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Some Remarks on Classical Lagrangian Symmetric Differential Expressions and their Composite Powers

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Dedicated to Dave Race, our long-time colleague and friend.

Abstract

In this paper, we prove some general results about Lagrangian symmetric ordinary differential expressions $\ell[\cdot]$ of order n when the coefficients of $\ell[\cdot]$ are sufficiently smooth. In particular, for a natural number j, we show that under increased smoothness conditions on the coefficients, the j^{th} composite power $\ell^j[\cdot]$ of $\ell[\cdot]$ is also Lagrangian symmetric. More generally, we show that if w is a symmetry factor for $\ell[\cdot]$, then w is also a symmetry factor for $\ell^j[\cdot]$. Several classical second-order examples are given to illustrate these results and their j^{th} composite powers, for each $j \in \mathbb{N}$, are given explicitly in Lagrangian symmetric form.

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1. Introduction

The general theory of linear, ordinary quasi-differential expressions and, in particular, Lagrangian symmetric quasi-differential expressions has been developed now over many years, starting with the original work of Shin, Glazman, Krein and Naimark; see [18]. In more recent years this subject was advanced by, amongst others, Everitt, Race and Zettl including the definition and properties of the analytical powers of quasi-differential expressions; see [5, 10–12, 20].

In this paper, we return to ask, and answer, some fundamental questions regarding Lagrangian symmetric expressions $\ell[\cdot]$ and their composite powers. Specifically, assuming the coefficients of $\ell[\cdot]$ are sufficiently smooth, are the composite powers of $\ell[\cdot]$ also Lagrangian symmetric? More generally, if $\ell[\cdot]$ is Lagrangian symmetrizable with symmetry factor $w(x)$ (see [14, 16]), are powers of $\ell[\cdot]$ also Lagrangian symmetrizable with the same symmetry factor? In this paper, we show that the answer to both of these questions is in the affirmative. These questions have become increasingly more important during the past few years in connection with the left-definite theory of certain ordinary differential operators. Indeed, as shown in [13], a complete left-definite spectral analysis of certain ordinary differential expressions requires explicit and detailed knowledge of their composite powers in Lagrangian symmetric form.

Since we will assume that the coefficients of our differential expressions are sufficiently smooth, the results that we present in this paper can be considered to be in the realm of the theory of classical ordinary differential expressions as opposed to the more general theory of quasi-differential expressions. Consequently, since the term "Lagrangian symmetric" is a much more general concept (see [5, 12, 20]), we will call expressions in this paper *classical Lagrangian symmetric* expressions or, in short, *classical symmetric* expressions.

As the reader will see later in this paper, it will be necessary to impose heavier differentiability assumptions on the coefficients of the classical symmetric expression in order to discuss various powers of this expression. However, as Everitt and Zettl show in [11] and Zettl shows in [20], this is not necessary when dealing with powers, or products, of general Lagrangian symmetric quasi-differential expressions. Nevertheless the classical theory illuminates many special properties of these powers which can be lost in the quasi-theory due to the generality of the latter methods.

The contents of this paper are as follows. In Section 2, we introduce the various notation that we use in the paper. We also define the differential expression $\ell[\cdot]$, its coefficients, and the domain of $\ell[\cdot]$. This will allow us, in this section, to discuss the classical adjoint expression $\ell^+[\cdot]$ of $\ell[\cdot]$. In Section 3, some important results on classical symmetry are proved; the authors feel that these results are perhaps known to experts in the area but definitive statements and proofs of these classical results are not to be found, as far as we know, in the literature. We use these results to show, in Section 4, that if $\ell[\cdot]$ is classical symmetric and has sufficiently smooth coefficients, then composite powers $\ell^j[\cdot]$ of $\ell[\cdot]$ are also classical symmetric. Section 5 of this paper deals with a review of results established by Littlejohn [14] and Littlejohn and Race [16] on classical Lagrangian symmetrizability of the differential expression $\ell[\cdot]$. Specifically, the authors

in these two papers establish necessary and sufficient conditions for the existence of a function w such that $w\ell[\cdot]$ is classical symmetric; in this case, we say that $\ell[\cdot]$ is classical symmetrizable with symmetry factor w. In Section 5, we generalize the main results obtained in Section 3 to classical symmetrizability. In Section 6, we show that if $\ell[\cdot]$ is classical symmetrizable with symmetry factor w, then under certain conditions on the coefficients of $\ell[\cdot]$, composite powers of $\ell[\cdot]$ are also classical symmetrizable with the same symmetry factor w. Lastly, in Section 7, several examples are discussed to illustrate these results.

2. Notations and Definitions

Throughout this paper, $I = (a, b)$, where $-\infty \leq a < b \leq \infty$, denotes an open interval of the real line $\mathbb{R}$. The set of positive integers is denoted by $\mathbb{N}$, the nonnegative integers is denoted by $\mathbb{N}_0$, and the field of complex numbers is denoted by $\mathbb{C}$. For a complex number $z = x + iy$, where $i = \sqrt{-1}$, the real part of z, $\mathrm{Re}(z)$, is x and the imaginary part of z, $\mathrm{Im}(z)$, is y. If w is a Lebesgue measurable function and is positive almost everywhere on I, $L^2(I; w)$ denotes the usual Hilbert space, with inner product

$$(f, g)_w := \int_I f\overline{g}w dx \quad (f, g \in L^2(I; w)), \tag{2.1}$$

of all Lebesgue measurable functions $f : I \to \mathbb{C}$ such that $\int_I |f|^2 w dx < \infty$. If $w = 1$ for almost all $x \in I$, we write this inner product as (f, g) and, in this case, denote the Hilbert space by $L^2(I)$.

The vector space $AC_{\mathrm{loc}}(I)$ consists of all Lebesgue measurable functions $f : I \to \mathbb{C}$ that are absolutely continuous on all compact subintervals of I. For $r \in \mathbb{N}$, $AC^{(r)}_{loc}(I)$ consists of all those functions f such that $f^{(j)} \in AC_{\mathrm{loc}}(I)$ for $j = 0, 1, \ldots, r-1$, while $AC^{(0)}_{\mathrm{loc}}(I)$ denotes the space of all Lebesgue measurable functions that are Lebesgue integrable on all compact subintervals of I. The space $AC^{(\infty)}_{\mathrm{loc}}(I)$ consists of all those $f \in AC^{(r)}_{\mathrm{loc}}(I)$ for each $r \in \mathbb{N}_0$. Another important vector space for our consideration in this paper is $C_c(I)$, the set of all continuous functions $f : I \to \mathbb{C}$ having compact support in I. Furthermore, for $n \in \mathbb{N}$, $C^{(n)}_c(I)$ denotes the set of all functions $f : I \to \mathbb{C}$ such that $f^{(j)} \in C_c(I)$ for $j = 0, 1, \ldots, n$. It is well known that $C^{(n)}_c(I)$ is a dense subspace of $L^2(I; w)$; we refer the reader to [17, Chapter 1, Section 1] for an excellent account of specific density properties that $C^{(n)}_c(I)$ holds in $L^2(I; w)$. From these definitions, and the mean value theorem, it is not difficult to see that

$$C^{(n)}_c(I) \subset AC^{(n)}_{\mathrm{loc}}(I). \tag{2.2}$$

Another space that will be useful in our analysis in this paper is $\mathcal{P}$, the space of all polynomials $p : I \to \mathbb{C}$.

For $n \in \mathbb{N}$ and, for $k = 0, 1, \ldots, n$, let

$$a_k := \alpha_k + i\beta_k : I \to \mathbb{C} \quad \text{with} \quad a_k \in AC^{(k)}_{\mathrm{loc}}(I). \tag{2.3}$$

We define the ordinary differential expression $\ell[\cdot]$, and its domain $\mathcal{D}(\ell)$, by

$$\ell[y](x) := \sum_{k=0}^{n} a_k(x) y^{(k)}(x) \quad (y \in \mathcal{D}(\ell) := AC^{(n)}_{\mathrm{loc}}(I)). \tag{2.4}$$

Observe, for $y \in AC^{(n)}_{\mathrm{loc}}(I)$, that $\ell[y](x)$ is defined for almost all $x \in I$. The (formal) classical adjoint of $\ell[\cdot]$ and its domain are defined to be

$$\ell^{+}[y](x) := \sum_{k=0}^{n} (-1)^k \left(\overline{a_k(x)} y(x)\right)^{(k)} \quad (y \in \mathcal{D}(\ell^{+}) := AC^{(n)}_{\mathrm{loc}}(I)). \tag{2.5}$$

The differential expression $\ell[\cdot]$ is said to be *classical* (*Lagrangian*) *symmetric* if $\ell[y] = \ell^{+}[y]$ for all $y \in AC^{(n)}_{\mathrm{loc}}(I)$. We note that these expressions are also called "formally self-adjoint". Upon expanding the terms in (2.5), we see that

$$\ell^{+}[y](x) = \sum_{k=0}^{n} a_k^{+}(x) y^{(k)}(x),$$

where

$$a_k^{+}(x) := \sum_{j=0}^{n-k} (-1)^{n-j} \binom{n-j}{n-j-k} \overline{a}_{n-j}^{(n-j-k)}(x) \quad (k = 0, 1, \ldots, n).$$

From this, it is easy to see that $a_k^{+} \in AC^{(k)}_{loc}(I)$; that is to say, the coefficients of $\ell^{+}[\cdot]$ also satisfy the smoothness conditions in (2.3).

This adjoint expression naturally appears in the classical Green formula; indeed, for a compact subinterval $[\alpha, \beta] \subset I$, integration by parts shows that

$$\int_{\alpha}^{\beta} \ell[f](x)\overline{g}(x)dx = [f, g](x) \mid_{\alpha}^{\beta} + \int_{\alpha}^{\beta} f(x)\overline{\ell^{+}[g]}(x)dx \quad (f, g \in AC^{(n)}_{\mathrm{loc}}(I)), \tag{2.6}$$

where $[\cdot, \cdot](\cdot)$: $AC^{(n)}_{\mathrm{loc}}(I) \times AC^{(n)}_{\mathrm{loc}}(I) \times I \to \mathbb{C}$ is the symplectic form (also known as the bilinear concomitant) defined by

$$[f, g](x) := \sum_{j=1}^{n} \sum_{m=1}^{j} (-1)^{m+j} \left\{ \left(a_j(x)\overline{g}^{(j)}(x)\right)^{(j-m)} f^{(m-1)}(x) - \left(a_j(x) f^{(j)}(x)\right)^{(j-m)} \overline{g}^{(m-1)}(x) \right\}.$$

We note that, given $f, g \in AC^{(n)}_{\mathrm{loc}}(I)$, the terms in (2.6) may not, in general, have limits as $\alpha \to a^{+}$ and/or $\beta \to b^{-}$. However, if $f, g \in C_c^{(n)}(I)$, note that $[f, g](a)$ and $[f, g](b)$

both exist and are equal to 0 and both integrals in (2.6) converge as $\alpha \to a^+$ and/or $\beta \to b^-$.

We refer the reader to the authoritative texts of Coddington and Levinson [1, Chapter 3], Dunford and Schwartz [4, Chapter XIII], and Naimark [18, Chapter V] for detailed information on properties of formal ordinary differential expressions. Lesser known sources, but also highly recommended, are the books of Cole [2] and Locker [17].

In this present paper, we are assuming a certain degree of smoothness on the coefficients of our differential expressions. In this regard, we recommend the papers of Everitt [5], Everitt and Race [10], Everitt and Zettl [12], and Zettl [20], where more general assumptions on the coefficients of (quasi) differential expressions are discussed and various connections are made with expressions that we deal with in this paper.

3. Classical Symmetry

Lemma 3.1. Let

$$M[y](x) = \sum_{k=0}^{n} b_k(x) y^{(k)}(x) \quad (x \in I),$$

where $b_k : I \to \mathbb{C}$ and $b_k \in AC^{(k)}_{\text{loc}}(I)$ for $k = 0, 1, \ldots, n$.

(a) Suppose $M[f](x) = 0$ for all $f \in C_c^{(n)}(I)$ and all $x \in I$. Then, for $k = 0, 1, \ldots n$, $b_k(x) = 0$ for all $x \in I$.

(b) Suppose $M[p](x) = 0$ for all $p \in \mathcal{P}$ and all $x \in I$. Then, for $k = 0, 1, \ldots, n$, $b_k(x) = 0$ for all $x \in I$.

Proof. To prove (a), we assume the contrary; that is, suppose that, for some $j \in \{0, 1, \ldots, n\}$, $b_j(x_0) \neq 0$ for some $x_0 \in I$. Then (see [17, Example 1.12, page 12]) there exists $f \in C_c^{\infty}(I) \subset C_c^{(n)}(I)$ such that, for $k \in \{0, 1, \ldots, n\}$,

$$f^{(k)}(x_0) = \begin{cases} 0 & k \neq j \\ 1 & k = j. \end{cases} \tag{3.1}$$

Then

$$M[f](x_0) = \sum_{k=0}^{n} b_k(x_0) f^{(k)}(x_0) = b_j(x_0) \neq 0,$$

contradicting our assumption. The proof of part (b) is almost identical. Indeed, instead of using the above function f, let $p(x) = (x - x_0)^j / j!$. This polynomial also satisfies the interpolation condition (3.1). ∎

Corollary 3.2. Let $\ell[\cdot]$ be the differential expression defined in (2.4) with coefficients satisfying the conditions in (2.3). Suppose either

(a) $\ell[f](x) = \ell^+[f](x)$ for all $x \in I$ and for all $f \in C_c^{(n)}(I)$, or

(b) $\ell[p](x) = \ell^+[p](x)$ for all $x \in I$ and for all $p \in \mathcal{P}$.

Then $\ell[f](x) = \ell^+[f](x)$ for almost all $x \in I$ and for all $f \in AC^{(n)}_{\text{loc}}(I)$. Moreover, condition (a) holds if and only if condition (b) holds.

Proof. From our assumption,

$$M[f](x) = \ell[f](x) - \ell^+[f](x) := \sum_{k=0}^{n} b_k(x) f^{(k)}(x) = 0 \quad (x \in I)$$

for all $f \in C^{(n)}_c(I)$ or for all $f \in \mathcal{P}$. By Lemma 3.1, it follows that $b_k(x) = 0$ for all $x \in I$ and $k = 0, 1, \ldots, n$. Consequently, for any $f \in AC^{(n)}_{\text{loc}}(I)$, $M[f](x) = 0$ for a.e. $x \in I$ which proves the first statement of the theorem. Finally, if (a) holds, then $\ell[f] = \ell^+[f]$ for all $f \in AC^{(n)}_{\text{loc}}(I)$ which implies (b). Similarly, we see that (b) implies (a). ∎

Theorem 3.3. Let $\ell[\cdot]$ be the differential expression defined in (2.4) with coefficients satisfying the conditions in (2.3). Then the following statements are equivalent:

(a) $\ell[\cdot]$ is classical symmetric; that is, $\ell[f](x) = \ell^+[f](x)$ for almost all $x \in I$ and all $f \in AC^{(n)}_{\text{loc}}(I)$;

(b) $\ell[p](x) = \ell^+[p](x)$ for all $x \in I$ and all $p \in \mathcal{B}$, where $\mathcal{B}$ is a basis for $\mathcal{P}$;

(c) $\ell[p](x) = \ell^+[p](x)$ for all $x \in I$ and all $p \in \mathcal{P}$;

(d) $\ell[f](x) = \ell^+[f](x)$ for all $x \in I$ and all $f \in C^{(n)}_c(I)$;

(e) $\displaystyle\int_a^b \ell[f](x)\overline{g}(x)dx = \int_a^b f(x)\overline{\ell[g]}(x)dx$ for all $f, g \in C^{(n)}_c(I)$; that is to say,

$$(\ell[f], g) = (f, \ell[g]) \quad (f, g \in C^{(n)}_c(I)).$$

Proof. Since $\mathcal{P} \subset AC^{(n)}_{\text{loc}}(I)$, the implication (a)⇒(b) is clear. Since both $\ell[\cdot]$ and $\ell^+[\cdot]$ are linear, (b)⇒(c) is also clear. From Corollary 3.2, statements (c) and (d) are equivalent. Suppose the condition in (d) holds. Then, for $f, g \in C^{(n)}_c(I)$, we see from (2.6) that

$$\begin{aligned}\int_a^b \ell[f](x)\overline{g}(x)dx &= [f, g](x)\,|_a^b + \int_a^b f(x)\overline{\ell^+[g]}(x)dx \\ &= \int_a^b f(x)\overline{\ell^+[g]}(x)dx \text{ since } [f, g](a) = [f, g](b) = 0 \\ &= \int_a^b f(x)\overline{\ell[g]}(x)dx \text{ by assumption;}\end{aligned}$$

this establishes (e). To show that (e) implies (a), fix $g \in C_c^{(n)}(I)$. Then, from (2.6) and the fact that $[f, g](a) = [f, g](b) = 0$ for all $f \in C_c^{(n)}(I)$,

$$\int_a^b \ell[f](x)\overline{g}(x)dx = \int_a^b f(x)\overline{\ell[g]}(x)dx = \int_a^b f(x)\overline{\ell^+[g]}(x)dx \quad (f \in C_c^{(n)}(I))$$

and hence

$$\int_a^b f(x)(\overline{\ell[g] - \ell^+[g]})(x)dx = 0 \quad (f \in C_c^{(n)}(I)).$$

From the density of $C_c^{(n)}(I)$ in $L^2(I)$, we conclude that $\ell[g](x) = \ell^+[g](x)$ a.e. $x \in I$. However, since both $\ell[g]$ and $\ell^+[g]$ are continuous functions on I, we see that $\ell[g](x) = \ell^+[g](x)$ for all $x \in I$. Finally, since $g \in C_c^{(n)}(I)$ is arbitrary, we apply Corollary 3.2 (a) to obtain the required result. ■

Remark 3.4. Condition (e) above justifies calling a differential expression of the form (2.4) satisfying any of the equivalent conditions in Theorem 3.3 "symmetric".

It is well known that the most general classical symmetric differential expression $\ell[\cdot]$ of order n of the form (2.4), with coefficients satisfying (2.3), is given by

$$\ell[y] = \sum_{k=0}^{[n/2]} (-1)^k (\alpha_k y^{(k)})^{(k)} + i \sum_{k=0}^{[(n-1)/2]} \left\{ (\beta_k y^{(k)})^{(k+1)} + (\beta_k y^{(k+1)})^{(k)} \right\}.$$

In particular, if each coefficient a_k is real valued and $\ell[\cdot]$ is classical symmetric, then it is necessary for the order n of $\ell[\cdot]$ to be even; that is $n = 2r$. In this case, the most general expression of this type has the form

$$\ell[y] = \sum_{k=0}^{r} (-1)^k (\alpha_k y^{(k)})^{(k)}.$$

For the sake of completeness, we list in Table 1 the most general classical symmetric expressions (2.4) for $n = 1, 2, 3, 4$, under the coefficient assumptions given in (2.3).

Order	Expression
1	$i\beta_1 y' + \left(\alpha_0 + \frac{1}{2} i\beta_1'\right) y$
2	$\alpha_2 y'' + \left(\alpha_2' + i\beta_1\right) y' + \left(\alpha_0 + \frac{1}{2} i\beta_1'\right) y$
3	$i\beta_3 y^{(3)} + \left(\alpha_2 + \frac{3}{2} i\beta_3'\right) y'' + \left(\alpha_2' + i\beta_1\right) y' + \left(\alpha_0 + \frac{1}{2} i\beta_1' - \frac{1}{4}\beta_3^{(3)}\right) y$
4	$\alpha_4 y^{(4)} + \left(2\alpha_4' + i\beta_3\right) y^{(3)} + \left(\alpha_2 + \frac{3}{2} i\beta_3'\right) y'' + \left(\alpha_2' - \alpha_4^{(3)} + i\beta_1\right) y'$ $+ \left(\alpha_0 + \frac{1}{2} i\beta_1' - \frac{1}{4} i\beta_3^{(3)}\right) y$

Table 1: List of classical symmetric differential expressions

4. Composite Powers of Classical Symmetric Differential Expressions

In this section, we assume that the differential expression $\ell[\cdot]$, defined in (2.4), is classical symmetric. For fixed $m \in \mathbb{N}$, we show that under increased smoothness conditions on the coefficients $a_k(\cdot)$ $(k = 0, 1, \ldots, n)$ of $\ell[\cdot]$, the composite powers

$$\ell^1[y] := \ell[y], \ell^2[y] := \ell(\ell[y]), \ldots, \ell^m[y] := \ell(\ell^{m-1}[y])$$

are also classical symmetric. Since the highest derivative term appearing in the coefficient of $y^{(k)}$ in the $(nj)^{th}$-order differential expression $\ell^j[\cdot]$ $(j = 1, 2, \ldots, m)$ is

$$\begin{cases} a_k^{(nj-n)} & \text{if } k = 0, 1, \ldots, n \\ a_n^{(nj-n-r)} & \text{if } k = n + r \ (r = 1, 2, \ldots, nj - n), \end{cases}$$

we assume in this section that

$$a_k \in AC_{\text{loc}}^{(nm-n+k)}(I) \quad (k = 0, 1, \ldots, n). \tag{4.1}$$

Note that, from Section 2, the domain of $\ell^j[\cdot]$ $(j = 1, 2, \ldots, m)$ is defined to be

$$\mathcal{D}(\ell^j) = AC_{\text{loc}}^{(nj)}(I).$$

We are now in position to prove the main result of this section.

Theorem 4.1. Let $\ell[\cdot]$ be the n^{th} order differential expression given in (2.4) with coefficients satisfying the conditions in (4.1). Suppose $\ell[\cdot]$ is classical symmetric. Then, for any $j \in \{1, 2, \ldots, m\}$, $\ell^j[\cdot]$ is classical symmetric.

Proof. Since $C_c^{(nj)}(I) \subset AC_{\text{loc}}^{(nj)}(I)$, it suffices to show, using Theorem 3.3 with $j = 1, 2, \ldots, m$, that

$$(\ell^j[f], g) = (f, \ell^j[g]) \quad (f, g \in C_c^{(nj)}(I)),$$

where we recall that $(\cdot, \cdot)$ is the inner product in $L^2(I)$ given by

$$(f, g) = \int_a^b f(x)\overline{g}(x)dx \quad (f, g \in L^2(I)).$$

We prove this theorem by induction on j. The case $j = 1$ is evident so we begin with $j = 2$. Let $f, g \in C_c^{(2n)}(I) \subset \mathcal{D}(\ell^2)$. Then

$$\begin{aligned} (\ell^2[f], g) &= (\ell(\ell[f]), g) \text{ since } \ell[f] \in C_c^{(n)}(I) \subset \mathcal{D}(\ell) \\ &= (\ell[f], \ell[g]) \text{ since } g \in C_c^{(2n)}(I) \subset C_c^{(n)}(I) \subset \mathcal{D}(\ell) \text{ and } \ell[\cdot] \text{ is symmetric} \\ &= (f, \ell^2[g]) \text{ since } \ell[g] \in C_c^{(n)}(I) \subset \mathcal{D}(\ell) \text{ and } \ell[\cdot] \text{ is symmetric.} \end{aligned}$$

Hence, from Theorem 3.3, we see that $\ell^2[\cdot]$ is classical symmetric. Assume for some $j \in \{1, 2, \ldots, m-1\}$ that $\ell^j[\cdot]$ is classical symmetric. We now show that $\ell^{j+1}[\cdot]$ is classical symmetric. Let $f, g \in C_c^{((j+1)n)}(I)$; then

$$\begin{aligned}(\ell^{j+1}[f], g) &= (\ell(\ell^j[f]), g) \text{ since } \ell^j[f] \in C_c^{(n)}(I) \subset \mathcal{D}(\ell)\\ &= (\ell^j[f], \ell[g]) \text{ since } g \in C_c^{((j+1)n)}(I) \subset C_c^{(n)}(I) \subset \mathcal{D}(\ell) \text{ and}\\ &\quad\ \ell[\cdot] \text{ is symmetric}\\ &= (f, \ell^j(\ell[g])) \text{ since } f, \ell[g] \in C_c^{(nj)}(I) \subset \mathcal{D}(\ell^j) \text{ and } \ell^j[\cdot] \text{ is symmetric}\\ &= (f, \ell^{j+1}[g]).\end{aligned}$$

This completes the proof of the theorem. ■

The proof of the following corollary follows immediately from Theorem 4.1 and the definition of the space $AC_{\text{loc}}^{(\infty)}(I)$.

Corollary 4.2. If $\ell[\cdot]$, the n^{th} order differential expression defined in (2.4), is classical symmetric and each of the coefficients of $\ell[\cdot]$ belong to $AC_{\text{loc}}^{(\infty)}(I)$, then for any $j \in \mathbb{N}$, the j^{th} composite power $\ell^{(j)}[\cdot]$ is also classical symmetric.

5. Lagrangian Symmetrizability

It is well known that every real second-order differential expression

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) \quad (x \in I),$$

with sufficiently smooth coefficients and where $a_2(x) \neq 0$ for $x \in I$, can be made classical symmetric when multiplied by the integrating factor

$$\exp\left(\int_{x_0}^{x} \frac{a_1(t)dt}{a_2(t)}\right)/a_2(x) \quad (x_0 \in I \text{ is fixed; } x \in I).$$

However, for $n > 2$, not every differential expression $\ell[\cdot]$ of order n can be multiplied by a function to make it classical symmetric. Which differential expressions can be? This was a question that Littlejohn [14] and Littlejohn and Race [16] answered; we discuss these results in this section and then offer a generalization of Theorem 4.1.

Let $\ell[\cdot]$ be the n^{th} order differential expression defined in (2.4); in this section, we assume that the coefficients of $\ell[\cdot]$ satisfy the conditions

$$a_k := \alpha_k + i\beta_k : I \to \mathbb{C};\ \frac{a_k}{a_n} \in AC_{\text{loc}}^{(k)}(I) \quad (k = 0, 1, \ldots, n-1). \tag{5.1}$$

We say that a function $w : I \to \mathbb{C}$ is a *symmetry factor* for $\ell[\cdot]$ if w satisfies the conditions

$$\begin{cases}\text{(i) } wa_n \in AC_{loc}^{(n)}(I),\\ \text{(ii) } w(x)a_n(x) \neq 0\ (x \in I), \text{ and}\\ \text{(iii) } w\ell[y] = (w\ell)^+[y]\ (y \in AC_{loc}^{(n)}(I)); \text{ that is to say, } w\ell[\cdot] \text{ is classical symmetric.}\end{cases} \tag{5.2}$$

In this case, we also say that $\ell[\cdot]$ is *classical* (*Lagrangian*) *symmetrizable with symmetry factor* w.

Remark 5.1. Observe that, under the conditions given in (5.1), each function $wa_k = (wa_n)(a_k/a_n) \in AC^{(k)}_{\text{loc}}(I)$ for $k = 0, 1, \ldots, n$. In particular, the coefficients of $w\ell[\cdot]$ satisfy the conditions in (2.3).

In [16, Theorem 4.7], the authors prove the following theorem.

Theorem 5.2. Let $n = 2r$ or $n = 2r - 1$ and let $\ell[\cdot]$ be the differential expression of order n defined in (2.4) with the coefficients satisfying the conditions in (5.1). Suppose $w : I \to \mathbb{C}$ satisfies conditions (i) and (ii) in (5.2). In addition, assume that $i^n a_n$ is real valued on I. Then w is (necessarily) a real symmetry factor for $\ell[\cdot]$ (equivalently, w satisfies condition (iii) of (5.2)) if and only if w simultaneously satisfies the following systems of "symmetry" equations almost everywhere on I:

$$\sum_{j=0}^{n-2k-1} (-1)^{j+1} \binom{2k+j+1}{2k+1} (w\alpha_{2k+j+1})^{(j)} - w\alpha_{2k+1} = 0 \quad (k = 0, 1, \ldots, r-1) \tag{5.3}$$

and

$$\sum_{j=0}^{n-2k} (-1)^{j+1} \binom{2k+j}{2k} (w\beta_{2k+j})^{(j)} - w\beta_{2k} = 0 \quad (k = 0, 1, \ldots, r-1). \tag{5.4}$$

Moreover, up to a nonzero constant multiple, w is necessarily given by

$$w(x) = \begin{cases} \dfrac{1}{\alpha_n(x)} \exp\left(\dfrac{2}{n}\displaystyle\int_{x_0}^{x} \frac{\alpha_{n-1}(t)}{\alpha_n(t)} dt\right) & \text{if } n = 2r \\ \dfrac{1}{\beta_n(x)} \exp\left(\dfrac{2}{n}\displaystyle\int_{x_0}^{x} \frac{\beta_{n-1}(t)}{\beta_n(t)} dt\right) & \text{if } n = 2r-1, \end{cases} \tag{5.5}$$

where $x_0 \in I$ is arbitrary. In particular, we can choose w such that $w(x) > 0$ for all $x \in I$.

Remark 5.3. When $n = 2r - 1$ and $k = r - 1$, the corresponding symmetry equation in (5.3) disappears. Consequently, for either even or odd n, there are always a total of n symmetry equations appearing in (5.3) and (5.4).

Remark 5.4. The requirement that $i^n a_n$ be a real-valued function on I is not restrictive. Indeed, if $i^n a_n$ is not real valued, we can multiply the expression $\ell[\cdot]$ by $\pm i^n \overline{a}_n$. The new expression has leading coefficient $b_n = \pm i^n \overline{a}_n a_n$; note that $i^n b_n$ is now real valued.

If the coefficients of $\ell[\cdot]$ are real valued on I, it is easy to see that in order for $\ell[\cdot]$ to be classical symmetric, it is necessary that the order n of $\ell[\cdot]$ is even. In this case, the above theorem has a further simplification; see [16, Theorem 5.4].

Theorem 5.5. Let

$$\ell[y](x) = \sum_{k=0}^{2n} \alpha_k(x) y^{(k)}(x), \tag{5.6}$$

where $\alpha_k : I \to \mathbb{R}$ is such that $\alpha_k/\alpha_{2n} \in AC^{(k)}_{\text{loc}}(I)$ for $k = 0, 1, \ldots, 2n-1$. Suppose $w : I \to \mathbb{R}$ is such that $w\alpha_{2n} \in AC^{(2n)}_{\text{loc}}(I)$ and $w(x)\alpha_{2n}(x) \neq 0$ for all $x \in I$. Then w is a symmetry factor for $\ell[\cdot]$ if and only if w simultaneously satisfies the following system of symmetry equations almost everywhere on I:

$$\sum_{i=2k+1}^{2n} (-1)^i \binom{i-k-1}{k} (w(x)\alpha_i(x))^{(i-2k-1)} = 0 \quad (k = 0, 1, \ldots, n-1). \tag{5.7}$$

Furthermore, w is necessarily a nonzero constant multiple of

$$w(x) = \exp\left(\frac{1}{n}\int_{x_0}^{x} \frac{\alpha_{2n-1}(t)}{\alpha_{2n}(t)} dt\right) \Big/ \alpha_{2n}(x), \tag{5.8}$$

where $x_0 \in I$ is arbitrary. In particular, we can choose w such that $w(x) > 0$ for all $x \in I$.

Remark 5.6. As in Remark 5.1, the coefficients of $w\ell[\cdot]$, under the assumptions of Theorem 5.5, satisfy the conditions given in (2.3).

Remark 5.7. Observe that there are a total of n symmetry equations in (5.7) when the differential expression $\ell[\cdot]$ is of order $2n$ and has real coefficients; in fact, these symmetry equations are of orders $1, 3, \ldots, 2n-1$. The first-order symmetry equation is given by

$$\alpha_{2n}(x)w'(x) + (\alpha'_{2n}(x) - \alpha_{2n-1}(x))w(x) = 0. \tag{5.9}$$

In particular, we see that for second-order real differential expressions, there is only one symmetry equation to solve; this explains why second-order real differential expressions (with, of course, sufficiently smooth coefficients) are always classical symmetrizable and this also explains why higher order differential expressions may not be classical symmetrizable. Indeed, for $n > 1$, there may not be a simultaneous solution to the system (5.7).

Example 5.8. If the coefficient assumptions given in (5.1) are relaxed, it is possible that even a real second-order differential expression does not have a nontrivial continuous symmetry factor. Indeed, let $\ell[y](x) = \alpha_2(x)y''(x) + \alpha_1(x)y'(x)$, where

$$\alpha_2(x) = \begin{cases} x^2 & \text{if } -1 < x < 0 \\ 0 & \text{if } 0 \leq x < 1, \end{cases} \qquad \alpha_1(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } 0 \leq x < 1. \end{cases}$$

By solving (5.9), we obtain the symmetry factor

$$w(x) = \begin{cases} c/x^2 & \text{if } -1 < x < 0 \\ 0 & \text{if } 0 \leq x < 1. \end{cases}$$

In order for w to be continuous on $(-1, 1)$, it is necessary that $c = 0$; consequently, there is no nontrivial continuous symmetry factor to this second-order differential equation. Notice in this example that $\alpha_1/\alpha_2 \notin AC^{(1)}_{\text{loc}}(-1, 1)$.

Example 5.9. Let $\ell[y](x) = \alpha_2(x)y''(x) + \alpha_1(x)y'(x) + \alpha_0(x)y(x)$, where

$$\alpha_2(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 2x - 1 & \text{if } 1 \le x < 2, \end{cases} \qquad \alpha_1(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \le x < 2, \end{cases}$$

and $\alpha_0 : (0, 2) \to \mathbb{R}$ be such that $\alpha_0/\alpha_2 \in AC^{(0)}_{\text{loc}}(0, 2)$. Any symmetry factor of $\ell[\cdot]$ is necessarily a nonzero constant multiple of

$$w(x) = \begin{cases} 1/x & \text{if } 0 < x < 1 \\ 1/\sqrt{2x - 1} & \text{if } 1 \le x < 2. \end{cases}$$

Remark 5.10. We note that, in [14, 16], the authors obtained a different, but equivalent, set of n symmetry equations that involved the classical Bernoulli numbers.

Remark 5.11. Suppose the eigenvalue equation $\ell[y] = \lambda y$, where $\ell[\cdot]$ is given in (5.6), has a sequence of polynomial solutions $\{p_m\}_{m=0}^{\infty}$ $(\deg(p_m) = m)$ with corresponding eigenvalues $\{\lambda_m\}_{m=0}^{\infty}$. In addition, suppose $\{p_m\}_{m=0}^{\infty}$ is an orthogonal polynomial sequence. By solving the system (5.7) distributionally, Littlejohn [15] showed that an orthogonalizing weight distribution for $\{p_m\}_{m=0}^{\infty}$ is produced; see also [16] for further details.

Lastly, in this section, we generalize Theorem 3.3 from Section 3.

Theorem 5.12. Let $\ell[\cdot]$ be the n^{th} order differential expression given in (2.4) with coefficients satisfying the conditions given in (5.1). Suppose $w : I \to \mathbb{R}$ is such that w satisfies conditions (i) and (ii) of (5.2) and $w > 0$ on I. In addition, assume that $i^n a_n$ is real valued on I. Then the following statements are equivalent:

(a) $\ell[\cdot]$ is classical symmetrizable with symmetry factor w;

(b) w simultaneously satisfies the n symmetry equations given in (5.3) and (5.4);

(c) $(w\ell)[p] = (w\ell)^+[p]$ for all $p \in \mathcal{B}$, where $\mathcal{B}$ is a basis for $\mathcal{P}$;

(d) $(w\ell)[p] = (w\ell)^+[p]$ for all $p \in \mathcal{P}$;

(e) $(w\ell)[f] = (w\ell)^+[f]$ for all $f \in C_c^{(n)}(I)$;

(f) $(\ell[f], g)_w = (f, \ell[g])_w$ for all $f, g \in C_c^{(n)}(I)$, where $(\cdot, \cdot)_w$ is the inner product defined in (2.1).

Moreover, if any one of the conditions (a)–(f) hold, then w is necessarily a (positive) multiple of the function given in (5.5).

Proof. The equivalence of (a) and (b) follows from Theorem 5.2. The rest of the proof follows immediately from Theorem 3.3 (see Remark 5.1). ■

The next theorem gives an analogous result for real differentiable expressions linking Theorem 5.5 with the results from Section 3.

Theorem 5.13. Let $\ell[\cdot]$ be the $2n^{th}$ order real differential expression given in (5.6). Suppose the coefficients of $\ell[\cdot]$ and the function $w > 0$ satisfy the conditions given in Theorem 5.5. Then the following statements are equivalent:

(a) $\ell[\cdot]$ is classical symmetrizable with symmetry factor w;

(b) w simultaneously satisfies the n symmetry equations given in (5.7);

(c) $(w\ell)[p] = (w\ell)^+[p]$ for all $p \in \mathcal{B}$, where $\mathcal{B}$ is a basis for $\mathcal{P}$;

(d) $(w\ell)[p] = (w\ell)^+[p]$ for all $p \in \mathcal{P}$;

(e) $(w\ell)[f] = (w\ell)^+[f]$ for all $f \in C_c^{(2n)}(I)$;

(f) $(\ell[f], g)_w = (f, \ell[g])_w$ for all $f, g \in C_c^{(2n)}(I)$, where $(\cdot, \cdot)_w$ is the inner product defined in (2.1).

Moreover, if any one of the conditions (a)–(f) hold, then w is necessarily a (positive) multiple of the function given in (5.8).

The importance of classical symmetrizable real differential expressions is evident in the theory of self-adjoint differential operators; for example, see [18, Chapter V]. Indeed, this theory normally begins with a classical symmetrizable differential expression on I (with symmetry factor w) from which the minimal and maximal operators in $L^2(I; w)$ are defined, and then self-adjoint extensions of the minimal operator in $L^2(I; w)$ are determined. We remark that it is possible that a real differential expression may not be classical symmetrizable (in the terminology of this paper) and yet it can still generate a self-adjoint operator! Indeed, in [6], the authors produce a fourth-order real differentiable expression that is not classical symmetrizable but it still generates a self-adjoint operator in a certain Hilbert–Sobolev space. Specifically, the fourth-order nonclassical symmetrizable expression is given by

$$\ell_4[y](x) := (x^2-1)^2 y^{(4)} + 4x(x^2-1)y''' + 2(x-1)\left[(1+2A)x + 2A + 3\right] y'' + y \quad (x \in (-1, 1)).$$

In this case, the inner product is not of the form (2.1); indeed, the inner product is a Sobolev inner product of the form

$$\phi(f, g) = \int_{\mathbb{R}} f(x)\overline{g}(x)w_1(x)dx + \int_{\mathbb{R}} f'(x)\overline{g}'(x)w_2(x)dx.$$

With respect to this inner product, it is the case, however, that $\phi(\ell_4[p], q) = \phi(p, \ell_4[q])$ for all $p, q \in \mathcal{P}$; that is to say, $\ell_4[\cdot]$ is *Sobolev symmetrizable* with respect to this inner product.

6. Composite Powers of Classical Symmetrizable Differential Expressions

The last two results of this paper deal with generalizations of Theorem 4.1.

Theorem 6.1. Let $m \in \mathbb{N}$ and suppose $\ell[\cdot]$, the n^{th} order differential expression given in (2.4), is classical symmetrizable with positive symmetry factor $w : I \to \mathbb{R}$ and with coefficients $a_k : I \to \mathbb{C}$ satisfying the conditions:

(a) $\dfrac{a_k}{a_n} \in AC_{\text{loc}}^{(nm-n+k)}(I) \quad (k = 0, 1, \ldots, n-1)$;

(b) $wa_n \in AC_{\text{loc}}^{(nm)}(I)$;

(c) $w(x)a_n(x) \neq 0$ for all $x \in I$ and $i^n a_n$ is real valued on I.

Let $j \in \{1, 2, \ldots, m\}$. Then $\ell^j[\cdot]$, the j^{th} composite power of $\ell[\cdot]$, is classical symmetrizable with symmetry factor w. Moreover, if the conditions (a) and (b) are replaced by, respectively,

(a)$'$ $\dfrac{a_k}{a_n} \in AC_{\text{loc}}^{(\infty)}(I) \quad (k = 0, 1, \ldots, n-1)$;

(b)$'$ $wa_n \in AC_{\text{loc}}^{(\infty)}(I)$,

then for any $j \in \mathbb{N}$, the j^{th} composite power of $\ell[\cdot]$ is classical symmetrizable with symmetry factor w. Moreover, w is necessarily a multiple of the function given in (5.5).

Theorem 6.2. Let $m \in \mathbb{N}$ and suppose $\ell[\cdot]$, the $2n^{th}$ order real differential expression given in (5.6), is classical symmetrizable with positive symmetry factor $w : I \to \mathbb{R}$ and with coefficients $\alpha_k : I \to \mathbb{R}$ satisfying the conditions:

(a) $\dfrac{\alpha_k}{\alpha_{2n}} \in AC_{\text{loc}}^{(2nm-2n+k)}(I) \quad (k = 0, 1, \ldots, 2n-1)$;

(b) $w\alpha_{2n} \in AC_{\text{loc}}^{(2nm)}(I)$;

(c) $w(x)\alpha_{2n}(x) \neq 0$ for all $x \in I$.

Let $j \in \{1, 2, \ldots, m\}$. Then $\ell^j[\cdot]$, the j^{th} composite power of $\ell[\cdot]$, is classical symmetrizable with symmetry factor w. Moreover, if the conditions (a) and (b) are replaced by, respectively,

(a)′ $\dfrac{\alpha_k}{\alpha_{2n}} \in AC^{(\infty)}_{\mathrm{loc}}(I) \quad (k = 0, 1, \ldots, 2n-1)$;

(b)′ $w\alpha_{2n} \in AC^{(\infty)}_{\mathrm{loc}}(I)$,

then, for any $j \in \mathbb{N}$, the j^{th} composite power of $\ell[\cdot]$ is classical symmetrizable with symmetry factor w. Moreover, w is necessarily a multiple of the function given in (5.8).

7. Examples

We now give several examples to illustrate the main results in this paper. The methods that we use to compute the composite powers of the expressions below require explicit knowledge of the solutions of the associated differential equations. Indeed, in the examples considered below, the theory of special functions, and in particular the theory of orthogonal polynomials, plays a key role in determining these powers in classical symmetric form. We refer the reader to the classic text [19, Chapters IV and V] of Szegö for specific properties of the Jacobi, Laguerre and Hermite polynomials which we use in the following examples.

Example 7.1. Laguerre's classical second-order differential expression is defined to be

$$\begin{aligned}\ell_L[y](x) &:= -xy''(x) + (-1-\alpha+x)y'(x)\\ &= \frac{1}{x^\alpha e^{-x}}\left(-x^{\alpha+1}e^{-x}y'(x)\right)' \quad (x \in (0,\infty));\end{aligned} \tag{7.1}$$

here α is, in general, a complex parameter but, for classical reasons, we assume $\alpha > -1$. Notice, from (7.1), that $w(x) = x^\alpha e^{-x}$ is a symmetry factor for $\ell_L[\cdot]$. Consequently, since the coefficients of $\ell_L[\cdot]$ are polynomials, Theorem 6.2 guarantees that each integral composite power $\ell_L^n[\cdot]$ $(n \in \mathbb{N})$ is classical symmetrizable with symmetry factor $w(x) = x^\alpha e^{-x}$. What are these explicit powers? We now outline how to compute these powers; a more detailed analysis can be found in [13] together with their importance to left-definite spectral theory. The Laguerre equation is important in many areas of mathematics and applied mathematics primarily because it has the Laguerre polynomials $\{L_m^\alpha(x)\}_{m=0}^\infty$ as solutions; specifically

$$\ell_L[L_m^\alpha](x) = mL_m^\alpha(x) \quad (m \in \mathbb{N}_0), \tag{7.2}$$

where

$$L_m^\alpha(x) := \left(\frac{1}{\Gamma(\alpha+1)\binom{m+\alpha}{m}}\right)^{1/2} \sum_{j=0}^{m} \frac{(-1)^j}{j!}\binom{m+\alpha}{m-j}x^j \quad (m \in \mathbb{N}_0).$$

These polynomials form a complete orthonormal set in the Hilbert space $L^2((0,\infty); x^\alpha e^{-x})$ with inner product

$$(f,g)_L := \int_0^\infty f(x)\overline{g}(x)x^\alpha e^{-x}dx \quad (f, g \in L^2((0,\infty); x^\alpha e^{-x}));$$

that is to say,

$$(L_m^\alpha, L_r^\alpha)_L = \delta_{m,r} \quad (m, r \in \mathbb{N}_0), \tag{7.3}$$

where $\delta_{m,r}$ is the Kronecker delta symbol. Like the other so-called classical polynomials of Jacobi and Hermite, the derivatives of Laguerre polynomials are also Laguerre polynomials. Indeed, a key formula in deriving the n^{th} composite power of the Laguerre expression (7.1) is

$$\frac{d^j(L_m^\alpha(x))}{dx^j} = (-1)^j \left(\frac{m!}{(m-j)!}\right)^{1/2} L_{m-j}^{\alpha+j}(x) \quad (m, j \in \mathbb{N}_0). \tag{7.4}$$

Moreover, from (7.3) and (7.4), it follows that

$$\int_0^\infty \frac{d^j(L_m^\alpha(x))}{dx^j} \frac{d^j(L_r^\alpha(x))}{dx^j} x^{\alpha+j} e^{-x} dx = \frac{m!}{(m-j)!} \delta_{m,r} \quad (m, r \in \mathbb{N}_0). \tag{7.5}$$

Another result that is essential in the explicit determination of the classical symmetric form of $\ell_L^n[\cdot]$ $(n \in \mathbb{N})$ is the identity

$$\sum_{j=1}^{2n} S_n^{(j)} \frac{m!}{(m-j)!} = m^n \quad (n, m \in \mathbb{N},\ 1 \le m \le 2n), \tag{7.6}$$

where $S_n^{(j)}$ is the Stirling number of the second kind (see [3, Chapter V]), defined by

$$S_n^{(j)} := \sum_{k=0}^{j} \frac{(-1)^{k+j}}{j!} \binom{j}{k} k^n. \tag{7.7}$$

For each $n \in \mathbb{N}$, we see from (7.2) that

$$\ell_L^n[L_m^\alpha](x) = m^n L_m^\alpha(x) \quad (m \in \mathbb{N}_0; x \in (0, \infty));$$

consequently, from (7.3), it follows that

$$\int_0^\infty \ell_L^n[L_m^\alpha](x) L_r^\alpha(x) x^\alpha e^{-x} dx = m^n \delta_{m,r}. \tag{7.8}$$

Moreover, from (7.5), we see that

$$\sum_{j=1}^{2n} S_n^{(j)} \int_0^\infty (L_m^\alpha(x))^{(j)} (L_r^\alpha(x))^{(j)} x^{\alpha+j} e^{-x} dx = \sum_{j=1}^{2n} S_n^{(j)} \frac{m!}{(m-j)!} \delta_{m,r}. \tag{7.9}$$

Hence, combining (7.6), (7.8) and (7.9), we see that

$$\int_0^\infty \ell_L^n[L_m^\alpha](x) L_r^\alpha(x) x^\alpha e^{-x} dx = \sum_{j=1}^{2n} S_n^{(j)} \int_0^\infty (L_m^\alpha(x))^{(j)} (L_r^\alpha(x))^{(j)} x^{\alpha+j} e^{-x} dx. \tag{7.10}$$

Identity (7.10) suggests the form of $\ell_L^n[\cdot]$; indeed, we now prove that $\ell_L^n[y] = \widetilde{\ell}[y]$ $(y \in AC_{loc}^{(n)}(0,\infty))$, where

$$\widetilde{\ell}[y](x) := \frac{1}{x^\alpha e^{-x}} \sum_{j=1}^{2n} (-1)^j S_n^{(j)} \left(x^{\alpha+j} e^{-x} y^{(j)}(x) \right)^{(j)}. \tag{7.11}$$

Through integration by parts, we find that

$$\begin{aligned} &\int_0^\infty \widetilde{\ell}[L_m^\alpha](x) L_r^\alpha(x) x^\alpha e^{-x} dx \\ &\quad = \sum_{j=1}^{2n} \sum_{k=1}^{j} (-1)^{k+j+1} S_n^{(j)} \left(x^{\alpha+j} e^{-x} (L_m^\alpha(x))^{(j)} \right)^{(j-k)} (L_r^\alpha(x))^{(k-1)} \Big|_0^\infty \\ &\qquad + \sum_{j=1}^{2n} S_n^{(j)} \int_0^\infty (L_m^\alpha(x))^{(j)} (L_r^\alpha(x))^{(j)} x^{\alpha+j} e^{-x} dx. \end{aligned} \tag{7.12}$$

It is not difficult to check that, for $j \in \{1, 2, \ldots, 2n\}$ and $k \in \{1, 2, \ldots, j\}$,

$$\left(x^{\alpha+j} e^{-x} (L_m^\alpha(x))^{(j)} \right)^{(j-k)} = x^{\alpha+1} e^{-x} p_{j,k}(x)$$

for some polynomial $p_{j,k}(x)$; in particular, since $\alpha > -1$, we see that

$$\left(x^{\alpha+j} e^{-x} (L_m^\alpha(x))^{(j)} \right)^{(j-k)} \Big|_{x=0} = \left(x^{\alpha+j} e^{-x} (L_m^\alpha(x))^{(j)} \right)^{(j-k)} \Big|_{x=\infty} = 0,$$

and hence, from (7.12), we see that

$$\begin{aligned} &\int_0^\infty \widetilde{\ell}[L_m^\alpha](x) L_r^\alpha(x) x^\alpha e^{-x} dx \\ &\quad = \sum_{j=1}^{2n} S_n^{(j)} \int_0^\infty (L_m^\alpha(x))^{(j)} (L_r^\alpha(x))^{(j)} x^{\alpha+j} e^{-x} dx. \end{aligned} \tag{7.13}$$

Comparing (7.10) and (7.13), we see that

$$(\ell_L^n[L_m^\alpha] - \widetilde{\ell}[L_m^\alpha], L_r^\alpha)_L = 0 \quad (m, r \in \mathbb{N}_0).$$

From the completeness of the Laguerre polynomials $\{L_r^\alpha\}_{r=0}^\infty$ in $L^2((0,\infty); x^\alpha e^{-x})$, we see that

$$x^\alpha e^{-x} \ell_L^n[L_m^\alpha](x) = x^\alpha e^{-x} \widetilde{\ell}[L_m^\alpha](x) \quad (x \in (0,\infty),\ m \in \mathbb{N}_0),$$

and hence, from Lemma 3.1(b) and (7.11), we see that

$$\ell_L^n[y](x) = \widetilde{\ell}[y](x) = \frac{1}{x^\alpha e^{-x}} \sum_{j=1}^{2n} (-1)^j S_n^{(j)} \left(x^{\alpha+j} e^{-x} y^{(j)}(x)\right)^{(j)} \\ (x \in (0,\infty),\ n \in \mathbb{N}). \tag{7.14}$$

We remark that the formula in (7.14) is actually valid for all $\alpha \in \mathbb{R}$. Moreover, we note that the formula in (7.14) yields a new application of the classical Stirling numbers of the second kind.

Example 7.2. The classical second-order Hermite differential expression is given by

$$\ell_H[y](x) := -y'' + 2xy' \\ = \frac{1}{\exp(-x^2)} \left(-\exp(-x^2) y'(x)\right)' \quad (x \in (-\infty,\infty)). \tag{7.15}$$

For each $m \in \mathbb{N}_0$, the m^{th} degree Hermite polynomial $H_m(x)$ is a solution of

$$\ell_H[y](x) = 2my(x);$$

the sequence of Hermite polynomials $\{H_m\}_{m=0}^\infty$ forms a complete orthogonal set in the Hilbert space $L^2((-\infty,\infty); \exp(-x^2))$ with inner product

$$(f,g)_H = \int_{-\infty}^{\infty} f(x)\overline{g}(x) \exp(-x^2) dx.$$

From (7.15), we see that the function $w(x) = \exp(-x^2)$ is a symmetry factor for $\ell_H[\cdot]$; hence using Theorem 6.2, it follows that each integral power $\ell_H^n[\cdot]$ is also classical symmetrizable with symmetry factor $w(x) = \exp(-x^2)$. Similar to the analysis in Example 7.1, it can be shown that this n^{th} composite power is explicitly given by

$$\ell_H^n[y] = \frac{1}{\exp(-x^2)} \sum_{j=1}^{n} (-1)^j \left(2^{n-j} S_n^{(j)} \exp(-x^2) y^{(j)}(x)\right)^{(j)} \quad (n \in \mathbb{N}),$$

where $S_n^{(j)}$ is the Stirling number of the second kind, defined in (7.7). Further details on the construction of these powers of the Hermite expression (7.15) and their application to left-definite spectral theory can be found in [8].

Example 7.3. Jacobi's classical second-order differential expression is given by

$$\ell_J[y](x) := (x^2-1)y''(x) + (-\beta+\alpha+(\alpha+\beta+2)x)y'(x) \\ = \frac{1}{(1-x)^\alpha (1+x)^\beta} \left(-(1-x)^{\alpha+1}(1+x)^{\beta+1} y'(x)\right)' \quad (x \in (-1,1)); \tag{7.16}$$

here, we assume that $\alpha, \beta > -1$. For each $m \in \mathbb{N}_0$, the m^{th} degree Jacobi polynomial $P_m^{(\alpha,\beta)}(x)$ is a solution of

$$\ell_J[y](x) = m(m + \alpha + \beta + 1)y(x).$$

The sequence $\{P_m^{(\alpha,\beta)}\}_{m=0}^{\infty}$ of Jacobi polynomials forms a complete orthogonal set in the Hilbert space $L^2((-1, 1); (1 - x)^\alpha(1 + x)^\beta)$. Legendre polynomials ($\alpha = \beta = 0$), Tchebichef polynomials of the first kind ($\alpha = \beta = -1/2$), and Tchebichef polynomials of the second kind ($\alpha = \beta = 1/2$) are special cases of Jacobi polynomials. We see from (7.16) that $w(x) = (1 - x)^\alpha(1 + x)^\beta$ is a symmetry factor for $\ell_J[\cdot]$ and, hence, each composite power $\ell_J^n[\cdot]$ also has $w(x) = (1-x)^\alpha(1+x)^\beta$ as a symmetry factor. We refer the reader to [7] (see also [9]) for explicit details on the construction of these powers as well as the complete left-definite theory of the Jacobi expression (7.16). In [7], it is shown that

$$\ell_J^n[y](x) = \frac{1}{(1 - x)^\alpha(1 + x)^\beta} \sum_{j=1}^{2n} (-1)^j \left(P^{(\alpha,\beta)} S_n^j (1 - x)^{\alpha+j}(1 + x)^{\beta+j} y^{(j)}(x) \right)^{(j)},$$

where $\{P^{(\alpha,\beta)} S_n^j\}$ are the so-called Jacobi–Stirling numbers defined by

$$P^{(\alpha,\beta)} S_n^{(j)} := \sum_{r=0}^{j} (-1)^{r+j} \frac{\Gamma(\alpha + \beta + r + 1)\Gamma(\alpha + \beta + 2r + 2)\,[r(r + \alpha + \beta + 1)]^n}{r!(j - r)!\Gamma(\alpha + \beta + 2r + 1)\Gamma(\alpha + \beta + j + r + 2)}.$$

References

[1] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[2] Randal H. Cole. *Theory of ordinary differential equations*, Appleton-Century-Crofts Meredith Corp., New York, 1968.

[3] Louis Comtet. *Advanced combinatorics*, D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974, The art of finite and infinite expansions.

[4] Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part II*, Wiley Classics Library. John Wiley & Sons Inc., New York, 1988, Spectral theory. Selfadjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication.

[5] W.N. Everitt. Linear ordinary quasidifferential expressions, In *Proceedings of the 1983 Beijing symposium on differential geometry and differential equations*, pages 1–28, Beijing, 1986. Science Press.

[6] W.N. Everitt, K.H. Kwon, J.K. Lee, L.L. Littlejohn, and S.C. Williams. Self-adjoint operators generated from non-Lagrangian symmetric differential equations having orthogonal polynomial eigenfunctions, *Rocky Mountain J. Math.*, 31(3):899–937, 2001.

[7] W.N. Everitt, K.H. Kwon, L.L. Littlejohn, R. Wellman, and G.J. Yoon. Jacobi–Stirling numbers, Jacobi polynomials and the left-definite analysis of the classical Jacobi differential expression, *J. Comput. Appl. Math.*, 2007. To appear.

[8] W.N. Everitt, L.L. Littlejohn, and R. Wellman. The left-definite spectral theory for the classical Hermite differential equation, *J. Comput. Appl. Math.*, 121(1-2):313–330, 2000. Numerical analysis in the 20th century, Vol. I, Approximation theory.

[9] W.N. Everitt, L.L. Littlejohn, and R. Wellman. Legendre polynomials, Legendre-Stirling numbers, and the left-definite spectral analysis of the Legendre differential expression, *J. Comput. Appl. Math.*, 148(1):213–238, 2002. On the occasion of the 65th birthday of Professor Michael Eastham.

[10] W.N. Everitt and D. Race. Some remarks on linear ordinary quasidifferential expressions, *Proc. London Math. Soc. (3)*, 54(2):300–320, 1987.

[11] W.N. Everitt and A. Zettl. Products of differential expressions without smoothness assumptions, *Quaestiones Math.*, 3(1):67–82, 1978/79.

[12] W.N. Everitt and A. Zettl. Generalized symmetric ordinary differential expressions. I. The general theory, *Nieuw Arch. Wisk. (3)*, 27(3):363–397, 1979.

[13] L.L. Littlejohn and R. Wellman. A general left-definite theory for certain self-adjoint operators with applications to differential equations, *J. Differential Equations*, 181(2):280–339, 2002.

[14] Lance L. Littlejohn. Symmetry factors for differential equations, *Amer. Math. Monthly*, 90(7):462–464, 1983.

[15] Lance L. Littlejohn. On the classification of differential equations having orthogonal polynomial solutions, *Ann. Mat. Pura Appl. (4)*, 138:35–53, 1984.

[16] Lance L. Littlejohn and David Race. Symmetric and symmetrisable differential expressions, *Proc. London Math. Soc. (3)*, 60(2):344–364, 1990.

[17] John Locker. *Functional analysis and two-point differential operators*, volume 144 of *Pitman Research Notes in Mathematics Series*, Longman Scientific & Technical, Harlow, 1986.

[18] M.A. Naĭmark. *Linear differential operators. Part II: Linear differential operators in Hilbert space*, With additional material by the author, and a supplement by V. È. Ljance. Translated from the Russian by E.R. Dawson. English translation edited by W.N. Everitt. Frederick Ungar Publishing Co., New York, 1968.

[19] Gábor Szegő. *Orthogonal polynomials*, American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[20] Anton Zettl. Powers of real symmetric differential expressions without smoothness assumptions, *Quaestiones Math.*, 1(1):83–94, 1976.