x(\tau(t))$, $x^{\sigma}(t) = x(\sigma(t))$ and $x^{\xi}(t) = x(\xi(t))$. We apply results from the theory of lower and upper solutions for related dynamic equations along with some additional estimates on the positive solutions.

AMS subject classification: 34K11, 39A10, 39A99.
Keywords: Oscillation, second-order nonlinear dynamic equation, time scale, Riccati transformation technique.

1. Introduction and Preliminary Results

In recent years, there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on a time scale (i.e., a closed subset of the real line $\mathbb{R}$). This has lead to many attempts to harmonize the oscillation theory for the continuous and the discrete cases, to include them in one comprehensive theory, and to extend the results to more general time scales. We refer the reader to the papers

[1]This paper is dedicated to Professor Allan Peterson.
Received October 30, 2007; Accepted January 29, 2008
Communicated by Elvan Akin-Bohner

[1, 3–7, 9–16, 18, 19] and the references cited therein. To illustrate some of the results we mention the work of Zhang and Shanliang [19] who considered the equation with a delay

$$x^{\Delta\Delta}(t) + p(t)f(x(t-\tau)) = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

where $\tau \in \mathbb{R}$ and $t - \tau \in \mathbb{T}$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing ($f'(u) > k > 0$), and $uf(u) > 0$ for $u \neq 0$. By using comparison theorems they proved that the oscillation of (1.1) is equivalent to the oscillation of the nonlinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)f(x(\sigma(t))) = 0, \quad t \in \mathbb{T} \tag{1.2}$$

and established some sufficient conditions for oscillation by applying the results established in [16] for (1.2).

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume throughout this paper that our time scale is unbounded above. We assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Our main interest is to consider the general nonlinear dynamic equation

$$x^{\Delta\Delta}(t) + f(t, x^{\sigma}(t), x^{\tau}(t), x^{\xi}(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.3}$$

where $f = f(t, u, v, w)$ is continuous for $t \in \mathbb{T}$, and for all $u, v, w \in \mathbb{R}$. We shall assume that $\tau(t) \leq t \leq \sigma(t) \leq \xi(t)$ for all $t \in \mathbb{T}$ and that $\tau, \xi : \mathbb{T} \to \mathbb{T}$, are rd-continuous functions. We assume also that τ satisfies $\lim\limits_{t\to\infty} \tau(t) = \infty$.

Our attention is restricted to those solutions x of (1.3) which exist on some half-line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$ for any $t_0 \geq t_x$. A solution x of (1.3) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory. The theory of time scales was introduced by Stefan Hilger in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis (see [17]). Not only does this unify the theories of differential equations and difference equations, but it also extends these classical situations to cases "in between"– e.g., to the so-called q-difference equations. Moreover, the theory can be applied to other different types of time scales. Since its introduction, many authors have expounded on various aspects of this new theory, and we refer specifically to the paper by Agarwal et al. [2] and the references cited therein. A book on the subject of time scales by Bohner and Peterson [8] summarizes and organizes much of time scale calculus.

We note that (1.3) in its general form includes several types of differential and difference equations with delay or advanced arguments or both. In addition, different equations correspond to the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, and so (1.3) includes the nonlinear delay differential equation

$$x''(t) + p(t)f(x(t), x^{\tau}(t), x^{\xi}(t)) = 0. \tag{1.4}$$

When $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^{\Delta}(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and a special case of (1.3) is the nonlinear delay difference equation

$$\Delta^2 x(t) + p(t) f(x(t+1), x^{\tau}(t), x^{\xi}(t)) = 0, \tag{1.5}$$

where Δ denotes the forward difference operator. When $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$, $x^{\Delta}(t) = \Delta_h x(t) = (x(t+h) - x(t))/h$, $\int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)h$, and so a special case of (1.3) is the second-order delay difference equation with constant step size

$$\Delta_h^2(r(t)\Delta_h x(t)) + p(t) f(x(t+h), x^{\tau}(t), x^{\xi}(t)) = 0. \tag{1.6}$$

Finally, when $\mathbb{T} = \{t : t = q^k, k \in \mathbb{N}_0\}$, $q > 1$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $x^{\Delta}(t) = \Delta_q x(t) = (x(q\,t) - x(t))/(q-1)\,t$, $\int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} f(q^k)\mu(q^k)$, and so we obtain, as a special case, the second-order q-difference equation with variable step size

$$\Delta_q(r(t)\Delta_q^2 x(t)) + p(t) f(x(qt), x^{\tau}(t), x^{\xi}(t)) = 0. \tag{1.7}$$

Of course many more examples may be given, and we will illustrate some of these in the examples in Section 3.

As observed above, equation (1.3) includes the delay and advanced argument cases. Concerning the function $f = f(t, u, v, w)$ we will always assume that f satisfies the following condition (A):

$$f(t, u, v, w) = -f(t, -u, -v, -w)$$

and

$$f(t, u, v, w) > 0 \text{ if } u, v, w > 0, t \in \mathbb{T}.$$

In this paper we shall apply the method of upper and lower solutions as well as some dynamic inequalities to obtain some new oscillation criteria for the delay and advanced argument cases. In the continuous case ($\mathbb{T} = \mathbb{R}$) some results for the second order nonlinear delay equation were obtained in [12]. We begin with the following.

Lemma 1.1. Let $x \in C_{\mathrm{rd}}^2[t_0, \infty)$ satisfy $x(t) > 0$, $x^{\Delta}(t) > 0$, $x^{\Delta\Delta}(t) \leq 0$ for $t \geq T \geq t_0$. Then:

(a) for each $0 < k < 1$ there exists $T_k \geq T \geq t_0$ such that

$$x^{\tau}(t) := x(\tau(t)) \geq k x^{\sigma}(t)\frac{\tau(t)}{\sigma(t)}, \quad t \geq T_k,$$

and

(b)

$$x^{\xi}(t) := x(\xi(t)) \leq x^{\sigma}(t)\frac{\xi(t)}{k\sigma(t)}, \quad t \geq T_k.$$

Proof. (a) For $t > T \geq t_0$ we have

$$x(\sigma(t)) - x(\tau(t)) = \int_{\tau(t)}^{\sigma(t)} x^{\Delta}(s)\Delta s \leq x^{\Delta}(\tau(t))(\sigma(t) - \tau(t))$$

and so

$$x(\sigma(t) \leq x(\tau(t)) + x^{\Delta}(\tau(t))(\sigma(t) - \tau(t)). \tag{1.8}$$

Also we have

$$x(\tau(t)) - x(T) = \int_{T}^{\tau(t)} x^{\Delta}(s)\Delta s \geq x^{\Delta}(\tau(t))(\tau(t) - T) \tag{1.9}$$

and hence

$$\frac{x(\tau(t))}{x^{\Delta}(\tau(t))} \geq \frac{x(T)}{x^{\Delta}(\tau(t))} + (\tau(t) - T) \tag{1.10}$$

which implies

$$\frac{x^{\Delta}(\tau(t))}{x(\tau(t))} \leq \frac{1}{(\tau(t) - T) + \dfrac{x(T)}{x^{\Delta}(\tau(t))}} < \frac{1}{\tau(t) - T}. \tag{1.11}$$

Therefore, (1.8) and (1.11) imply

$$\begin{aligned}
\frac{x^{\sigma}(t)}{x(\tau(t))} &\leq 1 + \frac{x^{\Delta}(\tau(t))}{x(\tau(t))}(\sigma(t) - \tau(t)) \\
&\leq 1 + \frac{\sigma(t) - \tau(t)}{\tau(t) - T} \\
&= \frac{\sigma(t) - T}{\tau(t) - T}.
\end{aligned} \tag{1.12}$$

Now given any $0 < k < 1$, there exists T_k such that

$$\frac{\sigma(t) - T}{\tau(t) - T} < \frac{1}{k}\frac{\sigma(t)}{\tau(t))}, \quad t \geq T_k. \tag{1.13}$$

Consequently, we have from (1.12) and (1.13)

$$x^{\tau}(t) = x(\tau(t)) \geq kx^{\sigma}(t)\frac{\tau(t)}{\sigma(t)}, \quad t \geq T_K$$

and this completes the proof of (a).

The proof of (b) is similar. We have for $T < t \leq \sigma(t) \leq \xi(t)$

$$x(\xi(t)) - x(\sigma(t)) = \int_{\sigma(t)}^{\xi(t)} y^{\Delta}(s)\Delta s \leq x^{\Delta}(\sigma(t))(\xi(t) - \sigma(t))$$

and so we have

$$\frac{x(\xi(t))}{x(\sigma(t))} \leq 1 + \frac{x^{\Delta}(\sigma(t))}{x(\sigma(t))}(\xi(t) - \sigma(t)). \tag{1.14}$$

Also we have

$$x(\sigma(t)) \geq x(T) + x^{\Delta}(\sigma(t))(\sigma(t) - T)$$

so that

$$\frac{x(\sigma(t))}{x^{\Delta}(\sigma(t))} \geq \frac{x(T)}{x^{\Delta}(\sigma(t))} + (\sigma(t) - T) \geq k\sigma(t), \quad t \geq T_k,\ 0 < k < 1.$$

Hence, from (1.14) we have

$$\frac{x(\xi(t))}{x(\sigma(t))} \leq 1 + \frac{\xi(t) - \sigma(t)}{k\sigma(t)} = \frac{(k-1)\sigma(t) + \xi(t)}{k\sigma(t)} \leq \frac{\xi(t)}{k\sigma(t)}, \quad t \geq T_k.$$

This completes the proof of the lemma. ■

We shall also need the following lemma which is often referred to as the Riccati substitution technique.

Lemma 1.2. The linear equation

$$Lx \equiv x^{\Delta\Delta} + q(t)x^{\sigma} = 0$$

is nonoscillatory if and only if there is a function z satisfying the Riccati dynamic inequality

$$z^{\Delta} + q(t) + S(z)(t) \leq 0 \tag{1.15}$$

with $1 + \mu(t)z(t) > 0$ for large t, where

$$S(z) := \frac{z^2}{1 + \mu(t)z}.$$

2. Main Results

We begin with the following result for the case when $f(t, u, v, w)$ satisfies the following condition (B):

- For each fixed $t \in \mathbb{T}$ and $u, v > 0$, f is nonincreasing in w and for fixed $u, w > 0$, f is
- nondecreasing in v for $v > 0$ and for fixed $v, w > 0$ f is
- nondecreasing in u for $u > 0$.

We introduce the functions g and h defined by

$$g(t) := \frac{\tau(t)}{\sigma(t)}, \quad h(t) := \frac{\xi(t)}{\sigma(t)}. \tag{2.1}$$

Our first result is the following.

Theorem 2.1. Assume conditions (A) and (B) hold. Then all bounded solutions of (1.3) are oscillatory in case

$$\left| \int^{\infty} tf\left(t, \alpha, \alpha k g(t), \frac{\alpha}{k^2} h(t)\right) \Delta t \right| = \infty, \tag{2.2}$$

for all $\alpha \neq 0$ and some $0 < k < 1$, where g and h are given in (2.1).

Proof. If not, let u be a bounded nonoscillatory solution which, in view of condition (A), we may assume satisfies

$$u(t) > 0, u(\tau(t)) > 0, \quad t \geq T_1 \geq t_0. \tag{2.3}$$

Consequently, $u^{\Delta\Delta}(t) = -f(t, u^{\sigma}(t), u(\tau(t)), u(\xi(t))) < 0$ for $t \geq T_1$ and so $u^{\Delta}(t)$ is decreasing for $t \geq T_1$. Consequently, $u^{\Delta}(t) > 0$ for $t \geq T_1$. Indeed, if $u^{\Delta}(t_1) \leq 0$ for some $t_1 \geq T_1$, then $u^{\Delta}(t) \leq 0$ for all $t \geq t_1$. Now if $u^{\Delta}(t_1) < 0$, then $u(t) - u(t_1) = \displaystyle\int_{t_1}^{t} u^{\Delta}(s)\Delta s \leq u^{\Delta}(t_1)(t - t_1) \to -\infty$ as $t \to \infty$, which is a contradiction. Also, if $u^{\Delta}(t_1) = 0$ then $u^{\Delta}(t) \equiv 0$ for $t \geq t_1$, and so $u^{\Delta\Delta}(t) \equiv 0 = f\left(t, u^{\sigma}(t), u(\tau(t)), u(\xi(t))\right)$, which is again a contradiction. Hence, we conclude that for all $t \geq t_1$

$$u(t) > 0, \quad u^{\Delta}(t) > 0, \quad u(\tau(t)) > 0.$$

From Lemma 1.1, given $0 < k < 1$, there exists $T_k \geq t_1$ such that $u(\tau(t)) \geq kg(t)u^{\sigma}(t)$ and $u(\xi(t)) \leq \dfrac{1}{k}h(t)u^{\sigma}(t)$ for $t \geq T_k$. By the monotonicity assumption on f we have

$$\begin{aligned} 0 &= u^{\Delta\Delta}(t) + f(t, u^{\sigma}(t), u(\tau(t)), u(\xi(t))) \\ &\geq u^{\Delta\Delta}(t) + f\left(t, u^{\sigma}(t), kg(t)u^{\sigma}(t), \frac{1}{k}h(t)u^{\sigma}(t)\right). \end{aligned} \tag{2.4}$$

If we set $F(t, u^\sigma(t)) := f\left(t, u^\sigma(t), kg(t)u^\sigma(t), \frac{1}{k}h(t)u^\sigma(t)\right)$, then (2.4) shows that $\beta := u$ is an upper solution for the dynamic equation $u^{\Delta\Delta} + F(t, u^\sigma(t)) = 0$. Also, the constant function $\alpha := u(t_1)$ satisfies $\alpha^{\Delta\Delta}(t) + F(t, \alpha^\sigma(t)) > 0$, and so $\alpha(t)$ is a lower solution. Therefore, by [8, Theorem 6.54], if we choose a sequence $t_n \to \infty, t_n \in \mathbb{T}$, then for each $t_n \in \mathbb{T}, t_n > t_1$, the BVP

$$y^{\Delta\Delta} + F(t, u^\sigma(t)) = 0, \quad y(t_1) = A = u(t_1), \quad y(\sigma^2(t_n)) = B,$$

where $u(t_1) \le B \le u(\sigma^2(t_n))$, has a solution u_n with

$$u(t_1) \le u_n(t) \le u(t), \quad t_1 \le t \le \sigma^2(t_n).$$

By a standard diagonalization argument, we conclude that there is a subsequence of $\{u_n(t)\}$ which converges, uniformly on compact subsets of $\mathbb{T}$, to a solution y of

$$y^{\Delta\Delta} + F(t, y^\sigma(t)) = 0$$

which satisfies $0 < u(t_1) \le y(t) \le u(t)$ on $[t_1, \infty)$. It follows that $y(t) > 0, \quad y^\Delta(t) > 0$ and $y^{\Delta\Delta}(t) \le 0$. Therefore, since $y(t)$ is bounded, we have that $\lim_{t\to\infty} y(t) := L > 0$ exists. Integration for $t_1 < s < T$ implies

$$y^\Delta(T) - y^\Delta(s) + \int_s^T F(r, y^\sigma(r))\Delta r = 0.$$

Letting $T \to \infty$ we obtain

$$y^\Delta(s) \ge \int_s^\infty F(r, y^\sigma(r))\Delta r,$$

and so integrating again for $t_1 < \tilde{t} < t$, we obtain

$$\begin{aligned}
y(t) - y(\tilde{t}) &= \int_{\tilde{t}}^t y^\Delta(s)\Delta s \\
&\ge \int_{\tilde{t}}^t \int_s^\infty F(r, y^\sigma(r))\Delta r\Delta s \\
&= \int_{\tilde{t}}^t \int_{\tilde{t}}^r F(r, y^\sigma(r))\Delta s\Delta r + \int_t^\infty \int_{\tilde{t}}^t F(r, y^\sigma(r))\Delta s\Delta r \\
&= \int_{\tilde{t}}^t (r - \tilde{t})F(r, y^\sigma(r))\Delta r + \int_t^\infty (t - \tilde{t})F(r, y^\sigma(r))\Delta r \\
&\ge \int_{\tilde{t}}^t (r - \tilde{t})F(r, y^\sigma(r))\Delta r. \qquad (2.5)
\end{aligned}$$

From (2.5) we have

$$y(t) \ge y(\tilde{t}) + \int_{\tilde{t}}^t (r - \tilde{t})F(r, y^\sigma(r))\Delta r > \int_{\tilde{t}}^t (r - \tilde{t})F(r, y^\sigma(r))\Delta r.$$

Since $y(t) \le u(t) \le M$ for some $M > 0$, by letting $t \to \infty$, it follows that

$$\int_{\tilde{t}}^{\infty} (r - \tilde{t}) F(r, y^{\sigma}(r)) \Delta r < \infty,$$

which implies that $\int_{\tilde{t}}^{\infty} r F(r, y^{\sigma}(r)) < \infty$. For the same $0 < k < 1$ as in the first part of the proof, we may assume that for $t \ge \tilde{T}_k \ge T_k$ we have $y(t) \ge kL$. However, then using the monotonicity of f we have

$$\begin{aligned} f\left(t, y^{\sigma}(t), kg(t)y^{\sigma}(t), \frac{1}{k}h(t)y^{\sigma}(t)\right) &\ge f\left(t, y^{\sigma}(t), kg(t)y^{\sigma}(t), \frac{L}{k}h(t)\right) \\ &\ge f\left(t, kL, k^2 Lg(t), \frac{L}{k}h(t)\right). \end{aligned}$$

Therefore, with $\alpha := kL$ it follows that

$$\int_{\tilde{T}_k}^{\infty} rf\left(r, \alpha, k\alpha g(r), \frac{\alpha}{k^2}h(r)\right) \Delta r < \infty,$$

a contradiction to our assumption (2.2). This completes the proof. ■

We next introduce the following condition ($\tilde{\text{B}}$) which replaces the nonincreasing assumption for the function f in the w variable by assuming f is nondecreasing in w for $w > 0$ and for fixed $t \in \mathbb{T}$ and $u, v > 0$.

- The function is said to satisfy condition ($\tilde{\text{B}}$) if for each fixed $t \in \mathbb{T}$ and $u, v > 0$, f is
- nondecreasing in w and for fixed $u, w > 0$, f is
- nondecreasing in v for $v > 0$ and for fixed $v, w > 0$ f is
- nondecreasing in u for $u > 0$. Of course, if f is
- independent of w, then conditions (B) and ($\tilde{\text{B}}$) coincide.

Then we have the following result which gives conditions under which all solutions of (1.3) are oscillatory:

Theorem 2.2. Assume conditions (A) and ($\tilde{\text{B}}$) hold. Then all solutions of (1.3) are oscillatory in case

$$\left| \int^{\infty} tf\,(t, \alpha, \alpha g(t), \alpha)\, \Delta t \right| = \infty, \tag{2.6}$$

for all $\alpha \neq 0$.

Proof. The proof is similar to that of Theorem 2.1. Here we use the fact that if u is an eventually positive solution of (1.3), then given $0 < k < 1$ there exists $T_k \geq T_1$ so that $u(\tau(t)) \geq kg(t)u^\sigma(t)$ and since $u^\Delta(t) > 0$ for $t \geq T$, $u(\xi(t)) \geq u^\sigma(t)$. Therefore, by the monotonicity assumption from ($\tilde{\text{B}}$), it follows that

$$\begin{aligned} 0 &= u^{\Delta\Delta}(t) + f(t, u^\sigma(t), u(\tau(t)), u(\xi(t)) \\ &\geq u^{\Delta\Delta}(t) + f(t, u^\sigma(t), kg(t)u^\sigma(t), u^\sigma(t)). \end{aligned}$$

So if we set $\tilde{F}(t, u^\sigma) =: f(t, u^\sigma, kg(t)u^\sigma, u^\sigma)$ and $\alpha = u(T) > 0$, then the rest of the proof is similar to that of Theorem 2.1. ∎

The next result may be regarded as a special case of Theorem 2.1.

Theorem 2.3. Assume f satisfies conditions (A) and (B) and suppose further that there exists $T \geq t_1$ such that

$$\inf\{g(t) : t \geq T\} = m_1 > 0 \quad \text{and} \quad \sup\{h(t) : t \geq T\} = m_2 < \infty, \tag{2.7}$$

where g and h are as defined in (2.1). Then all bounded solutions of (1.3) are oscillatory in case

$$\left| \int^\infty tf\left(t, \alpha, k\alpha m_1, \frac{\alpha m_2}{k^2}\right) \Delta t \right| = \infty, \tag{2.8}$$

for all $\alpha \neq 0$, and for some $0 < k < 1$.

Proof. Theorem 2.1 and the monotonicity assumption on f along with (2.7) imply the result. ∎

If we replace the assumption (B) by ($\tilde{\text{B}}$), then we may give a necessary and sufficient condition for the existence of a bounded nonoscillatory solution. In this case, we only need to assume the first part of (2.7) along with the assumption that $\dfrac{\mu(t)}{t}$ is bounded for all large t (which is equivalent to $\dfrac{\sigma(t)}{t}$ bounded).

Theorem 2.4. Assume f satisfies (A) and ($\tilde{\text{B}}$) and that $\dfrac{\mu(t)}{t}$ is bounded for all large t. Assume further that

$$\liminf g(t) := m_1 > 0. \tag{2.9}$$

Then equation (1.3) has a bounded nonoscillatory solution if and only if

$$\left| \int^\infty \sigma(t) f\,(t, \alpha, \alpha, \alpha)\, \Delta t \right| < \infty, \tag{2.10}$$

for some $\alpha \neq 0$.

Proof. We note that since $\frac{\mu(t)}{t}$ is bounded for all large t, it follows that

$$\left|\int^{\infty} \sigma(t) f\left(t, \alpha, \alpha, \alpha\right) \Delta t\right| < \infty$$

iff

$$\left|\int^{\infty} t f\left(t, \alpha, \alpha, \alpha\right) \Delta t\right| < \infty.$$

Now Theorem 2.4 and the monotonicity assumption of f along with (2.9) show that (2.10) is necessary for the existence of a bounded nonsocillatory solution. Conversely, if (2.10) holds, assume to be specific that $\alpha > 0$ and choose $T \geq t_1$ so that $\tau(t) \geq t_1$ if $t \geq T$ and such that

$$\int_T^{\infty} \sigma(t) f(t, \alpha, \alpha, \alpha) \Delta t < \frac{\beta}{2},$$

where $0 < \beta < \alpha$. Define $y_0(t) \equiv \beta$ for $t \geq t_1$ and

$$y_{n+1}(t) = \begin{cases} \beta - \displaystyle\int_T^{\infty} (\sigma(s) - T) f(s, y_n^{\sigma}(s), y_n(\tau(s)), y_n(\xi(s)))\Delta s, & t < T, \\ \beta - \displaystyle\int_t^{\infty} (\sigma(s) - t) f(s, y_n^{\sigma}(s), y_n(\tau(s)), y_n(\xi(s)))\Delta s, & t \geq T. \end{cases}$$

Then, since $t_1 \leq \tau(t) \leq t \leq \sigma(t) \leq \xi(t)$ for $t \geq T$, it follows by induction that

$$\frac{\beta}{2} \leq y_n(t) \leq \beta, \quad t \geq T \text{ and all } n \geq 0.$$

Since the sequences $\{y_n(t)\}$ and $\{y_n^{\Delta}(t)\}$ are both uniformly bounded for $t \geq T$, the Ascoli–Arzela theorem along with a standard diagonalization argument yields a subsequence of $\{y_n(t)\}$ which converges uniformly on compact subintervals of $[T, \infty) \cap \mathbb{T}$ to a solution y of (1.3) satisfying $\beta/2 \leq y(t) \leq \beta$. This proves the theorem. ■

It may be shown that the converse of Theorem 2.1 is not true without assumptions like (2.7). (See [12, Remark 3.5]).

To extend Theorems 2.1 and 2.3 to unbounded solutions, we let $\phi(u)$ denote a continuous nondecreasing function of u satisfying $u\phi(u) > 0$, $u \neq 0$ with

$$\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty.$$

We will say that $f(t, u, v, w)$ satisfies condition (H) provided there exists an $0 < \alpha < 1$ such that for any $c \neq 0$ and for all $t \geq T$

$$\inf_{|u| \geq c} \frac{f(t, u, \alpha g(t) u, \frac{1}{\alpha} h(t) u)}{\phi(u)} \geq k \left| f\left(t, c, \alpha g(t) c, \frac{1}{\alpha} h(t) c\right)\right|$$

for some positive constant k.

We may now prove the following result:

Theorem 2.5. Assume f satisfies conditions (A), (B), and (H). Then all solutions of (1.3) are oscillatory in case (2.2) holds for all $\alpha \neq 0$.

Proof. If (2.2) holds for all $\alpha \neq 0$, let u be a nonoscillatory solution of (1.3) with $u(t) > 0, u(\tau(t)) > 0$ for $t \geq T$. As in the proof of Theorem 2.1, given $0 < \alpha < 1$ from condition (H) there exists $T_\alpha \geq T$ such that

$$u^{\Delta\Delta} + f\left(t, u^\sigma(t), \alpha g(t) u^\sigma(t), \frac{1}{\alpha} h(t) u^\sigma(t)\right) \leq 0, \quad t \geq T_\alpha. \tag{2.11}$$

Hence we obtain a solution y of

$$y^{\Delta\Delta} + f\left(t, y^\sigma(t), \alpha g(t) y^\sigma(t), \frac{1}{\alpha} h(t) y^\sigma(t)\right) = 0 \tag{2.12}$$

with $0 < u(T_\alpha) \leq y(t) \leq u(t), t \geq T_\alpha$. We next define

$$G(u) := \int_{u_0}^{u} \frac{ds}{\phi(s)},$$

where $u_0 := y(T_\alpha) > 0$. Observe that $G'(u) = 1/\phi(u)$. By the Pötzsche chain rule,

$$(G(y(t))^\Delta = \left(\int_0^1 \frac{dh}{\phi(y_h(t))}\right) y^\Delta(t) \geq \left(\int_0^1 \frac{dh}{\phi(y^\sigma(t))}\right) y^\Delta(t) = \frac{y^\Delta(t)}{\phi(y^\sigma(t))}$$

since $y_h(t) := y(t) + h\mu(t) y^\Delta(t) \leq y^\sigma(t)$ and ϕ is nondecreasing so that $\dfrac{1}{\phi(y_h(t))} \geq \dfrac{1}{\phi(y^\sigma(t))}$. Consequently,

$$(G(y(t)))^\Delta \geq \frac{y^\Delta(t)}{\phi(y^\sigma(t))}. \tag{2.13}$$

Furthermore, since $y(t) > 0$ and $y^\Delta(t)$ is nonincreasing, $\lim_{t\to\infty} y^\Delta(t) = L_1$ with $0 \leq L_1 < \infty$. Now integrating (2.12) for $t \geq \tilde{T} \geq T_\alpha$ gives

$$0 = y^\Delta(t) - y^\Delta(\tilde{T}) + \int_{\tilde{T}}^{t} f\left(s, y^\sigma(s), \alpha g(s) y^\sigma(s), \frac{1}{\alpha} h(s) y^\sigma(s)\right) \Delta s$$

and letting $t \to \infty$ in the above, we obtain

$$\begin{aligned}
y^\Delta(\tilde{T}) &= L_1 + \int_{\tilde{T}}^{\infty} f\left(s, y^\sigma(s), \alpha g(s) y^\sigma(s), \frac{1}{\alpha} h(s) y^\sigma(s)\right) \Delta s \\
&\geq \int_{\tilde{T}}^{\infty} f\left(s, y^\sigma(s), \alpha g(s) y^\sigma(s), \frac{1}{\alpha} h(s) y^\sigma(s)\right) \Delta s \\
&> \int_{\tilde{T}}^{t} f\left(s, y^\sigma(s), \alpha g(s) y^\sigma(s), \frac{1}{\alpha} h(s) y^\sigma(s)\right) \Delta s.
\end{aligned}$$

Now multiplying by $\left(\phi(y^\sigma(\tilde{T}))\right)^{-1}$, we obtain

$$\begin{aligned}
\frac{y^\Delta(\tilde{T})}{\phi(y^\sigma(\tilde{T}))} &\geq \frac{1}{\phi(y^\sigma(\tilde{T}))}\int_{\tilde{T}}^{t} f\left(s, y^\sigma(s), \alpha g(s)y^\sigma(s), \frac{1}{\alpha}h(s)y^\sigma(s)\right)\Delta s \\
&\geq \int_{\tilde{T}}^{t} \frac{f(s, y^\sigma(s), \alpha g(s)y^\sigma(s), \frac{1}{\alpha}h(s)y^\sigma(s))}{\phi(y^\sigma(s))}\Delta s \\
&\geq \int_{\tilde{T}}^{t} kf\left(s, c, \alpha g(s)c, \frac{1}{\alpha}h(s)c\right)\Delta s \qquad (2.14)
\end{aligned}$$

for sufficiently large $\tilde{T}$ (by condition (H)) where $c := u(T_\alpha) > 0$. Observe that since $y^\Delta(t) > 0$, we have $\lim_{t\to\infty} y(t) = L_2$ with $0 \leq L_2 \leq \infty$ and so

$$\lim_{t\to\infty} G(y(t)) = \lim_{t\to\infty}\int_{u_0}^{y(t)} \frac{du}{\phi(u)} = \int_{u_0}^{L_2}\frac{du}{\phi(u)} := L < \infty. \qquad (2.15)$$

We have therefore as $t \to \infty$

$$\int_{T_\alpha}^{\infty} (G(y(s)))^\Delta \Delta s = L - G(y(T_\alpha)) < \infty.$$

We integrate (2.14) for $t \geq \tilde{T}$ and use (2.13) to obtain

$$\begin{aligned}
\int_{\tilde{T}}^{t} (G(y(s)))^\Delta \Delta s &\geq \int_{\tilde{T}}^{t} \frac{y^\Delta(s)}{\phi(y^\sigma(s))}\Delta s \\
&\geq \int_{\tilde{T}}^{t}\int_{\tilde{T}}^{s} kf\left(r, c, \alpha g(r)c, \frac{1}{\alpha}h(r)c\right)\Delta r\Delta s \\
&= k\int_{\tilde{T}}^{t}(s-\tilde{T})f\left(s, c, \alpha g(s)c, \frac{1}{\alpha}h(s)c\right)\Delta s. \qquad (2.16)
\end{aligned}$$

However, the left side of (2.16) is bounded as $t \to \infty$ whereas the right side is unbounded by assumption (2.2). This contradiction shows that all solutions of (1.3) are oscillatory. ■

We may also prove an analog of Theorem 2.5 by replacing condition (B) by ($\tilde{\text{B}}$) and condition (H) by ($\tilde{\text{H}}$), which we state as follows: We say that f satisfies condition ($\tilde{\text{H}}$) provided there exists an $0 < \alpha < 1$ such that for any $c \neq 0$ and for all $t \geq T$

$$\inf_{|u|\geq c} \frac{f(t, u, \alpha g(t)u, u)}{\phi(u)} \geq k|f(t, c, \alpha g(t)c, c)|$$

for some positive constant k, where $\phi(u)$ denotes a continuous nondecreasing function of u satisfying $u\phi(u) > 0$, $u \neq 0$ with

$$\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty.$$

We then may establish the following result.

Theorem 2.6. Assume f satisfies conditions (A), ($\tilde{\text{B}}$), and ($\tilde{\text{H}}$). Then all solutions of (1.3) are oscillatory in case (2.6) holds for all $\alpha \neq 0$.

We omit the proof since it is similar to the proof of Theorem 2.5.

3. Examples

We would like to illustrate some of the results above by means of several examples. We first consider the linear case when the equation contains an advanced argument.

Example 3.1. Consider the linear functional dynamic equation

$$y^{\Delta\Delta} + p(t)y^{\sigma}(t) + q(t)y^{\tau}(t) + r(t)y^{\xi}(t) = 0, \tag{3.1}$$

where $p(t), q(t), r(t) > 0, t \geq t_0$. If we set

$$Q(t) := p(t) + q(t)\frac{\tau(t)}{\sigma(t)} + r(t), \tag{3.2}$$

then (3.1) is oscillatory in case

$$y^{\Delta\Delta} + \lambda Q(t)y^{\sigma} = 0 \tag{3.3}$$

is oscillatory for some $0 < \lambda < 1$. To see this, suppose that u is a nonoscillatory solution of (3.1) with $u(t) > 0, u^{\tau}(t) > 0,\ \ t \geq T$. Then by Lemma 1.1, for $\lambda < k < 1$ there is a $T_k \geq T$ such that

$$u^{\Delta\Delta}(t) + \left(p(t) + kq(t)\frac{\tau(t)}{\sigma(t)} + r(t)\right)u^{\sigma}(t) \leq 0,\ t \geq T_k. \tag{3.4}$$

Then with $z(t) := \dfrac{u^{\Delta(t)}}{u(t)}$, we see that $z(t)$ satisfies the Riccati dynamic inequality (1.15) with $q(t)$ replaced by $p(t) + kq(t)\dfrac{\tau(t)}{\sigma(t)} + r(t)$. By Lemma 1.2, this means that the equation

$$y^{\Delta\Delta} + \left(p(t) + kq(t)\frac{\tau(t)}{\sigma(t)} + r(t)\right)y^{\sigma}(t) = 0 \tag{3.5}$$

is nonoscillatory and so by the Sturm comparison theorem, (3.3) is also nonoscillatory. This contradiction shows that (3.1) is oscillatory. If we apply a specific oscillation criterion, we conclude that (3.1) is oscillatory if

$$\liminf\ t\int_t^\infty \left(p(s)+q(s)\frac{\tau(s)}{\sigma(s)}+r(s)\right)\Delta s > \frac{1}{4}$$

(see [15, Example 3.4]).

Example 3.2. Let $f(t,u,v,w) := p(t)u^{\gamma_1}+q(t)v^{\gamma_2}$, where $\gamma_1,\gamma_2 > 0$ and are the quotients of odd positive integers. We assume also that $p(t), q(t) > 0$ for all large t and are rd-continuous. Clearly condition (A) holds so that with $g(t) = \dfrac{\tau(t)}{\sigma(t)}$ we conclude from Theorem 2.1 that all bounded solutions of

$$y^{\Delta\Delta}+p(t)(y^\sigma(t))^{\gamma_1}+q(t)(y^\tau(t))^{\gamma_2}=0 \tag{3.6}$$

are oscillatory if

$$\int^\infty t\left(p(t)+q(t)(g(t))^{\gamma_2}\right)\Delta t=\infty. \tag{3.7}$$

If $\gamma_1,\gamma_2 > 1$, then with $\phi(u) = u^\gamma$, where $1 < \gamma < \min\{\gamma_1,\gamma_2\}$, it is not difficult to show that $f(t,u,v) = p(t)u^{\gamma_1}+q(t)v^{\gamma_2}$ satisfies condition (H). Therefore, from Theorem 2.5, we conclude that all solutions of (3.6) are oscillatory provided

$$\int^\infty t\left(p(t)\alpha^{\gamma_1}+q(t)(\alpha g(t))^{\gamma_2}\right)\Delta t=\infty \tag{3.8}$$

for all $\alpha \neq 0$. Moreover, (3.8) holds for all $\alpha \neq 0$ if and only if

$$\int^\infty tp(t)\Delta t+\int^\infty tq(t)(g(t))^{\gamma_2}\Delta t=\infty. \tag{3.9}$$

Conversely, if $\liminf g(t) := m_1 > 0$, then (3.9) is necessary for all solutions of (3.6) to be oscillatory.

We next give an illustration of the situation when f involves an advanced argument.

Example 3.3. Suppose that

$$f(t,u,v,w) := \frac{p(t)u^{\gamma_1}+q(t)v^{\gamma_2}}{1+r(t)w^2},$$

where $r(t) \geq 0$ is rd-continuous and $\gamma_1,\gamma_2, p(t), q(t) > 0$. From Theorem 2.1, we conclude that all bounded solutions of

$$y^{\Delta\Delta}+\frac{p(t)(y^\sigma(t))^{\gamma_1}+q(t)(y^\tau(t))^{\gamma_2}}{1+r(t)(y^\xi(t))^2}=0 \tag{3.10}$$

are oscillatory in case

$$\int^{\infty} \frac{t\,(p(t)\alpha^{\gamma_1} + q(t)(\alpha g(t))^{\gamma_2})}{\alpha^2 + r(t)(h(t))^2} \Delta t = \infty \tag{3.11}$$

for all $\alpha \neq 0$. Moreover, (3.10) has a bounded nonoscillatory solution iff

$$\int^{\infty} t f(t, \alpha, \alpha, \alpha) \Delta t < \infty$$

for some $\alpha \neq 0$.

The results in the last two examples may be regarded as extensions of some oscillation criteria due to Atkinson [5].

References

[1] R.P. Agarwal, M. Bohner, and S.H. Saker, Oscillation of second order delay dynamic equations, *Can. Appl. Math. Q.*, 13(1):1–17, 2005.

[2] Ravi Agarwal, Martin Bohner, Donal O'Regan, and Allan Peterson, Dynamic equations on time scales: a survey, *J. Comput. Appl. Math.*, 141(1-2):1–26, 2002. Dynamic equations on time scales.

[3] Ravi P. Agarwal, Donal O'Regan, and S.H. Saker, Oscillation criteria for second-order nonlinear neutral delay dynamic equations, *J. Math. Anal. Appl.*, 300(1):203–217, 2004.

[4] Elvan Akin-Bohner, Martin Bohner, and Samir H. Saker, Oscillation criteria for a certain class of second order Emden-Fowler dynamic equations, *Electron. Trans. Numer. Anal.*, 27:1–12 (electronic), 2007.

[5] F.V. Atkinson, On second-order non-linear oscillations, *Pacific J. Math.*, 5:643–647, 1955.

[6] M. Bohner and S.H. Saker, Oscillation criteria for perturbed nonlinear dynamic equations, *Math. Comput. Modelling*, 40(3-4):249–260, 2004.

[7] M. Bohner and S.H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, *Rocky Mountain J. Math.*, 34(4):1239–1254, 2004.

[8] Martin Bohner and Allan Peterson, *Dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2001. An introduction with applications.

[9] Yeter Şahiner, Oscillation of nonlinear dynamic equations on time scales, *Nonlinear Anal.*, 63(5-7):1073–1080, 2005.

[10] L. Erbe, Oscillation criteria for second order linear equations on a time scale, *Canad. Appl. Math. Quart.*, 9(4):345–375 (2002), 2001.

[11] L. Erbe, A. Peterson, and S.H. Saker, Kamenev-type oscillation criteria for second-order linear delay dynamic equations, *Dynam. Systems Appl.*, 15(1):65–78, 2006.

[12] Lynn Erbe, Oscillation criteria for second order nonlinear delay equations, *Canad. Math. Bull.*, 16:49–56, 1973.

[13] Lynn Erbe and Allan Peterson, Riccati equations on a measure chain, In *Dynamic systems and applications, Vol. 3 (Atlanta, GA, 1999)*, pages 193–199. Dynamic, Atlanta, GA, 2001.

[14] Lynn Erbe and Allan Peterson, Boundedness and oscillation for nonlinear dynamic equations on a time scale, *Proc. Amer. Math. Soc.*, 132(3):735–744 (electronic), 2004.

[15] Lynn Erbe and Allan Peterson, Comparison theorems of Hille–Wintner type for dynamic equations on time scales, *Proc. Amer. Math. Soc.*, 133:3243–3253, 2007.

[16] Lynn Erbe, Allan Peterson, and S.H. Saker, Oscillation criteria for second-order nonlinear dynamic equations on time scales, *J. London Math. Soc. (2)*, 67(3):701–714, 2003.

[17] Stefan Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.*, 18(1-2):18–56, 1990.

[18] S.H. Saker, Oscillation of nonlinear dynamic equations on time scales, *Appl. Math. Comput.*, 148(1):81–91, 2004.

[19] B.G. Zhang and Zhu Shanliang, Oscillation of second-order nonlinear delay dynamic equations on time scales, *Comput. Math. Appl.*, 49(4):599–609, 2005.