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The Marshall Differential Analyzer Project: A Visual Interpretation of Dynamic Equations

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Abstract

The Marshall Differential Analyzer Team is currently constructing a primarily mechanical four integrator differential analyzer from Meccano type components. One of the primary goals of the project is to build a machine that can be used by local mathematics and science educators to teach their charges to analyze a particular function from the perspective of the relationship between its derivatives. In addition, the machine will be used to study nonlinear problems of interest to researchers in the broad field of dynamic equations. This work chronicles the project as well as describes in general how a differential analyzer models a differential equation. In particular, a description of the first phase of construction, our mini two integrator differential analyzer we call Lizzie is presented.

AMS subject classification: 34, 39.
Keywords: Differential analyzer, dynamic equations.

1. Introduction

The Marshall Differential Analyzer Team is a collection of undergraduate and graduate students gathered together with a common goal of studying and constructing a mechanical differential analyzer, a machine designed and first built in the late 1920's to solve nonlinear differential equations that could not be solved by other methods. The first machine was designed and built by Vannevar Bush at M.I.T. and the idea spread to England

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in the early 1930's when Douglas Hartree of the University of Manchester visited Bush to see his machine and returned with the idea to create a working model of Bush's machine from Meccano components (the British version of Erector Set). Arthur Porter, an undergraduate student majoring in physics, took Hartree's idea and ran with it. He constructed the first Manchester machine primarily from Meccano components and used it to solve a variety of problems including the determination of the atomic structure for the chromium atom and a variety of problems in control theory. After completing his Ph.D. thesis, Porter was selected for the Commonwealth Fund Fellowship affording him the opportunity to travel to M.I.T. and work for two years with Dr. Bush on the Rockefeller Differential Analyzer, certainly an opportunity that helped pave the way for what has proved to be an illustrious career that included the Admiralty Research Laboratories in England, industrial work at Ferranti Electric in Canada, and work in the academic realm at the University of Saskatchewan and the University of Toronto.

The Marshall University connection to this historical work began with a visit to the London Science Museum in the summer of 2004 by Lawrence and Brooks. Inspired by a static display of a portion of the finely machined model built for the University of Manchester by the Metropolitan-Vickers Company, Lawrence returned to Marshall with a basic plan to involve students in an in-depth study of the machine. In particular, Lawrence was intrigued by the physical model of a mathematical equation that the machine offers the observer and the educational benefits that could result from this feature.

After some investigation into the history of the machine and those involved in its construction and uses, the Marshall DA Team discovered that a working model of a machine very similar to Porter's had been recently built by electronics engineer Tim Robinson in California. Some discussion with Mr. Robinson revealed that Dr. Arthur Porter, age 93 at this point in time, was currently residing in North Carolina. In the spring of 2005, the Team visited Dr. Porter and learned from the master about the construction of the first machine built in England and the variety of critical problems that Porter and others solved using it. The students (as well as Lawrence and Brooks) were amazed and intrigued and certainly felt that the opportunity to speak with Porter was perhaps a once-in-a-lifetime opportunity. Later that same spring, Dr. and Mrs. Porter visited the Marshall University Campus for an invited lecture about his experiences with the differential analyzer and the prestigious group of scientists with whom he worked and contributed ground breaking results to various fields of science. Again, the audience was intrigued and inspired. The initial inspiration for the Project was Dr. Arthur Porter. More details about his amazing career can be found in his memoirs, [7].

Our first view of a mechanical differential analyzer in motion was at the home of Tim and Lisa Robinson in Boulder Creek, California in the fall of 2005. The Team was thrilled at the opportunity to see Mr. Robinson's four integrator machine construct exclusively of Meccano components. The magnitude of the task of building such a machine of our own literally came to life as we watched the wheels and gears turn according to the mathematics that guided them. Mr. Robinson is the technical advisor and lead designer for the Marshall Differential Analyzer Project. He is our mentor and guru.

Since our first contacts with Dr. Porter and Tim Robinson the discussions about the differential analyzer have come to life in the form of a small two integrator machine. Ultimately, the goal is to build a large machine the size of Porter's first Manchester model from Meccano type components. To study the mechanics required for this large construction project, the mini differential analyzer, known to all as Lizzie, was built with direction from an article found in Meccanoman magazine, [5]. The Team obtained valuable information about the effects of various constants that are inherent to the machine as a result of its particular structure through gear ratios and about the important and troublesome issue of creating enough torque to allow the motion to move through a long string of gears.

In Section 2 we will describe the mechanics of integration as it is modeled by the differential analyzer. The major components of the machine are also described in some detail. Section 3 offers the reader an example how the machine is programmed for a particular differential equation and some discussion of the scope of the complexity of the problems that a differential analyzer can handle. We will conclude this work with some of the experiences the team has had during their travels with Lizzie and plans for the future.

2. Mathematics in Motion

2.1. The Mechanical Integrator

A differential analyzer is a machine designed to model differential equations of up to a limited order prescribed by the general structure of the particular machine. The primary component of the machine is the **mechanical integrator**, composed of a horizontal disk that turns a vertical wheel which is connected to a rod positioned parallel to a line passing through the center of the disk. The turning disk moves under the wheel on a carriage along a track changing the distance the wheel is positioned from the center of the disk. The number of turns of the rod connected to the wheel is determined by the position of the wheel on the disk. (See Figure 1.)

For the sake explanation, assume first that the carriage remains stationary and that the disk turns the wheel at a constant rate. The connection between the wheel and the disk can be viewed as a pair of meshed gears with the number of "teeth" proportional to the relative sizes of the wheel and the circle inscribed by the wheel on the disk as it is turned by the disk. (We will consider this a connection between say a spur gear and a pinion.) In fact, the ratio of the number of "teeth" of the two imagined gears reduces to the ratio between the radius of the circle inscribed by the turning wheel and the radius of the wheel. Denote by b the radius of the circle inscribed by the wheel on the disk, call it gear B, and a the radius of the wheel, call it gear A. Then one turn of gear B yields b/a turns of gear A. Now expand this idea by allowing the number of teeth in gear B to change continuously. This is physically accomplished by moving the carriage along the track by way of a screwed rod known as a lead screw turning in a threaded stationary guide known as a boss.

Clearly, when $a = b$, each turn of B yields a turn of A. Let x denote the number of

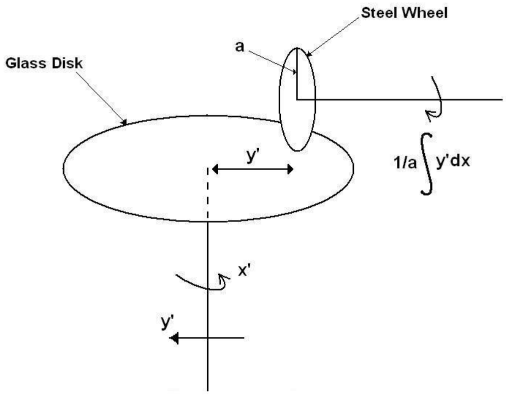


Figure 1

turns of gear B and $y(x)$ the radius of gear B at turn x. Then the total distance the wheel rides on the disk for a prescribed number of turns of the disk is the sum of the arc lengths at varying positions of the wheel on the disk. This is equivalent to

$$\sum_{i=1}^{n} y(x_i)2\pi\,\Delta x_i.$$

To calculate the number of turns of the wheel we divide this sum by $2\pi a$, yielding

$$\frac{1}{a}\sum_{i=1}^{n} y(x_i)\Delta x_i.$$

This sum has all the characteristics of a Riemann sum. As we reduce size of the portions of a turn where the number of turns of the wheel is calculated and allow it to get as small as possible our sum becomes an integral with integrand $y(x)$ multiplied by the constant $1/a$,

$$\frac{1}{a}\int y(x)dx.$$

This mechanical connection and motion gives us a beautiful physical representation of integration, where the value of the integral is measured in rotations of the rod that carries the wheel. When the wheel passes through the center of the disk, the zero position, the motion of the wheel reverses direction. One direction represents positive values and the other negative values.

Now how can we use this form of mechanical integration to find a function knowing information about its derivatives? Consider a simple example: Let the position of the wheel on the disk of Integrator I, measured from the center of the disk, represent the value of the second derivative of the function we seek. Then the number of turns of the wheel on this integrator will represent the value of the first derivative of our desired function (or perhaps a constant multiple of that value). If this motion from the wheel of the first integrator is fed into a second integrator, Integrator II, so that it determines the position of the wheel on the disk, then the motion of this wheel, defines the function we are seeking. So, in general, if the position of the wheel on the disk of a given integrator defines the nth derivative of a desired function, then the rotations of the wheel riding on this disk represents the $(n-1)$st derivative of the desired function. After the motion passes through n integrators we arrive at the function we are interested in. In Section 3 we will look at a particular differential equation and describe how the machine is configured to model the equation and how we can extract a solution from the motion produced.

Note: Note as you read this work that there are many gear ratios that must be taken into account when setting up a particular equation on a particular machine.

2.2. The Torque Amplifier

The motion of the wheel on the integrator is created by the friction between the wheel and the surface of the disk. The torque that is produced by this friction is not adequate enough to allow the motion from the wheel to continue through the various components of the machine, particularly if the set-up requires the use of several integrators. Increasing the friction between the wheel and the disk may at first pondering seem a plausible solution to this crucial issue. However, a delicate balance between enough friction to create the desired torque and little enough friction to allow the disk to move linearly under the wheel must be maintained. The torque needs a boost and this is the task of the **torque amplifier**.

Through the years of development and construction of differential analyzers, a variety of torque amplifiers have been designed and put into use. Porter designed a mechanical torque amplifier for the first Manchester machine that essentially used the same concept as a capstan used to move large boats. Two cylindrical drums spinning in opposite directions are wound with a slightly flexible friction bands. Each band is connected on one end to an input arm connected to the rod that carries the motion from the integrator wheel (input rod) and on the other end to output arm connected to the rod that will carry the motion to the next component on the machine (output rod). (The drums are positioned coaxially with the input and output shafts.) When the input rod turns, the cord is tightened on the associated revolving cylinder amplifying the torque and transferring this motion the output rod. Diagrams of this type of construction can be found in [2] and [6].

This type of torque amplifier is an incredible piece of engineering. However, constructing and maintaining such a component is at best very difficult. The proper amount of tension on the friction bands to allow slippage on the drums at the appropriate times as well as the ability for the bands to grab the drums when appropriate requires fine tuning. The real source of wonder for the differential analyzer is its mechanical components that are connected in such a way to model a mathematical equation. The torque amplifier is a necessary component of the machine but it does not affect the mathematics being modeled so the Team has decided (with valuable advice from Porter and Robinson) to use a servo-type motor to amplify torque on the Marshall Differential Analyzer.

2.3. Other Important Components

Practical use of the differential analyzer requires the ability to combine the output of two or more integrators. For this purpose, we use mechanical adders. The **adder** is a gearbox made of bevel gears connected in such an way to allow the motion from two independent rods to be combined to create the sum of the two motions on a single rod. (Diagrams of the construction of an adder can be found in [2] and [6].) This setup is similar to that of a differential on an automobile. In the case of the adder the motion of two rods drives the motion of a third rod where in the differential case the motion of one driving rod splits and drives two rods or shafts. The adders are located in an avenue of interconnection that runs the length of the machine. The amplified motion from each of the integrators can be connected to an adder to be combined with the the motion of any of the other integrators.

In many cases when our forefathers modeled differential equations on a differential analyzer they were in search of a first glimpse at the solution of the differential equation. It is an amazing even today to watch the machine plot a solution as you note the impact of the each of the integrators involved in the setup. For the purpose of plotting a solution have an **output table**.

The table is fed by two sources of motion that move a pen in the horizontal and the vertical directions. The DA operator has the the freedom to choose the source for the motion in each direction. For example, if a plot of the solution of the differential equation is desired, the operator connects the motion of the independent variable (from the motor that turns the disks) to the horizontal axis and the rod representing the solution values to the vertical axis. In a similar way any derivative represented by a turning rod can be connected to the vertical axis resulting in a plot of that particular derivative. In general, the operator can plot the relation of his/her choice.

It is important to note that the output table is an accessory and not a necessity for operation. The machine is setup to model a particular differential equation according to the interconnections of the integrators and the machine will continue to run happily according to this setup (within the physical limitations of the machine). If a given output is desired the operator taps off the appropriate rods that represent the desired information.

The scope of the problems that can be studied using this fantastic machine can be broadly expanded with the addition of an input table. When the equation to be modeled includes a function that can be described by a function curve, the impact of this data can be entered into the machine using an input table. A graph of the function is placed on the flat surface of the input table. The operator maintains the vertical positioning of a pointer device as the pointer moves along the domain of the function. The source of the horizontal movement is the rod that represents the domain values of the function. For example, if the function depends the independent variable of the solution to the differential equation, then the horizontal motion comes from the motor that drives the disks.

3. Programming and Capabilities of the Machine

The most popular question asked when our mini differential analyzer is viewed for the first time is "How do you know what equation is being solved?" From a broader and proactive perspective, the question becomes "How do we program the machine to model the desired equation?" In this section we describe the machine setup for an equation that describes dampened harmonic motion,

$$y'' = -y - \frac{1}{4}y'.$$

Following the lead of the master, Vannevar Bush, our first step is to draw a schematic that models the equation through the interconnections of the integrators. (See Figure 2.)

The independent variable, x for each of the integrators is run by a motor. Note that since we have a second order equation, we need two integrators to model this equation. Starting at Integrator I, let the distance the wheel sits from the center of the disk on Integrator I denote the value of the function y''. Following the path of the motion of the wheel on the schematic, note that this motion, y', is fed into Integrator II at the lead screw. Then the motion that leaves Integrator II through the wheel represents the value of y.

At this point we have a source for y and y'. Recall that our model equates $-y - \frac{1}{4}y'$ with y''. So starting with y' we obtain $-\frac{1}{4}y'$ by gearing down the motion and changing the sign with a train of gears. Similarly a sign change is applied to y through a gear chain and $-\frac{1}{4}y'$ and $-y$ are combined using an adder. The resulting motion is connected to the lead screw on Integrator I. This connection equates y'', our original designation for the input of Integrator I, with the motion $-y - \frac{1}{4}y'$ and we have our model!

If the operator is interested in a particular set of initial conditions, this information can be incorporated into the setup through the initial positioning of the wheels on the integrator disks.

Ideally the integrator disk as well as the integrator carriage (lead screw) can be fed by any motion produced by the machine. This offers the operator some creative freedom. Suppose we need to integrate a product of two functions, denoted by $f_1(x)$ and $f_2(x)$. (Note that these functions could be the output of an integrator or fed into the machine by way of an input table.) From the identity

$$\int f_1(x)f_2(x)dx = \int f_2(x)(\int f_1(x)dx)$$

we can determine a machine setup for the evaluation of this integral. If the motion that turns the integrator disk is fed by $\int f_1(x)dx$ and the lead screw is fed by the function $f_2(x)$, the output is the desired integral of the product of the two functions.

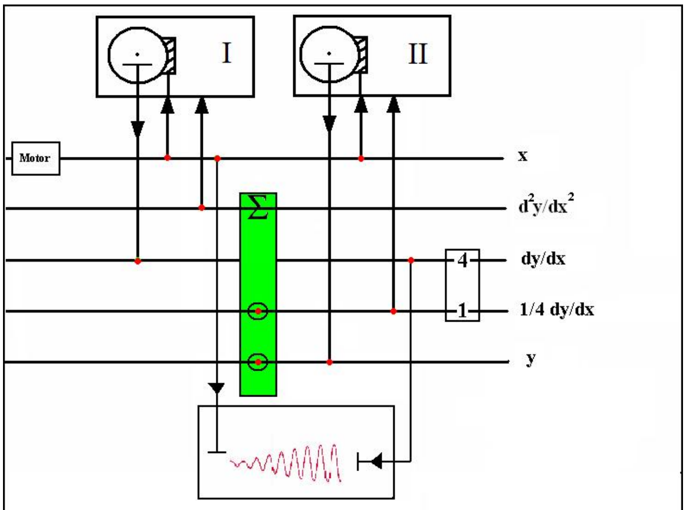


Figure 2

This idea can be directly applied to the calculation of the square of a function. Suppose we want to find the integral of the square P. We start by doubling the value of P with a gear ratio and then note that

$$\int 2PdP = P^2$$

In this particular case we feed both the lead screw and the integrator disk with the motion of P.

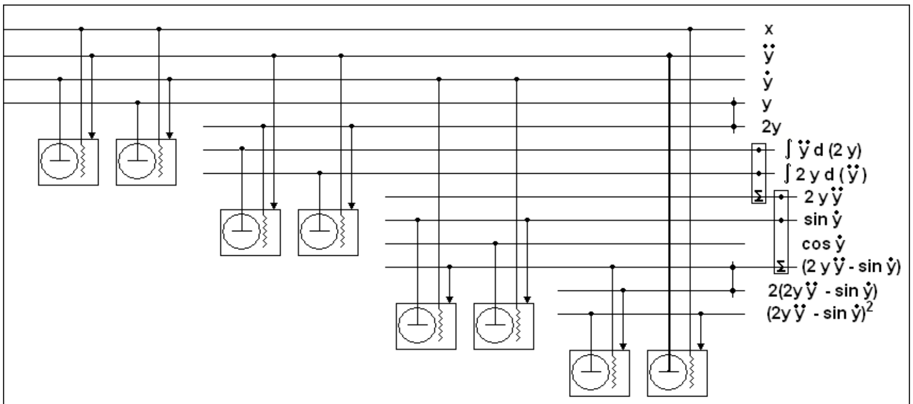
In [2], Dr. Bush presents an intriguing problem that uses this idea of integrating a product extensively. A schematic of the third order extremely nonlinear differential equation

$$y''' = (2yy'' - \sin(y'))^2$$

can be found in Figure 3. A study of this schematic reveals much about the power of the machine. Note how the sine and cosine functions are created using the relationship between the two functions.

4. Lizzie's Travels

Since the completion of the construction of our two integrator mini differential analyzer in Spring of 2006, the Team has had the opportunity to travel with Lizzie and talk about this historical concept of mechanical integration. For her first outing Lawrence and Brooks took her along for an invited lecture at at the University of Missouri - Rolla at the invitation of Dr. Martin Bohner. After the lecture we moved her to the lovely home of Dr. Bohner and Dr. Akin–Bohner for the occasion of a wonderful gala, where she entertained those who were interested.



$$\dddot{y} = (2y\ddot{y} - \sin\dot{y})^2$$

Figure 3

Lizzie's next presentation was at an art show to benefit the Ohio Valley Environmental Coalition in our hometown of Huntington, West Virginia in November of 2006. It was at this show, where at least 400 people visited our display to watch Lizzie model a problem, that the Team fist discovered the broad range ages and educational backgrounds of those intrigued by the machine. The ages ranged from from five to about sixty-five and educational training from elementary school to terminal degrees in a variety of fields. This first outing was an exciting experience for all.

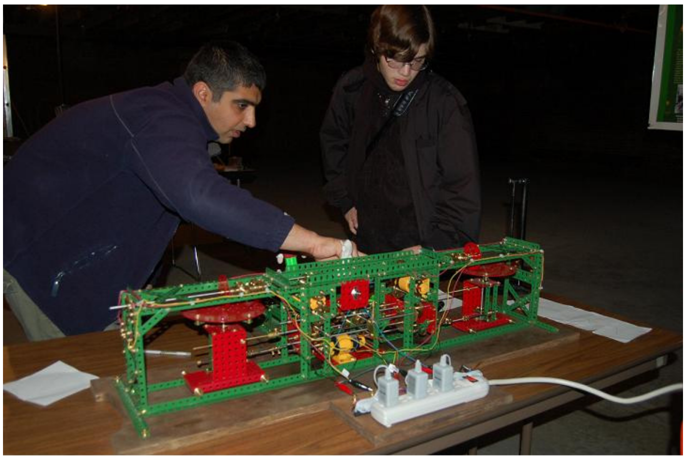
In January of 2007 the Team submitted a poster for the Undergraduate Research Poster Session at the Joint Meetings of the AMS and MAA at the first such meeting after Hurricane Katrina in New Orleans, Louisiana. The Team traveled with Lizzie by van to the conference and displayed her in front of their poster for visitors and judges to investigate. This was her first presentation for a large group of mathematicians and the response was quite rewarding.

In April of the same year, the Team was chosen for the Council of Undergraduate Research's "Poster's on the Hill" in Washington, D.C.. During this visit to "the Hill" the Team had the opportunity to demonstrate the use of Lizzie for modeling equations to Senator John D. Rockefeller, IV. He was very impressed with the Team's work and we were equally pleased to have the opportunity to make the presentation to our Senator.

From her first glimpse of the static display of the Manchester machine at the London Museum, Lawrence had a strong feeling that the machine could continue have an impact on research as well as on mathematics education. Recently a Calculus I class visited her lab and learned about modeling equations with the machine. The response to the visit was very enthusiastic. The Team's goal for the visit was to introduce to these bright young minds to a different perspective on the study of rates of change of a function and how it affects the function's behavior. When the full scale four integrator model is complete, the Team will train in-service and pre-service teachers to oversee the use of the

The Marshall Differential Analyzer Team's visit to the Washington, D.C. office of Senator John D. Rockefeller, IV (D - West Virginia).

Figure 4



Senior Mathematics major Saeed Keshavarzian describes the path of motion for Journalism major, Luke Williams, viewing Lizzie for the first time.

Figure 5

machine by their own students. In 1931 the master, Vannevar Bush, wrote, "Experience is necessary of course, in order to use the device effectively. This is actually one of the most attractive aspects of the machine; one acquires an entirely new appreciation of the innate nature of a differential equation as that experience is gained".

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