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Oscillation of Damped Second Order Nonlinear Delay Differential Equations of Emden–Fowler Type

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Abstract

In this paper we establish some new oscillation criteria for a second order nonlinear delay differential equation of Emden–Fowler type with damping term. These results extend and improve some of the well-known results in the nondelay case. Our results in the delay case are new and can be applied to new classes of equations which are not covered by the known criteria for oscillation. Some selected examples are provided.

AMS subject classification: 34C10, 34K11, 34K40.
Keywords: Oscillation, nonlinear delay differential equation, Emden–Fowler equation, damping term.

1. Introduction

In this paper we are concerned with the oscillation of all solutions of the second order nonlinear delay differential equation with damping term

$$(ax')'(t) + p(t)x'(t) + q(t)|x(g(t))|^{\lambda}\mathrm{sgn}x(g(t)) = 0 \quad \text{for} \quad t \in [t_0, \infty), \tag{1.1}$$

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where

$$a, p, q, g \in C([t_0, \infty), \mathbb{R}^+),\ \lambda > 0,\ \lim_{t\to\infty} g(t) = \infty,\ a(t) > 0,\ g(t) \le t,\ g'(t) > 0. \tag{1.2}$$

Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for any $T \ge t_x$. We make a standing hypothesis that (1.1) does possess such solutions. A solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory. An equation itself is called oscillatory if all its solutions are oscillatory.

Oscillatory and nonoscillatory behavior of solutions for different classes of linear and nonlinear differential equations of second order of types

$$x''(t) + q(t)|x(t)|^{\lambda}\mathrm{sgn}x(t) = 0 \quad \text{for} \quad t \in [t_0, \infty), \tag{1.3}$$

$$(ax')'(t) + q(t)x(t) = 0 \quad \text{for} \quad t \in [t_0, \infty), \tag{1.4}$$

$$(ax')'(t) + q(t)f(x(t)) = 0 \quad \text{for} \quad t \in [t_0, \infty), \tag{1.5}$$

$$(ax')'(t) + p(t)x'(t) + q(t)f(x(t)) = 0 \quad \text{for} \quad t \in [t_0, \infty) \tag{1.6}$$

has been studied by many authors, we choose to refer to these in [4, 9–11, 13, 17, 25, 26, 28–33, 35–50]. Discrete and dynamic versions of these equations have been studied as well, see e.g., [3, 5–8]. The averaging function method is one of the most important techniques in studying oscillations. Using this technique, many oscillation criteria have been found which involve the behavior of the integral of the coefficients. In the study of the oscillatory behavior of the Emden–Fowler equation (1.3), many criteria involve the behavior of the integral of q. In the linear case, i.e., when $\lambda = 1$, six of the more important oscillation criteria for (1.3) are

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(v)\mathrm{d}v\mathrm{d}s = \infty \qquad \text{(Wintner [44])} \tag{C_1}$$

$$\int_{t_0}^{\infty} q(t) = \infty \qquad \text{(Leighton [35])} \tag{C_2}$$

$$\begin{cases} \text{(i)} \quad \liminf\limits_{t\to\infty} \dfrac{1}{t} \displaystyle\int_{t_0}^{t} \int_{t_0}^{s} q(v)\mathrm{d}v\mathrm{d}s > -\infty \\ \\ \text{(ii)} \quad \limsup\limits_{t\to\infty} \dfrac{1}{t} \displaystyle\int_{t_0}^{t} \int_{t_0}^{s} q(v)\mathrm{d}v\mathrm{d}s = \infty \end{cases} \qquad \text{(Hartman [28])} \tag{C_3}$$

$$\lim_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n q(s)\mathrm{d}s = \infty \quad \text{for some} \quad n \in \mathbb{N} \setminus \{1\} \qquad \text{(Kamenev [31])} \tag{C_4}$$

$$\begin{cases} \text{(i)} \quad \limsup\limits_{t\to\infty} \dfrac{1}{t^n}\displaystyle\int_{t_0}^{t}(t-s)^n q(s)\mathrm{d}s < \infty \\ \text{(ii)} \quad \limsup\limits_{t\to\infty} \dfrac{1}{t^n}\displaystyle\int_{t_0}^{t}(t-s)^n q(s)\mathrm{d}s > A(u) \quad \text{for all} \quad u \geq t_0 \qquad \text{(Yan [48])} \\ \text{(ii)} \quad \displaystyle\int_{t_0}^{\infty} A_+^2(t)\mathrm{d}t = \infty, \quad \text{where} \quad A_+(t) = \max\{A(t), 0\} \end{cases} \quad (C_5)$$

$$\begin{cases} \limsup\limits_{t\to\infty} \dfrac{1}{H(t,t_0)}\displaystyle\int_{t_0}^{t}\left[H(t,s)q(s) - \frac{h^2(t,s)}{4}\right]\mathrm{d}s = \infty, \\ \text{where } H : D := \{(t,s) :\ t \geq s \geq t_0\} \to \mathbb{R} \text{ is continuous with} \\ H(t,t) = 0 \text{ for } t \geq t_0 \quad \text{and} \quad H(t,s) > 0 \text{ for } t > s \geq t_0 \\ \text{and has a continuous and positive partial derivative on } D \text{ with} \\ -\dfrac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \quad \text{for all} \quad (t,s) \in D, \\ \text{where } h : D \to \mathbb{R} \text{ is a continuous function.} \end{cases} \quad \text{(Philos [40])} \quad (C_6)$$

However, if $q(t) = \dfrac{\nu}{t^2}$, then (1.3) reduces to the well-known Euler equation

$$u''(t) + \frac{\nu}{t^2}u(t) = 0 \quad \text{for} \quad t \geq 1. \tag{1.7}$$

Note that none of the above mentioned oscillation criteria can be applied to (1.7). In fact, the Euler equation (1.7) is oscillatory if $\nu > \dfrac{1}{4}$ and nonoscillatory if $\nu \leq \dfrac{1}{4}$ (see [36]). Butler [9] proved that the Wintner criterion (C_1) is sufficient for the oscillation of (1.3) in the superlinear case, i.e., when $\lambda > 1$. In the sublinear case, i.e., $0 < \lambda < 1$, Kamenev [29] relaxed condition (C_1) to (C_3) (ii). In [42], Willett showed that the condition (C_3) (ii) is not sufficient for the oscillation of (1.3) when $\lambda \geq 1$. However, if (C_3) holds, then (1.3) with $\lambda > 1$ is oscillatory (see Wong [45]). Butler [9] proved that in the sublinear case the condition

$$\liminf_{T\to\infty} \frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t} q(s)\mathrm{d}s\mathrm{d}t < \limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t} q(s)\mathrm{d}s\mathrm{d}t \tag{C_7}$$

is a sufficient condition for the oscillation of (1.3). Wong [47] proved that for the sublinear case

$$\limsup_{T\to\infty} \frac{1}{T}\int_{t_0}^{T}\left[\int_{t_0}^{t} q(s)\mathrm{d}s\right]^2 \mathrm{d}t = \infty \tag{C_8}$$

implies oscillation of (1.3). Recently Li and Yan [38] proved that for the superlinear case, if

$$\begin{cases} \limsup\limits_{t\to\infty} \dfrac{1}{t^\alpha} \displaystyle\int_{t_0}^{t} (t-s)^\alpha q(s)\mathrm{d}s = \infty \quad \text{for some} \quad \alpha > 1, \\ \liminf\limits_{t\to\infty} \dfrac{1}{t^\beta} \displaystyle\int_{t_0}^{t} (t-s)^\beta q(s)\mathrm{d}s > -\infty \quad \text{for some} \quad \beta \geq 1, \end{cases} \tag{C_9}$$

then every solution of (1.3) oscillates. Coles [10,11], Kamenev [29,30] and Willett [43] extended the criteria in the linear case by considering weighted averages of the integral of q of the form

$$A(t,t_0,\phi,q) = \frac{1}{\int_{t_0}^{t}\phi(s)\mathrm{d}s} \int_{t_0}^{t} \phi(\tau) \int_{t_0}^{\tau} q(s)\mathrm{d}\tau\mathrm{d}t,$$

where ϕ is positive and locally integrable, but not integrable, on $[t_0,\infty)$, and proved that the condition

$$\lim_{t\to\infty} A(t,t_0,\phi,q) = \infty \tag{C_{10}}$$

is a sufficient condition for the oscillation of (1.3) when $\lambda > 1$ (for the definition of locally integrable functions see Willett [43]). Leighton [35] considered (1.4) and proved that if

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} p(t)\mathrm{d}t = \infty, \tag{C_{11}}$$

then every solution oscillates. Willett [43] used the transformation

$$\tau = \left(\int_t^{\infty} \frac{\mathrm{d}s}{a(s)} \right)^{-1} \quad \text{and} \quad u(t) = \tau^{-1} y(t)$$

and established a new version of Leighton's criterion and obtained the following oscillation criteria: If

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} < \infty \quad \text{and} \quad \int_{t_0}^{\infty} p(t) \left(\int_t^{\infty} \frac{\mathrm{d}s}{a(s)} \right)^2 \mathrm{d}t = \infty, \tag{C_{12}}$$

then every solution of (1.4) oscillates. Note that the oscillation criteria given by Leighton and Willett cannot be applied to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^2 u'(t)) + \mu u(t) = 0 \quad \text{for} \quad t > 0, \tag{1.8}$$

where μ is a positive constant. Yu [50], Li [36], Kong [33], Li and Yeh [37] used the generalized Riccati technique and gave several sufficient conditions for oscillation of (1.4), which can be applied to (1.8). In fact Li [36] proved that every solution of (1.8) oscillates if $\mu > \dfrac{1}{4}$.

In the study of the differential equation (1.5), many criteria for oscillation exist which involve the behavior of the integral of q; however the common restrictions, namely $q(t) > 0$, $f'(u) > 0$ and $\int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} = \infty$ on the functions q, f and a are required. As examples of this study we cite the papers of Bhatia [4], Grace and Lalli [17], Graef, Rankin and Spikes [25] and Graef and Spikes [26].

Presence of the damping term in (1.6) requires modification of approaches to the study of the oscillatory properties of solutions. A number of oscillation criteria for (1.6) can be found, for example, in the papers by Grace [13–15], Grace and Lalli [16–24], Kirane and Rogovchenkov [32], Li, Zhang, and Fe [39], Saker, Pang and Agarwal [41], Yeh [49]; however the common restriction $f'(u) \geq k > 0$ is required, which is not applicable to the differential equation (1.1) because of the above restriction on the function f.

Considerably less is known about the oscillation of the nonlinear delay differential equation of Emden–Fowler type (1.1) when $g(t) = t$. Grace and Lalli [24] and Grace [14] used the averaging technique that was used in [23], and extended the condition (C_{10}) to (1.1). The results of Grace and Lalli [24] and Grace [14] are formulated as follows:

Let $\Phi(t, t_0)$ denote the class of positive and locally integrable functions, but not integrable, which contains all the bounded functions for $t \geq t_0$. Let $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$, $\phi \in \Phi$, and for any $T \geq t_0$, define

$$\alpha(t,T) = \int_T^t \phi(s)\mathrm{d}s, \quad \bar{\eta}(t) = \frac{1}{a(t)\rho(t)}, \quad \gamma(t) = a(t)\rho'(t) - p(t)\rho(t),$$
$$\nu(t,T) = \frac{1}{\phi(t)} \int_{t^*}^t \frac{\phi^2(s)}{\bar{W}(s,T)}\mathrm{d}s,\ t^* \geq T, \quad \bar{W}(t,T) = \bar{\eta}(t)\left(\int_T^t \bar{\eta}(s)\mathrm{d}s\right)^{-1},$$

and

$$A_\phi(t,T) = \frac{1}{\alpha(t,T)} \int_T^t \phi(s) \int_T^s \rho(u)q(u)\mathrm{d}u\mathrm{d}s.$$

Theorem 1.1. (see [14, 24]) Let $g(t) = t$. Suppose there exist functions $\phi \in \Phi(t, t_0)$ and $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that

$$\gamma(t) \geq 0 \quad \text{and} \quad \gamma'(t) \leq 0 \quad \text{for} \quad t \geq t_0,$$
$$\liminf_{t\to\infty} \int_{t_0}^t \rho(s)q(s)\mathrm{d}s > -\infty, \quad \int_{t_0}^{\infty} \bar{\eta}(s)\mathrm{d}s = \infty,$$

and

$$\int_T^{\infty} \frac{\alpha^\mu(s,T)}{\nu(s,T)}\mathrm{d}s = \infty \quad \text{for some} \quad T \geq t_0 \quad \text{and} \quad \mu \in (0,1).$$

If

$$\lim_{t\to\infty} A_\phi(t,T) = \infty, \tag{C$_{15}$}$$

then every solution of (1.1) oscillates for all $\lambda > 1$.

Theorem 1.2. (see [14, 24]) Let the assumptions of Theorem 1.1 hold, and let the differentiable function ρ be defined by

$$\rho(t) = \exp\left(\int_{t_0}^{t} \frac{p(s)}{a(s)}\mathrm{d}s\right) \quad \text{for} \quad t \geq t_0.$$

Then every solution of (1.1) oscillates for all $\lambda > 0$.

For the oscillation of other various functional differential equations, we refer the reader to the monographs [1, 2, 12, 27, 34].

In this paper we are interested in extending and improving Theorems 1.1 and 1.2 to a broad class of second order nonlinear delay differential equations of the form (1.1) by using a technique similar to that exploited by Grace [15], Philos [40], Saker, Pang and Agarwal [41] and Yan [48] to derive some new oscillation criteria for (1.1). Our criteria cover new classes of equations to which the above known results do not apply. Throughout this paper, we assume that the coefficients are all positive functions. The problem when p and q take different signs still is open due to the delay. The relevance of our theorems becomes clear due to a number of selected examples. It is interesting to study the second order Emden–Fowler delay differential equation because it has several physical applications, see, e.g., [46].

2. Main Results

In this section we extend and improve Theorems 1.1 and 1.2 to (1.1) with delay, establish some new oscillation criteria for oscillation of (1.1), and present some Kamenev-type and Philos-type theorems for oscillation. It will be convenient to make the following notations for the remainder of this paper. In the sequel, when we write a functional inequality we will assume that it holds for all sufficient large values of t.

Let $\Phi(t,t_0)$ denote the class of positive and locally integrable functions, but not integrable, which contains all the bounded functions for $t \geq t_0$. Let $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$, $\phi \in \Phi(t,t_0)$, and for any $T \geq t_0$, let

$$\alpha(t,T) = \int_T^t \phi(s)\mathrm{d}s, \quad \eta(t) = \frac{a(g(t))}{a^2(t)\rho(t)g'(t)}, \quad \gamma(t) = a(t)\rho'(t) - p(t)\rho(t),$$

$$W(t,T) = \frac{g'(t)}{a(g(t))\rho(t)}\left\{1 + \left(\int_T^{g(t)} \eta(s)\mathrm{d}s\right)^{\frac{1}{2}}\right\}^{-2},$$

and

$$A_\phi(t,T)=\frac{1}{\alpha(t,T)}\int_T^t \phi(s)\int_T^s \rho(u)q(u)\mathrm{d}u\mathrm{d}s.$$

Theorem 2.1. Assume (1.2) and suppose there exists $\rho\in C^1([t_0,\infty),\mathbb{R}^+)$ such that

$$\gamma(t)\geq 0,\quad \left(\frac{\gamma(a\circ g)}{ag'}\right)'(t)\leq 0,\quad (\rho p)'(t)\leq 0\quad \text{for}\quad t\geq t_0, \tag{2.1}$$

$$\int_{t_0}^\infty \rho(s)q(s)\mathrm{d}s=\infty,\quad \int_{t_0}^\infty \frac{1}{\rho(s)a(s)}\int_{t_0}^s \rho(\tau)q(\tau)\mathrm{d}\tau\mathrm{d}s=\infty. \tag{2.2}$$

If

$$\lim_{t\to\infty} A_\phi(t,t_0)=\infty\quad \text{for some}\quad \phi\in\Phi(t,t_0), \tag{2.3}$$

then every solution x of (1.1) oscillates or satisfies $x(t)\to 0$ as $t\to\infty$ for all $\lambda>1$.

Proof. Let x be a nonoscillatory solution of (1.1), and, without loss of generality, we assume that $x(t)>0$ and $x(g(t))>0$ for $t\geq t_1$. Define the function

$$w(t)=\rho(t)\frac{a(t)x'(t)}{x^\lambda(g(t))}\quad \text{for}\quad t\geq t_1. \tag{2.4}$$

Then it follows from (1.1) that for $t\geq t_1$

$$w'(t)=-\rho(t)q(t)+\gamma(t)\frac{x'(t)}{x^\lambda(g(t))}-\lambda a(t)\rho(t)g'(t)\frac{x'(t)x'(g(t))}{x^{\lambda+1}(g(t))}. \tag{2.5}$$

Now note that (1.1) and $q(t)>0$ imply for $t\geq t_1$

$$(ax')'(t)+p(t)x'(t)<0,\quad \text{i.e.,}\quad \frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left(\int_{t_1}^t \frac{p(s)}{a(s)}\mathrm{d}s\right)a(t)x'(t)\right\}<0.$$

Thus

$$\exp\left(\int_{t_1}^t \frac{p(s)}{a(s)}\mathrm{d}s\right)a(t)x'(t)$$

is eventually decreasing and hence eventually of one sign, and therefore $x'(t)$ is eventually of one sign. This allows us to consider the following two possible cases.

Case I: $x'>0$ on $[t_1,\infty)$ for some $t_1\geq t_0$. Since $x'(t)>0$ and $g(t)\leq t$, we have

$$x(t)\geq x(g(t))\quad \text{for}\quad t\geq t_1. \tag{2.6}$$

Also from (1.1) since $x'(t)>0$, we have $(ax')'(t)<0$ for $t\geq t_1$ so that the function ax' is nonincreasing, and hence

$$a(t)x'(t)<a(g(t))x'(g(t))\quad \text{for}\quad t\geq t_1. \tag{2.7}$$

Combining (2.5) and (2.7), we get for $t \geq t_1$

$$w'(t) \leq -\rho(t)q(t) + \gamma(t)\frac{x'(t)}{x^{\lambda}(g(t))} - \frac{\lambda a^2(t)\rho(t)g'(t)}{a(g(t))}\left(\frac{x'(t)}{x^{\delta}(g(t))}\right)^2, \tag{2.8}$$

where $\delta = \dfrac{\lambda+1}{2}$. Also by using (2.6), (2.7) and the fact that $x'(t) > 0$ for $t \geq t_1$, (2.8) implies that for $t \geq t_1$

$$w'(t) \leq -\rho(t)q(t) + \frac{\gamma(t)a(g(t))}{a(t)g'(t)}\frac{x'(g(t))g'(t)}{x^{\lambda}(g(t))} - \frac{\lambda a^2(t)\rho(t)g'(t)}{a(g(t))}\left(\frac{x'(t)}{x^{\delta}(t)}\right)^2.$$

Integrating this inequality from t_1 to t provides

$$\begin{aligned} w(t) \;\leq\; & w(t_1) - \int_{t_1}^{t}\rho(s)q(s)\mathrm{d}s + \int_{t_1}^{t}\frac{\gamma(s)a(g(s))}{a(s)g'(s)}\frac{x'(g(s))g'(s)}{x^{\lambda}(g(s))}\mathrm{d}s \\ & - \int_{t_1}^{t}\frac{\lambda a^2(s)\rho(s)g'(s)}{a(g(s))}\left(\frac{x'(s)}{x^{\delta}(s)}\right)^2\mathrm{d}s. \end{aligned} \tag{2.9}$$

By the second mean value theorem for integrals (observe that $\gamma(a\circ g)/(ag')$ is nonnegative and decreasing by (2.1)), for $t \geq t_1$ there exists $\zeta \in [t_1, t]$ so that

$$\begin{aligned} \int_{t_1}^{t}\frac{\gamma(s)a(g(s))}{a(s)g'(s)}\frac{x'(g(s))g'(s)}{x^{\lambda}(g(s))}\mathrm{d}s &= \frac{\gamma(t_1)a(g(t_1))}{a(t_1)g'(t_1)}\int_{t_1}^{\zeta}\frac{x'(g(s))g'(s)}{x^{\lambda}(g(s))}\mathrm{d}s \\ &= \frac{\gamma(t_1)a(g(t_1))}{a(t_1)g'(t_1)}\int_{x(g(t_1))}^{x(g(\zeta))}u^{-\lambda}\mathrm{d}u \\ &= \frac{\gamma(t_1)a(g(t_1))}{(\lambda-1)a(t_1)g'(t_1)}\left[x^{1-\lambda}(g(t_1)) - x^{1-\lambda}(g(\zeta))\right] \\ &< \frac{\gamma(t_1)a(g(t_1))}{(\lambda-1)a(t_1)g'(t_1)}x^{1-\lambda}(g(t_1)) =: K. \end{aligned} \tag{2.10}$$

Thus, for $t \geq t_1$, we find from (2.9) that

$$w(t) + \int_{t_1}^{t}\rho(s)q(s)\mathrm{d}s + \int_{t_1}^{t}\frac{\lambda a^2(s)\rho(s)g'(s)}{a(g(s))}\left(\frac{x'(s)}{x^{\delta}(s)}\right)^2\mathrm{d}s \leq K + w(t_1) =: L, \tag{2.11}$$

and hence, since $w(t) > 0$, we have

$$\int_{t_1}^{t}\frac{\lambda a^2(s)\rho(s)g'(s)}{a(g(s))}\left(\frac{x'(s)}{x^{\delta}(s)}\right)^2\mathrm{d}s \leq L - w(t) - \int_{t_1}^{t}\rho(s)q(s)\mathrm{d}s \leq L.$$

Thus for $N := L/\lambda > 0$, we have

$$\int_{t_1}^{t}\frac{a^2(s)\rho(s)g'(s)}{a(g(s))}\left(\frac{x'(s)}{x^{\delta}(s)}\right)^2\mathrm{d}s \leq N \quad \text{for} \quad t \geq t_1.$$

By the Schwarz inequality

$$\begin{aligned}\left|\int_{t_1}^{t}\frac{x'(s)}{x^{\delta}(s)}\mathrm{d}s\right|^2 &\leq \left(\int_{t_1}^{t}\frac{a(g(s))\mathrm{d}s}{a^2(s)\rho(s)g'(s)}\right)\left(\int_{t_1}^{t}\frac{a^2(s)\rho(s)g'(s)}{a(g(s))}\left(\frac{x'(s)}{x^{\delta}(s)}\right)^2\mathrm{d}s\right)\\ &\leq N\int_{t_1}^{t}\frac{a(g(s))\mathrm{d}s}{a^2(s)\rho(s)g'(s)}\\ &= N\int_{t_1}^{t}\eta(s)\mathrm{d}s.\end{aligned}$$

Hence

$$|x^{1-\delta}(t)-x^{1-\delta}(t_1)|\leq(1-\delta)N^{\frac{1}{2}}\left(\int_{t_1}^{t}\eta(s)\mathrm{d}s\right)^{\frac{1}{2}}$$

and therefore

$$|x^{1-\delta}(t)|\leq M\left\{1+\left(\int_{t_1}^{t}\eta(s)\mathrm{d}s\right)^{\frac{1}{2}}\right\}\quad\text{for all}\quad t\geq t_1,$$

where $M:=\max\{|x^{1-\delta}(t_1)|,(1-\delta)\sqrt{N}\}$. Furthermore, as $\lim\limits_{t\to\infty}g(t)=\infty$, we can assume that there exists $T\geq t_1$ such that $g(t)\geq T$ for all $t\geq T$. Hence

$$|x^{1-\delta}(g(t))|\leq M\left\{1+\left(\int_{T}^{g(t)}\eta(s)\mathrm{d}s\right)^{\frac{1}{2}}\right\}\quad\text{for all}\quad t\geq T.$$

Substitution of this inequality into (2.8) and using the definition of the function $W(t,T)$ yields

$$w'(t)\leq-\rho(t)q(t)+\frac{\gamma(t)}{\rho(t)a(t)}w(t)-\frac{\lambda W(t,T)}{M^2}w^2(t)\quad\text{for}\quad t\geq T.\tag{2.12}$$

Now, by integrating, we easily obtain

$$w(t)+\int_{T}^{t}\frac{\lambda W(s,T)}{M^2}w^2(s)\mathrm{d}s\leq w(T)-\int_{T}^{t}\rho(s)q(s)\mathrm{d}s.\tag{2.13}$$

Multiplying (2.13) by $\phi(t)$ and integrating from T to t provides

$$0\leq\int_{T}^{t}\phi(s)w(s)\mathrm{d}s+\frac{\lambda}{M^2}\int_{T}^{t}\phi(s)\int_{T}^{s}W(u,T)w^2(u)\mathrm{d}u\mathrm{d}s\leq\alpha(t,T)[w(T)-A_{\phi}(t,T)].$$

Using the condition (2.3), we see that the right-hand side of the last inequality tends to $-\infty$ as $t\to\infty$, a contradiction.

Case II: $x' < 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$. In this case we have $\lim_{t\to\infty} x(t) = c \geq 0$. We assert that $c = 0$. If not, then $x(g(t)) > c > 0$ for $t \geq t_2 > t_1$. Then $x^\lambda(g(t)) \geq c^\lambda$ for $t \geq t_2$. Define the function u by

$$u(t) = \rho(t)a(t)x'(t) \quad \text{for} \quad t \geq t_2.$$

Differentiating u and using (1.1) provides

$$u'(t) < -c^\lambda \rho(t)q(t) - \rho(t)p(t)x'(t) \quad \text{for} \quad t \geq t_2.$$

Hence for all $t \geq t_2$, we have

$$u(t) \leq u(t_2) - c^\lambda \int_{t_2}^{t} \rho(s)q(s)\mathrm{d}s - \int_{t_2}^{t} \rho(s)p(s)x'(s)\mathrm{d}s.$$

By the second mean value theorem for integrals (observe that ρp is nonnegative and decreasing by (2.1)), for any $t \geq t_2$ there exists $\zeta \in [t_2, t]$ with

$$\int_{t_2}^{t} \rho(s)p(s)(-x'(s))\mathrm{d}s = \rho(t_2)p(t_2)\int_{t_2}^{\zeta}(-x'(s))\mathrm{d}s \leq \rho(t_2)p(t_2)x(t_2).$$

So, for every $t \geq t_2$

$$u(t) \leq C - c^\lambda \int_{t_2}^{t} \rho(s)q(s)\mathrm{d}s,$$

where $C = u(t_2) + \rho(t_2)p(t_2)x(t_2)$. Therefore, from condition (2.2), we can choose $t_3 \geq t_2$ such that for all $t \geq t_3$

$$u(t) \leq \frac{c^\lambda}{2}\int_{t_2}^{t} \rho(s)q(s)\mathrm{d}s - c^\lambda \int_{t_2}^{t} \rho(s)q(s)\mathrm{d}s = -\frac{c^\lambda}{2}\int_{t_2}^{t} \rho(s)q(s)\mathrm{d}s,$$

i.e.,

$$x'(t) \leq -\frac{c^\lambda}{2}\frac{1}{\rho(t)a(t)}\int_{t_2}^{t} \rho(s)q(s)\mathrm{d}s.$$

Integrating the last inequality and using (2.2), we get $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction. Then $c = 0$ and hence $\lim_{t\to\infty} x(t) = 0$.

The case where x is eventually negative is dealt with similarly. We conclude that every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$. This completes the proof. ■

Theorem 2.2. Assume (1.2) and suppose ρ defined by

$$\rho(t) = \exp\left(\int_{t_0}^{t} \frac{p(s)}{a(s)}\mathrm{d}s\right) \quad \text{for} \quad t \geq t_0 \tag{2.14}$$

satisfies (2.2), (2.3), and

$$(ap' + p^2)(t) \le 0 \quad \text{for all} \quad t \ge t_0. \tag{2.15}$$

Then every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$ for all $\lambda > 0$.

Proof. Clearly, $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$. From the definition of ρ, it follows that $\gamma(t) = 0$ for $t \ge t_0$. Thus the first two conditions in (2.1) are satisfied. The last condition in (2.1) is satisfied due to (2.15). By Theorem 2.1, the claim holds for all $\lambda > 1$. However, the part of the proof (2.10) of Theorem 2.1, where $\lambda > 1$ was used, is not needed anymore due to $\gamma(t) = 0$ for all $t \ge t_0$, and thus the claim holds for all $\lambda > 0$. ■

Remark 2.3. Theorems 2.1 and 2.2 extend and improve Theorems 1.1 and 1.2.

The following examples illustrate our main results.

Example 2.4. Let $\lambda > 1$ and consider the delay differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{t}}x'(t)\right) + \frac{1}{2t\sqrt{t}}x'(t) + \frac{1}{\sqrt{t}}\left(\frac{2+\cos t}{2t}\right)x^\lambda(t/2)\mathrm{sgn}x(t/2) = 0 \quad \text{for} \quad t \ge \frac{\pi}{2}. \tag{2.16}$$

Here $a(t) = \dfrac{1}{\sqrt{t}}$, $p(t) = \dfrac{1}{2t\sqrt{t}}$, $g(t) = \dfrac{t}{2}$, $g'(t) = \dfrac{1}{2} > 0$, $q(t) = \dfrac{1}{\sqrt{t}}\left(\dfrac{2+\cos t}{2t}\right)$, and $t_0 = \dfrac{\pi}{2}$. We choose $\rho(t) = t$ and $\phi(t) = \dfrac{1}{t}$. Then

$$\alpha(t, \pi/2) = \ln\frac{2t}{\pi}, \quad \gamma(t) = \frac{1}{2\sqrt{t}} \ge 0, \quad \frac{\gamma(t)a(g(t))}{a(t)g'(t)} = \sqrt{\frac{2}{t}}, \quad \rho(t)p(t) = \frac{1}{2\sqrt{t}},$$

$$\int_{t_0}^{s} \rho(u)q(u)\mathrm{d}u = \int_{t_0}^{s} \frac{2+\cos u}{2\sqrt{u}}\mathrm{d}u \ge \int_{t_0}^{s} \frac{\mathrm{d}u}{2\sqrt{u}} = \sqrt{s} - \sqrt{t_0},$$

$$\int_{t_0}^{t} \frac{1}{\rho(s)a(s)} \int_{t_0}^{s} \rho(\tau)q(\tau)\mathrm{d}\tau\mathrm{d}s \ge \int_{t_0}^{t} \frac{1}{\sqrt{s}}(\sqrt{s} - \sqrt{t_0})\mathrm{d}s = t - 2\sqrt{t_0 t} + t_0,$$

and

$$A_\phi(t, t_0) \ge \frac{1}{\ln\frac{2t}{\pi}} \int_{t_0}^{t} \frac{1}{s}(\sqrt{s} - \sqrt{t_0})\mathrm{d}s = \frac{2\sqrt{t} - 2\sqrt{t_0}}{\ln\frac{2t}{\pi}} - \sqrt{t_0}.$$

Hence all conditions of Theorem 2.1 are satisfied, and thus every solution x of (2.16) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$, for all $\lambda > 1$. Note that none of the oscillation criteria mentioned in Section 1 can be applied to (2.16).

Example 2.5. Consider the differential equation

$$x''(t) + \frac{1}{t}x'(t) + \frac{1}{t^2}x(t) = 0 \quad \text{for} \quad t \ge 1. \tag{2.17}$$

Here $a(t)=1$, $p(t)=\dfrac{1}{t}$, $g(t)=t$, $g'(t)=1>0$, $q(t)=\dfrac{1}{t^2}$, and $t_0=1$. We choose $\rho(t)=\exp\left(\displaystyle\int_1^t \frac{p(s)}{a(s)}\mathrm{d}s\right)=t$ and $\phi(t)=\dfrac{1}{t}$. Then

$$\gamma(t)=0, \quad \alpha(t,1)=\ln t, \quad \bar{\eta}(t)=\frac{1}{t}, \quad \bar{W}(t,1)=\frac{1}{t\ln t}$$

and

$$\nu(t,1)=\frac{t(\ln t)^2}{2} \quad \text{for} \quad t\geq t_0=1.$$

Now we can easily prove that for $\mu\in(0,1)$

$$\int_2^t \frac{\alpha^\mu(s,1)}{\nu(s,1)}\mathrm{d}s=\int_2^t \frac{2\,(\ln s)^\mu}{s(\ln s)^2}\mathrm{d}s=2\int_2^t \frac{1}{s}\,(\ln s)^{\mu-2}\,\mathrm{d}s \not\to \infty \quad \text{as} \quad t\to\infty.$$

Hence Theorem 1.2 cannot be applied to (2.17) when $\phi(t)=\dfrac{1}{t}$. But one can easily prove by Theorem 2.2 that every solution x of (2.17) oscillates or satisfies $x(t)\to 0$ as $t\to\infty$. One such solution of (2.17) is $x(t)=\sin(\ln t)$.

The following theorems improve the above results.

Theorem 2.6. Assume (1.2) and let ρ be such that (2.1) and (2.2) hold. If for some $T\geq t_0$ and for all $M>0$

$$\limsup_{t\to\infty}\int_{t_0}^t \left(\rho(s)q(s)-\frac{M^2}{4}\left(\frac{\gamma(s)}{a(s)\rho(s)\sqrt{\lambda W(s,T)}}\right)^2\right)\mathrm{d}s=\infty, \tag{2.18}$$

then every solution x of (1.1) oscillates or satisfies $x(t)\to 0$ as $t\to\infty$ for all $\lambda>1$.

Proof. Let x be an eventually positive solution of (1.1), say $x(t)>0$ and $x(g(t))>0$ for $t\geq t_1$. The case where x is eventually negative is dealt with similarly and is omitted. As in the proof of Theorem 2.1, we can check that x' has a fixed sign eventually. Condition (2.3) is not needed in the case $x'(t)<0$ eventually, and therefore this case leads to a contradiction as in Case II of the proof of Theorem 2.1. Now we consider the case $x'(t)>0$ for $t\geq t_1$. Again, defining w as in (2.4), we obtain (2.12) as condition (2.3) was not used in this part of the proof of Theorem 2.1. Define

$$Q(t)=\rho(t)q(t)-\frac{M^2}{4}\frac{\gamma^2(t)}{a^2(t)\rho^2(t)\lambda W(t,T)}.$$

Then from (2.12), we have

$$\begin{aligned} w'(t)+Q(t) &\leq \frac{\gamma(t)}{\rho(t)a(t)}w(t)-\frac{\lambda W(t,T)}{M^2}w^2(t)-\frac{M^2}{4}\frac{\gamma^2(t)}{a^2(t)\rho^2(t)\lambda W(t,T)} \\ &= -\left(\sqrt{\frac{\lambda W(t,T)}{M^2}}w(t)-\frac{M}{2}\left(\frac{\gamma(t)}{a(t)\rho(t)\sqrt{\lambda W(t,T)}}\right)\right)^2\leq 0. \end{aligned} \tag{2.19}$$

Then integrating the inequality (2.19) from T to t provides

$$\int_T^t \left(\rho(s)q(s) - \frac{M^2}{4} \left(\frac{\gamma(s)}{a(s)\rho(s)\sqrt{\lambda W(s,T)}} \right)^2 \right) \mathrm{d}s \leq w(T) - w(t) \leq w(T),$$

which contradicts (2.18) and completes the proof. ■

Next, we present some oscillation results for (1.1) by using integral average conditions of the Kamenev-type (C_4).

Theorem 2.7. Let $n \in \mathbb{N}_0$. Assume (1.2) and let ρ be such that (2.1) and (2.2) hold. If for some $T \geq t_0$ and for all $M > 0$

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^t (t-s)^n \left(\rho(s)q(s) - \frac{M^2}{4} \left(\frac{\gamma(s)}{a(s)\rho(s)\sqrt{\lambda W(s,T)}} \right)^2 \right) \mathrm{d}s = \infty, \quad (2.20)$$

then every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$ for all $\lambda > 1$.

Proof. Proceeding just as in the proof of Theorem 2.3, we arrive at the inequality (2.19). Then

$$\begin{aligned} 0 &> \int_{t_0}^t (t-s)^n \left[w'(s) + Q(s) \right] \mathrm{d}s \\ &= n \int_{t_0}^t (t-s)^{n-1} w(s) \mathrm{d}s - w(t_0)(t-t_0)^n + \int_{t_0}^t (t-s)^n Q(s) \mathrm{d}s \\ &\geq \int_{t_0}^t (t-s)^n Q(s) \mathrm{d}s - w(t_0)(t-t_0)^n \end{aligned}$$

so that

$$\frac{1}{t^n} \int_{t_0}^t (t-s)^n Q(s) \mathrm{d}s \leq w(t_0) \left(\frac{t-t_0}{t} \right)^n,$$

which contradicts (2.20) and completes the proof. ■

Theorem 2.8. Let $n \in \mathbb{N}_0$. Assume (1.2) and suppose ρ defined by (2.14) satisfies (2.2) and (2.15). If

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^t (t-s)^n \rho(s) q(s) \mathrm{d}s = \infty, \quad (2.21)$$

then every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$ for all $\lambda > 0$.

Proof. As in the proof of Theorem 2.2, it follows that $\gamma(t) = 0$ for $t \geq t_0$ and that (2.1) is automatically satisfied. Moreover, (2.20) reduces to (2.21). By Theorem 2.7,

the claim holds for all $\lambda > 1$. However, as in the last part of the proof of Theorem 2.2, it follows that the claim actually holds for all $\lambda > 0$. ■

Next we present some new oscillation results for (1.1) by using integral average conditions of the Philos-type (C_6). Following Philos [40], we introduce a class of functions $\mathcal{R}$. Let

$$D_0 = \{(t,s) : \ t > s \geq t_0\} \quad \text{and} \quad D = \{(t,s) : \ t \geq s \geq t_0\}.$$

The function $H \in C(D,R)$ is said to belong to the class $\mathcal{R}$ if

(I) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ on D_0;

(II) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable such that

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \quad \text{for all} \quad (t,s) \in D_0.$$

Theorem 2.9. Assume (1.2). Suppose $H \in \mathcal{R}$ and let ρ be such that (2.1) and (2.2) hold. If for some $T \geq t_0$ and for all $M > 0$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)q(s) - \frac{M^2 Q^2(t,s)}{4\lambda W(s,T)} \right] \mathrm{d}s = \infty, \tag{2.22}$$

where

$$Q(t,s) = h(t,s) - \frac{\gamma(s)}{\rho(s)a(s)}\sqrt{H(t,s)},$$

then every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$ for all $\lambda > 1$.

Proof. Proceeding just as in the proof of Theorem 2.1, we arrive at (2.12). In order to simplify notation, we denote

$$\gamma_1(s) = \frac{\gamma(s)}{\rho(s)a(s)} \quad \text{and} \quad W_1(s) = \frac{\lambda W(s,T)}{M^2}.$$

Then from (2.12) for all $t \geq T$, we have

$$\begin{aligned}
0 \ \geq \ & \int_{t_0}^{t} H(t,s)\left[w'(s) + \rho(s)q(s) - \gamma_1(s)w(s) + W_1(s)w^2(s)\right]\mathrm{d}s \\
= \ & -H(t,t_0)w(t_0) + \int_{t_0}^{t} h(t,s)\sqrt{H(t,s)}w(s)\mathrm{d}s \\
& + \int_{t_0}^{t} H(t,s)\left[\rho(s)q(s) - \gamma_1(s)w(s) + W_1(s)w^2(s)\right]\mathrm{d}s \\
= \ & \int_{t_0}^{t} H(t,s)\rho(s)q(s)\mathrm{d}s - H(t,t_0)w(t_0) \\
& + \int_{t_0}^{t} \left\{ \sqrt{H(t,s)}\left[h(t,s) - \gamma_1(s)\sqrt{H(t,s)}\right] w(s) + H(t,s)W_1(s)w^2(s) \right\}\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_0}^{t} H(t,s)\rho(s)q(s)\mathrm{d}s - H(t,t_0)w(t_0) \\
&\quad + \int_{t_0}^{t} \left\{ \left[\sqrt{H(t,s)W_1(s)}w(s) + \frac{1}{2}\frac{Q(t,s)}{\sqrt{W_1(s)}} \right]^2 - \frac{1}{4}\frac{Q^2(t,s)}{W_1(s)} \right\} \mathrm{d}s \\
&\geq \int_{t_0}^{t} \left\{ H(t,s)\rho(s)q(s) - \frac{1}{4}\frac{Q^2(t,s)}{W_1(s)} \right\} \mathrm{d}s - H(t,t_0)w(t_0)
\end{aligned}
$$

so that

$$
\frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ H(t,s)\rho(s)q(s) - \frac{1}{4}\frac{Q^2(t,s)}{W_1(s)} \right\} \mathrm{d}s \leq w(t_0),
$$

which contradicts (2.22) and completes the proof. ■

Theorem 2.10. Assume (1.2). Let $H \in \mathcal{R}$ and suppose ρ defined by (2.14) satisfies (2.2) and (2.15). If for some $T \geq t_0$ and for all $M > 0$

$$
\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ H(t,s)\rho(s)q(s) - \frac{M^2h^2(t,s)}{4\lambda W(s,T)} \right\} \mathrm{d}s = \infty, \tag{2.23}
$$

then every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$ for all $\lambda > 0$.

Proof. The proof follows immediately as in the proof of Theorem 2.8 by noting that (2.22) reduces to (2.23) when $\gamma(t) = 0$. ■

The next two results follow easily from Theorem 2.9 and Theorem 2.10.

Theorem 2.11. If the assumptions of Theorem 2.9 hold except that (2.22) is replaced by

$$
\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)\mathrm{d}s = \infty \text{ and } \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{Q^2(t,s)}{W(s,T)}\mathrm{d}s < \infty,
$$

then every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$ for all $\lambda > 1$.

Theorem 2.12. If the assumptions of Theorem 2.10 hold except that (2.23) is replaced by

$$
\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)\mathrm{d}s = \infty \text{ and } \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{h^2(t,s)}{W(s,T)}\mathrm{d}s < \infty,
$$

then every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$ for all $\lambda > 0$.

The following two oscillation criteria treat the case when it is not possible to verify easily conditions (2.22) and (2.23).

Theorem 2.13. Assume that all the assumptions of Theorem 2.9 hold except (2.22). Further, let $H \in \mathcal{R}$ and assume that

$$0 < \inf_{s \geq t_0} \left[\liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty. \tag{2.24}$$

Let $\psi \in C([t_0, \infty), \mathbb{R})$ be such that for $t \geq t_0$ and $T \geq t_0$

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{Q^2(t,s)}{W_1(s)} \mathrm{d}s < \infty, \qquad \limsup_{t \to \infty} \int_{t_0}^{t} \psi_+^2(s) W_1(s) \mathrm{d}s = \infty, \tag{2.25}$$

and

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ H(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right\} \mathrm{d}s \geq \psi(T),$$

where $Q(t,s)$ is as in Theorem 2.9 and $\psi_+(t) = \max\{\psi(t), 0\}$. Then every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$ for all $\lambda > 1$.

Proof. We proceed as in the proof of Theorem 2.9, and the remainder of the proof is similar to that of [41, Theorem 5.2] and hence is omitted. ■

Theorem 2.14. Assume that all the assumptions of Theorem 2.9 hold except (2.22). Let $H \in \mathcal{R}$ and assume that (2.24) holds. Suppose there exists a function $\psi \in C([t_0, \infty), \mathbb{R})$ such that for $t > t_0$ and $T \geq t_0$ (2.25) holds,

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s) \mathrm{d}s < \infty,$$

and

$$\limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{4W_1(s)} \right) \mathrm{d}s \geq \psi(T),$$

where $Q(t,s)$ is as in Theorem 2.9 and $\psi_+(t) = \max\{\psi(t), 0\}$. Then every solution x of (1.1) oscillates or satisfies $x(t) \to 0$ as $t \to \infty$ for all $\lambda > 1$.

Proof. The proof is similar to that of [41, Theorem 5.2] and hence is omitted. ■

Remark 2.15. For the choice $H(t,s) = (t-s)^n$ and $h(t,s) = n(t-s)^{(n-2)/2}$, the Philos-type condition reduces to the Kamenev-type condition. Other choices of H include

$$H(t,s) = \left(\ln \frac{t}{s} \right)^n \quad \text{and} \quad h(t,s) = \frac{n}{s} \left(\ln \frac{t}{s} \right)^{n/2-1}$$

or more generally,

$$H(t,s) = \left(\int_s^t \frac{\mathrm{d}u}{\theta(u)} \right)^n \quad \text{and} \quad h(t,s) = \frac{n}{\theta(s)} \left(\int_s^t \frac{\mathrm{d}u}{\theta(u)} \right)^{n/2-1},$$

where $n \in \mathbb{N} \setminus \{1\}$ and $\theta \in C([t_0, \infty), \mathbb{R}^+)$ satisfies $\lim_{t\to\infty} \int_{t_0}^{t} \frac{\mathrm{d}u}{\theta(u)} = \infty$, and

$$H(t,s) = (e^t - e^s)^n \quad \text{and} \quad h(t,s) = ne^s \left(e^t - e^s\right)^{(n-2)/2}.$$

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