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Lyapunov Exponents of Polynomials with Respect to Certain Weighted Lyubich Measures

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Abstract

In this paper, we consider the structurally stable family of monic, centred, hyperbolic polynomials of degree $d \geq 2$, restricted on their respective Julia sets and compute the Lyapunov exponents of every member in the family with respect to certain weighted Lyubich measures. In particular, we show a certain well-behavedness of some coefficients of the Lyapunov exponents, that quantifies the non-well-behavedness in a system.

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1 Introduction

The theory of Lyapunov exponents plays a significant role in the theory of dynamical systems. The Lyapunov exponent is a major characteristic computable quantity that describes the well-behavedness or otherwise of typical trajectories in dynamical systems. The relationship between the Lyapunov exponent and exponential divergence of typical trajectories, as studied by Benettin *et al* in [4], have been widely used to understand various other properties of such systems. This study has been continued by various people in different fields of mathematics as well as physics. The relationship between Lyapunov exponent and other computable quantities such as topological entropy, index of exponential stability and supersymmetry are well-explored topics amidst mathematicians and physicists, for example, [4, 8, 11, 12, 17]. Important works of recent times

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include [1, 2, 6, 10, 13]. In particular, the relationship between the Lyapunov exponent and the degree of stochasticity for typical trajectories like Kolmogorov entropy has been calculated by Pesin in [11].

The concept of using symbolic spaces is a powerful tool in calculating the Lyapunov exponent for any dynamical system. This has been used in the study of dynamical systems, by various authors, including [7, 9, 11–13, 17]. The authors, in [14], make use of this concept and compute the Lyapunov exponent of quadratic and cubic polynomials satisfying a few technical assumptions, with respect to various measures associated to the family of Bernoulli measures on appropriate symbolic spaces. In fact, they establish the dependence of the Lyapunov exponent on the coefficients of the polynomials and their relationship to the Hausdorff dimension of the respective Julia set.

A surprising, yet powerful result by the authors in [14] was regarding the second derivative of the Lyapunov exponent of a monic, centred, hyperbolic quadratic polynomial parameterised by a complex co-efficient restricted on its Julia set, with respect to the real part and the imaginary part of the parameter, as well as the second derivative of the Lyapunov exponent of a monic, centred, hyperbolic cubic polynomial parameterised by two complex coefficients restricted on its Julia set, with respect to the respective real part and the imaginary part of the same parameter agreed in size, but varied in sign. This paper naturally has its genesis from there, to verify the same for a polynomial with similar properties, however of any arbitrary degree, say d. We will now state the main theorems of this paper.

Theorem 1.1. *Consider the family of monic, centred, hyperbolic polynomial maps consisting of* $P \equiv P_{(\kappa_{d-2}, \cdots, \kappa_0)}$*, of degree* $d > 1$*, with complex coefficients, given by*

$$\begin{aligned} P(z) &:= z^d + \kappa_{d-2} z^{d-2} + \cdots + \kappa_1 z + \kappa_0 \\ &= z^d + (\alpha_{d-2} + i\beta_{d-2}) z^{d-2} + \cdots + (\alpha_1 + i\beta_1) z + (\alpha_0 + i\beta_0), \end{aligned}$$

where $\kappa_r \in \mathbb{C}$ *satisfying* $|\kappa_r| < 1$ *for all* $0 \le r \le d-2$*, restricted on their respective Julia sets,* $\mathcal{J}_P$*. Let the coefficients* $(\kappa_{d-2}, \cdots, \kappa_1, \kappa_0) \in \mathcal{U}$*, a neighbourhood of the origin in* $\mathbb{C}^{d-1}$ *that preserves the hyperbolicity and structural stability of the considered family. Further, suppose that every critical orbit of* P *is bounded. If* Λ_μ *denotes the Lyapunov exponent of* P *with respect to a probability measure* μ *supported on* $\mathcal{J}_P$ *given by*

$$\Lambda_\mu(P) := -\int_{\mathcal{J}_P} \log|P'| d\mu, \tag{1.1}$$

then, irrespective of the strictly positive probability vector $\vec{p} = (p_1, p_2, \cdots, p_d)$ *based on which we define a probability measure* $\mu_{\vec{p}}$ *supported on* $\mathcal{J}_P$*, we have*

$$\frac{\partial^2 \Lambda_{\mu_{\vec{p}}}}{\partial\alpha_r \partial\alpha_s} = -\frac{\partial^2 \Lambda_{\mu_{\vec{p}}}}{\partial\beta_r \partial\beta_s}, \quad \textit{for every } 0 \le r, s \le d-2.$$

In the course of proving this theorem, we will evaluate the Lyapunov exponent of a monic, centred, hyperbolic polynomial P of degree d, up to terms with a certain order. Then, we also obtain the following theorem.

Theorem 1.2. *Consider the polynomial P that satisfies the hypotheses in theorem* (1.1) *with real coefficients and complex coefficients separately as given below:*

$$P_{\mathbb{R}}(z) = z^d + \alpha_{d-2} z^{d-2} + \cdots + \alpha_1 z + \alpha_0, \tag{1.2}$$
$$P_{\mathbb{C}}(z) = z^d + (\alpha_{d-2} + i\beta_{d-2})\, z^{d-2} + \cdots + (\alpha_1 + i\beta_1)\, z + (\alpha_0 + i\beta_0)\,. \tag{1.3}$$

Let $\Lambda_{\mu_{\vec{p}}}$ denote the appropriate Lyapunov exponent with respect to the probability measure $\mu_{\vec{p}}$ based on the strictly positive probability vector $\vec{p} = (p_1, p_2, \cdots, p_d)$. Then,

$$\frac{\partial \Lambda_{\mu_{\vec{p}}}(P_{\mathbb{R}})}{\partial \alpha_r} = \frac{\partial \Lambda_{\mu_{\vec{p}}}(P_{\mathbb{C}})}{\partial \alpha_r}; \qquad \frac{\partial^2 \Lambda_{\mu_{\vec{p}}}(P_{\mathbb{R}})}{\partial \alpha_r^2} = \frac{\partial^2 \Lambda_{\mu_{\vec{p}}}(P_{\mathbb{C}})}{\partial \alpha_r^2}.$$

This paper is structured as follows: In the following section namely section (2), we state the necessary definitions and build the necessary preliminaries that would give us the basic settings on which the results of this paper rest. In section (3), we define the equidistributed Lyubich measure and the weighted Lyubich measure, by according different weights on the preimage branches of generic points under the considered polynomial map. In section (4), we make use of a topological conjugacy and calculate certain coefficient functions (only as much as necessary), that will come in handy while we compute the Lyapunov exponents in section (5). With all the computations done, we prove our main theorems in section (6) and conclude the paper, with corollaries by interpreting the theorem for the specific cases when the polynomial is quadratic or cubic.

2 Basic Settings

In this section, we focus on building the basic settings of this paper by writing the essential terminologies, elementary definitions and a few results from the literature that we will require to prove the theorems.

We consider a monic centred polynomial, say P of degree $d > 1$, with complex coefficients defined on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, given by

$$P(z) = \kappa_d z^d + \kappa_{d-1} z^{d-1} + \kappa_{d-2} z^{d-2} + \cdots + \kappa_1 z + \kappa_0.$$

Here, monic means $\kappa_d \equiv 1$ and centred means $\kappa_{d-1} \equiv 0$. Further, we consider the remaining coefficients to be complex; $\kappa_r = \alpha_r + i\beta_r$ with $\alpha_r, \beta_r \in \mathbb{R}$ for $0 \le r \le d-2$. It is then obvious that the critical points of P are determined by the $(d-2)$ coefficients $\kappa_{d-2}, \cdots, \kappa_1$ while each of the critical orbit is determined by the $(d-1)$ coefficients $\kappa_{d-2}, \cdots, \kappa_1, \kappa_0$. In particular, the origin in $\mathbb{C}^{d-1}$ is the point that yields the polynomial $P(z) = z^d$. The critical point of $P(z) = z^d$ (of multiplicity $d-1$) is the fixed point of the polynomial at $0 \in \mathbb{C}$, and thus has a bounded critical orbit. We reserve the notation $Q(z)$ to represent this polynomial, throughout this paper, where all, but the leading coefficient, are 0; *i.e.*, $Q(z) \equiv z^d$.

For a polynomial P given by,

$$P(z) \;=\; z^d + \kappa_{d-2} z^{d-2} + \cdots + \kappa_1 z + \kappa_0, \tag{2.1}$$

with complex coefficients, we define its filled Julia set $\mathcal{K}_P$ and its basin of attraction to the point at infinity $\mathcal{A}_P(\infty)$ respectively as

$$\mathcal{K}_P \;:=\; \{z \in \mathbb{C} : P^n(z) \nrightarrow \infty \text{ for any } n\}\,; \qquad \mathcal{A}_P(\infty) \;:=\; \{z \in \mathbb{C} : P^n(z) \to \infty\}\,.$$

It is, of course, obvious from the definition that the completely P-invariant sets $\mathcal{K}_P$ and $\mathcal{A}_P(\infty)$ dichotomise the Riemann sphere, $\overline{\mathbb{C}}$. The common topological boundary between the sets $\mathcal{K}_P$ and $\mathcal{A}_P(\infty)$ is then defined to be the *Julia set* of the polynomial, denoted by $\mathcal{J}_P$. For example, suppose $\kappa_r \equiv 0$, $\forall 0 \le r \le d-2$, then for the polynomial $Q(z) = z^d$, we have

$$\mathcal{K}_Q \;=\; \{z \in \mathbb{C} : |z| \le 1\}\,; \qquad \mathcal{A}_Q(\infty) \;=\; \{z \in \mathbb{C} : |z| > 1\}\,;$$

$$\implies \qquad \mathcal{J}_Q \;=\; \{z \in \mathbb{C} : |z| = 1\}\,.$$

Interested readers may know that the Julia set of a polynomial map P of degree $d > 1$ is also defined as the closure of the set of all periodic points that satisfy a repelling condition,

$$\mathcal{J}_P \;=\; \overline{\left\{z_0 \in \mathbb{C} : P^m z_0 = z_0 \text{ for some } m \in \mathbb{Z}_+ \text{ and } \left|(P^m)'(z_0)\right| > 1\right\}}.$$

Alternatively, $\mathcal{J}_P$ is the set of points where the family of iterates of P, *i.e.*, $\{P^n\}_{n\ge 1}$ does not form a normal family (in the sense of Montel). The various definitions elucidate that $\mathcal{J}_P$ is a non-empty, compact, completely P-invariant metric space. For more properties of the Julia set, one may refer [3, 9].

In this paper, we shall be interested in restricting the parameter space of the polynomial $(\kappa_{d-2}, \cdots, \kappa_1, \kappa_0) \in \mathbb{C}^{d-1}$ to that connected component containing the origin, where any two polynomials restricted on their respective hyperbolic Julia sets are conjugate to each other, *i.e.*, suppose $\mathcal{U} \subseteq \mathbb{C}^{d-1}$ is a neighbourhood of $\mathbf{0} \in \mathbb{C}^{d-1}$, then, for any two points $(\kappa_{d-2}, \cdots, \kappa_1, \kappa_0)$, $(\eta_{d-2}, \cdots, \eta_1, \eta_0) \in \mathcal{U}$ describing the polynomials P_1 and P_2 respectively given by,

$$P_1(z) \;=\; z^d + \kappa_{d-2} z^{d-2} + \cdots + \kappa_1 z + \kappa_0 \quad \text{and} \quad P_2(z) \;=\; z^d + \eta_{d-2} z^{d-2} + \cdots + \eta_1 z + \eta_0,$$

there exists a conjugacy, say $\Phi_{(P_1,P_2)} : \mathcal{J}_{P_1} \longrightarrow \mathcal{J}_{P_2}$ that satisfies $\Phi_{(P_1,P_2)} \circ P_1 = P_2 \circ \Phi_{(P_1,P_2)}$. Further, by *hyperbolicity*, we mean that there exists constants $C > 0$ and $\lambda > 1$ such that for any $z \in \mathcal{J}_P$, we have $|\,(P^n)'(z)| \ge C\lambda^n$ for all $n \ge 1$. The conjugacy $\Phi_{(P_1,P_2)}$ is naturally dependent on the polynomials P_1 and P_2, in other words, the respective coefficients. In particular, we shall be interested in the conjugacy between the unit circle $\mathbb{S}^1$ which is the Julia set of $Q(z) = z^d$ described by $\mathbf{0} \in \mathcal{U} \subset \mathbb{C}^{d-1}$ and the Julia set of the monic centred hyperbolic polynomial map P as written in equation

(2.1), described by $(\kappa_{d-2}, \cdots, \kappa_1, \kappa_0) \in \mathcal{U} \subset \mathbb{C}^{d-1}$, namely $\Phi_{(Q,P)} : \mathbb{S}^1 = \mathcal{J}_Q \longrightarrow \mathcal{J}_P$ that satisfies

$$\Phi_{(Q,P)}\left(z^d\right) - P\left(\Phi_{(Q,P)}(z)\right) \;=\; 0. \tag{2.2}$$

Here, we keep the polynomial Q fixed and focus on the dependence of the conjugacy on the coefficients of P. Thus, we suppress Q in the notation of the conjugacy and denote the same merely by Φ_P. It is then a result from [5, 9, 17] that has been interpreted in theorems (4.1) and (4.2) in [14] that the conjugacy Φ_P depends analytically on each of the parameters, κ_r for $d-2 \geq r \geq 0$.

The rationale behind working merely with the set of all monic, centred polynomial maps P, as given in equation (2.1) with complex coefficients $\kappa_r = \alpha_r + i\beta_r$ is due to the fact that any polynomial of degree d can, by an affine change of coordinates, be written as in equation (2.1). For the sake of computations in this paper, we also demand that the coefficients κ_r in equation (2.1) satisfy $|\kappa_r|^2 = \alpha_r^2 + \beta_r^2 < 1$ for $d-2 \geq r \geq 0$.

3 Weighted Lyubich Measures

In this section, we shall focus on the space of P-invariant probability measures supported on the compact metric space $\mathcal{J}_P$, denoted by $\mathcal{M}_P$, *i.e.*,

$$\mathcal{M}_P \;:=\; \left\{\mu : \mathscr{B}\left(\mathcal{J}_P\right) \longrightarrow [0,1] \;:\; \mu(E) = \mu(P^{-1}E)\; \forall E \in \mathscr{B}\left(\mathcal{J}_P\right)\right\}.$$

Here, $\mathscr{B}\left(\mathcal{J}_P\right)$ denotes the σ-algebra of subsets of $\mathcal{J}_P$.

Owing to the density of preimages of any point $\zeta \in \mathcal{J}_P$, it can be observed that the sequence of measures

$$\mu_n^{(\zeta)} \;:=\; \frac{1}{d^n} \sum_{P^n\omega = \zeta} \delta_\omega, \quad \text{where } \delta_\omega \text{ is the Dirac delta measure at the point } \omega, \tag{3.1}$$

converges to some measure $\mu \in \mathcal{M}_P$ called the *Lyubich measure*, independent of ζ, in the weak*-topology, see for example [15]. For example, the polynomial map $Q(z) = z^d$ has the unit circle, $\mathbb{S}^1$ in $\mathbb{C}$ as its Julia set and the Lyubich measure can be thought of as the Haar measure on $\mathbb{S}^1$. It is then obvious that the support of the Lyubich measure is the Julia set.

An effective implication of the Lyubich measure is that if the hyperbolic $\mathcal{J}_P$ is divided into d mutually disjoint equal arcs, then every arc contains one and only one preimage of any generic point $\zeta \in \mathcal{J}_P$. Thus, by according equal weightage to every preimage branch, we obtain an equilibrium distribution. However, suppose we accord different weights to the different preimage branches of P, we obtain a distorted distribution, as we now explain.

Let $\{P_1, P_2, \cdots, P_d\}$ denote the preimage branches of the polynomial map P. For some strictly positive probability vector, $\vec{p}$, *i.e.*, $\vec{p} = (p_1, p_2, \cdots, p_d)$ with $p_j > 0,\ \forall j$

and $\sum_{j=1}^{d} p_j = 1$, we define a quantity called the weighted Lyubich measure as follows. Consider any n-lettered word $\theta = (\theta_1, \theta_2, \cdots, \theta_n) \in \{1, 2, \cdots, d\}^n$. We define the appropriate n-th order preimage branch of P as $P_\theta = P_{\theta_n} \circ P_{\theta_{n-1}} \circ \cdots \circ P_{\theta_1}$. Then, for any generic point $\zeta \in \mathcal{J}_P$, we define a sequence of measures with respect to $\vec{p}$ namely $\left\{ \left(\mu_{\vec{p}}^{(\zeta)} \right)_n \right\}_{n \geq 1}$ as

$$\left(\mu_{\vec{p}}^{(\zeta)} \right)_n := \sum_{\theta \,:\, P_\theta \omega = \zeta} p_{\theta_n} p_{\theta_{n-1}} \cdots p_{\theta_1} \delta_\omega. \tag{3.2}$$

Then, owing to the uniform distribution of the preimage branches in $\mathcal{J}_P$, when p_j's are not equally distributed in the probability vector $\vec{p}$, some preimage branches gain prominence over the other ones. However, as n increases, the sections of the Julia set, $\mathcal{J}_P$, that gain prominence get tinier and tinier. In any case, this sequence of measures $\left\{ \left(\mu_{\vec{p}}^{(\zeta)} \right)_n \right\}_{n \geq 1}$ converges to some measure $\mu_{\vec{p}} \in \mathcal{M}_P$ called the *weighted Lyubich measure*, independent of ζ, in the weak*-topology.

This family of weighted Lyubich measures is interesting to work with, especially when one of the p_j's is extremely close to 1, leaving the remainder of the p_k's to be arbitrarily close to 0. In such a case, the section of the Julia set corresponding to the preimage branch that gains prominence in the Julia set, eventually reduces to a point measure.

We conclude this section by defining a quantity called *pressure* for a real-valued continuous function, say f, defined on $\mathcal{J}_P$, in accordance with thermodynamic formalism. This will come in handy, when we compute the Lyapunov exponent, later.

$$\mathfrak{P}(f) := \sup_{\mu \in \mathcal{M}_P} \left\{ h_\mu(P) + \int_{\mathcal{J}_P} f d\mu \right\}, \tag{3.3}$$

where $h_\mu(P)$ represents the measure theoretic entropy of P with respect to the measure μ. For more properties of pressure and entropy, interested readers are referred to [16].

In this paper, we are interested in the real-valued continuous function $-\log|P'|$ defined on $\mathcal{J}_P$. Hence,

$$\begin{aligned} \mathfrak{P}\left(-\log|P'|\right) &= \sup_{\mu \in \mathcal{M}_P} \left\{ h_\mu(P) - \int_{\mathcal{J}_P} \log|P'| d\mu \right\} \\ &= \sup_{\mu \in \mathcal{M}_P} \left\{ h_\mu(P) + \Lambda_\mu(P) \right\} \\ &= \sup_{\nu \in \mathcal{M}_Q} \left\{ h_\nu(Q) - \log d - \int_{\mathbb{S}^1} \log|\Phi(z)| d\nu \right\}, \end{aligned} \tag{3.4}$$

where, in order to obtain the last equality, we make an effective use of the conjugacy equation, as given in (2.2).

4 Coefficient Functions

In this section, we make necessary preparations to calculate the Lyapunov exponent, as defined in equation (1.1) of a monic, centred, hyperbolic polynomial P, as given in equation (2.1), that will help us obtain expressions, as required in the main theorems. Since we know from section (2) that the conjugacy Φ_P analytically depends on the coefficients κ_r for $d-2 \geq r \geq 0$, it is only reasonable to consider

$$\Phi_P(z) \;=\; z + \sum_{\xi_{d-2}+\cdots+\xi_1+\xi_0 \geq 1} \phi_{(\xi_{d-2},\cdots,\xi_1,\xi_0)}(z)\kappa_{d-2}^{\xi_{d-2}}\cdots\kappa_1^{\xi_1}\kappa_0^{\xi_0}.$$

Then, substituting this expression for Φ in equation (2.2) yields

$$\begin{aligned}
& z^d + \sum_{\xi_{d-2}+\cdots+\xi_1+\xi_0 \geq 1} \phi_{(\xi_{d-2},\cdots,\xi_1,\xi_0)}(z^d)\kappa_{d-2}^{\xi_{d-2}}\cdots\kappa_1^{\xi_1}\kappa_0^{\xi_0} \\
- & \left(z + \sum_{\xi_{d-2}+\cdots+\xi_1+\xi_0 \geq 1} \phi_{(\xi_{d-2},\cdots,\xi_1,\xi_0)}(z)\kappa_{d-2}^{\xi_{d-2}}\cdots\kappa_1^{\xi_1}\kappa_0^{\xi_0}\right)^d \\
- & \kappa_{d-2}\left(z + \sum_{\xi_{d-2}+\cdots+\xi_1+\xi_0 \geq 1} \phi_{(\xi_{d-2},\cdots,\xi_1,\xi_0)}(z)\kappa_{d-2}^{\xi_{d-2}}\cdots\kappa_1^{\xi_1}\kappa_0^{\xi_0}\right)^{d-2} \\
- & \cdots \\
- & \kappa_1\left(z + \sum_{\xi_{d-2}+\cdots+\xi_1+\xi_0 \geq 1} \phi_{(\xi_{d-2},\cdots,\xi_1,\xi_0)}(z)\kappa_{d-2}^{\xi_{d-2}}\cdots\kappa_1^{\xi_1}\kappa_0^{\xi_0}\right) - \kappa_0 \\
= & \; 0.
\end{aligned} \tag{4.1}$$

Using the above equation, it is possible to obtain the expression for all the functions $\phi_{(\xi_{d-2},\cdots,\xi_1,\xi_0)}(z)$ for the various $\xi_r \geq 0$ satisfying $\sum_{r=1}^{d-2}\xi_r \geq 1$. However, since the theorems (1.1) and (1.2) only deal with the first and the second derivatives of the Lyapunov exponent, we are only interested in the cases when $\sum_{r=1}^{d-2}\xi_r = 1$ and when $\sum_{r=1}^{d-2}\xi_r = 2$. The former case is achieved only when one and only one of the $\xi_r = 1$ and the remaining $\xi_s = 0$ while the latter case is achieved when either one and only one of the $\xi_r = 2$ and the remaining $\xi_s = 0$ or when two of the $\xi_r = 1$ and the remaining $\xi_s = 0$.

For ease of writing and the readers' convenience, we introduce the following notations.

$$\begin{aligned}
\phi_r(z) &= \phi_{(\xi_{d-2},\cdots,\xi_1,\xi_0)}(z) \text{ when } \xi_r = 1 \text{ and } \xi_s = 0,\ \forall d-2 \geq s \geq 0 \text{ with } s \neq r; \\
\phi_{r^2}(z) &= \phi_{(\xi_{d-2},\cdots,\xi_1,\xi_0)}(z) \text{ when } \xi_r = 2 \text{ and } \xi_s = 0,\ \forall d-2 \geq s \geq 0 \text{ with } s \neq r;
\end{aligned}$$

$$\begin{aligned}\phi_{rs}(z) &= \phi_{(\xi_{d-2},\cdots,\xi_1,\xi_0)}(z) \text{ when } \xi_r = \xi_s = 1 \text{ with } r < s \text{ and} \\ &\qquad \xi_t = 0,\ \forall d-2 \geq t \geq 0 \text{ with } t \neq r \text{ and } t \neq s. \end{aligned} \tag{4.2}$$

Thus, using these notations as in (4.2) and making necessary computations as far as necessary in equation (4.1), we obtain

$$\phi_r(z) = -z \sum_{\kappa_1 \geq 1} \frac{1}{d^{\kappa_1}} \frac{1}{z^{d^{\kappa_1} - r d^{\kappa_1 - 1}}} ; \tag{4.3}$$

$$\begin{aligned}\phi_{r^2}(z) = -z\Bigg[&\frac{d(d-1)}{2} \sum_{\kappa_3 \geq 1} \frac{1}{d^{\kappa_3}} \sum_{\kappa_2 \geq 1} \sum_{\kappa_1 \geq 1}^{\kappa_2} \frac{1}{d^{\kappa_2 + 1}} \\ &\times \frac{1}{z^{d^{\kappa_3} - d^{\kappa_3 - 1} + d^{\kappa_3 - 1}(d^{\kappa_1} - r d^{\kappa_1 - 1} + d^{\kappa_2 - \kappa_1 + 1} - r d^{\kappa_2 - \kappa_1} - d + 1)}} \\ &- r \sum_{\kappa_3 \geq 1} \frac{1}{d^{\kappa_3}} \sum_{\kappa_1 \geq 1} \frac{1}{d^{\kappa_1}} \frac{1}{z^{d^{\kappa_3} - d^{\kappa_3 - 1} + d^{\kappa_3 - 1}(d^{\kappa_1} - r d^{\kappa_1 - 1} - r + 1)}}\Bigg] ; \end{aligned} \tag{4.4}$$

$$\begin{aligned}\phi_{rs}(z) = -z\Bigg[&d(d-1) \sum_{\kappa_3 \geq 1} \frac{1}{d^{\kappa_3}} \sum_{\kappa_2 \geq 1} \sum_{\kappa_1 \geq 1}^{\kappa_2} \frac{1}{d^{\kappa_2 + 1}} \\ &\times \frac{1}{z^{d^{\kappa_3} - d^{\kappa_3 - 1} + d^{\kappa_3 - 1}(d^{\kappa_1} - r d^{\kappa_1 - 1} + d^{\kappa_2 - \kappa_1 + 1} - s d^{\kappa_2 - \kappa_1} - d + 1)}} \\ &- r \sum_{\kappa_3 \geq 1} \frac{1}{d^{\kappa_3}} \sum_{\kappa_1 \geq 1} \frac{1}{d^{\kappa_1}} \frac{1}{z^{d^{\kappa_3} - d^{\kappa_3 - 1} + d^{\kappa_3 - 1}(d^{\kappa_1} - s d^{\kappa_1 - 1} - r + 1)}} \\ &- s \sum_{\kappa_3 \geq 1} \frac{1}{d^{\kappa_3}} \sum_{\kappa_1 \geq 1} \frac{1}{d^{\kappa_1}} \frac{1}{z^{d^{\kappa_3} - d^{\kappa_3 - 1} + d^{\kappa_3 - 1}(d^{\kappa_1} - r d^{\kappa_1 - 1} - s + 1)}}\Bigg]. \end{aligned} \tag{4.5}$$

5 Computation of Lyapunov Exponents

We know from equation (3.4) that in order to calculate the Lyapunov exponent of P with respect to the measure $\mu_{\vec{p}}$, it is sufficient for us to evaluate $\int_{\mathbb{S}^1} \log|\Phi_P(z)| d\mu_{\vec{p}}$. We now undertake the necessary computations, here.

$$\begin{aligned} &- \int_{\mathbb{S}^1} \log|\Phi_P(z)| d\mu_{\vec{p}} \qquad (5.1) \\ = &- \int_{\mathbb{S}^1} \sum_{r=0}^{d-2} \mathrm{Re}\left(\kappa_r \overline{z} \phi_r\right) d\mu_{\vec{p}} \\ &- \int_{\mathbb{S}^1} \sum_{r=0}^{d-2} \left[\mathrm{Re}\left(\kappa_r^2 \overline{z} \phi_{r^2}\right) - \frac{1}{2}\left\{\mathrm{Re}\left(\kappa_r \overline{z} \phi_r\right)\right\}^2 + \frac{1}{2}\left\{\mathrm{Im}\left(\kappa_r \overline{z} \phi_r\right)\right\}^2\right] d\mu_{\vec{p}} \\ &- \int_{\mathbb{S}^1} \sum_{r=0}^{d-3} \sum_{r<s=1}^{d-2} \Big[\mathrm{Re}\left(\kappa_r \kappa_s \overline{z} \phi_{rs}\right) - \left\{\mathrm{Re}\left(\kappa_r \overline{z} \phi_r\right) \times \mathrm{Re}\left(\kappa_s \overline{z} \phi_s\right)\right\} \end{aligned}$$

$$+ \left\{ \mathrm{Im}\left(\kappa_r \overline{z} \phi_r\right) \times \mathrm{Im}\left(\kappa_s \overline{z} \phi_s\right) \right\} \Big] d\mu_{\vec{p}}$$

$$+ O\left(\text{terms where } \sum_{r=0}^{d-2} \xi_r \geq 3 \right). \tag{5.2}$$

We first observe that each of the integrals in the right hand side of equation (5.1) evaluates to 0 when the measure of integration is the equidistributed Lyubich measure, as defined in section (3). We now evaluate the integrals with respect to the weighted Lyubich measure, also defined in section (3), in particular when one of the $p_j \uparrow 1$ for some $d \geq j \geq 1$. We separate the cases of the polynomial, $P_{\mathbb{R}}$, as written in equation (1.2) with only real coefficients, *i.e.*, $\kappa_r = \alpha_r$ and $\beta_r \equiv 0 \, \forall d-2 \geq r \geq 0$ and $P_{\mathbb{C}}$, as written in equation (1.3) with complex coefficients, *i.e.*, $\kappa_r = \alpha_r + i\beta_r, \ \forall d-2 \geq r \geq 0$.

Computations for $P_{\mathbb{R}}$

We urge the reader to observe that when the coefficients of the polynomial are all real, we take the terms containing $\mathrm{Re}\left(\kappa_r\right) = \alpha_r$ out of the integral, as a multiplicative factor. Further, in this case, we have $\mathrm{Im}\left(\kappa_r\right) = 0$. Thus,

$$\begin{aligned}
\int \mathrm{Re}\left(\overline{z}\phi_r\right) d\mu_{\vec{p}} \quad &\to \quad -\frac{1}{d-1} \ ; \\
\int \mathrm{Re}\left(\overline{z}\phi_{r^2}\right) d\mu_{\vec{p}} \quad &\to \quad -\frac{d-2r}{2(d-1)^2} \ ; \\
\frac{1}{2}\int \left[\mathrm{Re}\left(\overline{z}\phi_r\right)\right]^2 d\mu_{\vec{p}} \quad &\to \quad \frac{1}{2}\frac{1}{(d-1)^2} \ ; \\
\int \mathrm{Re}\left(\overline{z}\phi_{rs}\right) d\mu_{\vec{p}} \quad &\to \quad -\frac{d-r-s}{(d-1)^2} \ ; \\
\int \left[\mathrm{Re}\left(\overline{z}\phi_r\right) \times \mathrm{Re}\left(\overline{z}\phi_s\right)\right] d\mu_{\vec{p}} \quad &\to \quad \frac{1}{(d-1)^2}.
\end{aligned}$$

Computations for $P_{\mathbb{C}}$

In this case, $\kappa_r = \alpha_r + i\beta_r$ for all $d-2 \geq r \geq 0$. Then, the computations yield

$$\begin{aligned}
\int \mathrm{Re}\left(\kappa_r \overline{z}\phi_r\right) d\mu_{\vec{p}} \quad &\to \quad -\frac{1}{d-1}\alpha_r \ ; \\
\int \mathrm{Re}\left(\kappa_r^2 \overline{z}\phi_{r^2}\right) d\mu_{\vec{p}} \quad &\to \quad -\frac{d-2r}{2(d-1)^2}\left(\alpha_r^2 - \beta_r^2\right) \ ; \\
\frac{1}{2}\int \left[\mathrm{Re}\left(\kappa_r \overline{z}\phi_r\right)\right]^2 d\mu_{\vec{p}} \quad &\to \quad \frac{1}{2}\frac{1}{(d-1)^2}\alpha_r^2 \ ;
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\int\left[\operatorname{Im}\left(\kappa_r\overline{z}\phi_r\right)\right]^2 d\mu_{\vec{p}} &\to \frac{1}{2}\frac{1}{(d-1)^2}\beta_r^2\ ;\\
\int \operatorname{Re}\left(\kappa_r\kappa_s\overline{z}\phi_{rs}\right) d\mu_{\vec{p}} &\to -\frac{d-r-s}{(d-1)^2}\left(\alpha_r\alpha_s-\beta_r\beta_s\right)\ ;\\
\int\left[\operatorname{Re}\left(\kappa_r\overline{z}\phi_r\right)\times\operatorname{Re}\left(\kappa_s\overline{z}\phi_s\right)\right] d\mu_{\vec{p}} &\to \frac{1}{(d-1)^2}\left(\alpha_r\alpha_s\right)\ ;\\
\int\left[\operatorname{Im}\left(\kappa_r\overline{z}\phi_r\right)\times\operatorname{Im}\left(\kappa_s\overline{z}\phi_s\right)\right] d\mu_{\vec{p}} &\to \frac{1}{(d-1)^2}\left(\beta_r\beta_s\right).
\end{aligned}$$

Thus, from the computations of the Lyapunov exponent, we have, as $p_j \uparrow 1$, for some $d \geq j \geq 1$, that

$$\begin{aligned}
\Lambda_{\mu_{\vec{p}}}(P_{\mathbb{R}}) \ \to\ & -\log d + \sum_{r=0}^{d-2}\frac{1}{d-1}\alpha_r + \sum_{r=0}^{d-2}\frac{d-2r+1}{2(d-1)^2}\alpha_r^2\\
& + \sum_{r=0}^{d-3}\sum_{r<s=1}^{d-2}\frac{d-r-s+1}{(d-1)^2}\alpha_r\alpha_s. \qquad (5.3)
\end{aligned}$$

$$\begin{aligned}
\Lambda_{\mu_{\vec{p}}}(P_{\mathbb{C}}) \ \to\ & -\log d + \sum_{r=0}^{d-2}\frac{1}{d-1}\alpha_r + \sum_{r=0}^{d-2}\frac{d-2r+1}{2(d-1)^2}\alpha_r^2 - \sum_{r=0}^{d-2}\frac{d-2r+1}{2(d-1)^2}\beta_r^2\\
& + \sum_{r=0}^{d-3}\sum_{r<s=1}^{d-2}\frac{d-r-s+1}{(d-1)^2}\alpha_r\alpha_s - \sum_{r=0}^{d-3}\sum_{r<s=1}^{d-2}\frac{d-r-s+1}{(d-1)^2}\beta_r\beta_s. \qquad (5.4)
\end{aligned}$$

6 Proofs of the Main Results

In this concluding section, we write the proof of the main theorems (1.1) and (1.2), by appealing to the computations that we have done in section (5).

Proof of Theorem 1.1. We obtain the following expressions, by directly differentiating equation (5.4), with respect to the corresponding variable.

$$\frac{\partial^2\Lambda_{\mu_{\vec{p}}}}{\partial\alpha_r^2} = \frac{d-2r+1}{2(d-1)^2}, \qquad \frac{\partial^2\Lambda_{\mu_{\vec{p}}}}{\partial\beta_r^2} = -\frac{d-2r+1}{2(d-1)^2},$$

$$\frac{\partial^2\Lambda_{\mu_{\vec{p}}}}{\partial\alpha_r\partial\alpha_s} = \frac{d-r-s+1}{(d-1)^2}, \qquad \frac{\partial^2\Lambda_{\mu_{\vec{p}}}}{\partial\beta_r\partial\beta_s} = -\frac{d-r-s+1}{(d-1)^2} \quad \text{for } d-2 \geq s > r \geq 0.$$

Thus, we have

$$\frac{\partial^2\Lambda_{\mu_{\vec{p}}}}{\partial\alpha_r\partial\alpha_s} = -\frac{\partial^2\Lambda_{\mu_{\vec{p}}}}{\partial\beta_r\partial\beta_s}, \quad \forall d-2 \geq s > r \geq 0.$$

The proof is complete. □

Proof of Theorem 1.2. We obtain the following expressions, by directly differentiating equation (5.4) and (5.3), with respect to the corresponding variable.

$$\begin{aligned} \frac{\partial \Lambda_\mu(P_{\mathbb{R}})}{\partial \alpha_r} &= \frac{\partial \Lambda_\mu(P_{\mathbb{C}})}{\partial \alpha_r} = \frac{1}{d-1} + \frac{d-2r+1}{2(d-1)^2}\alpha_r + \sum_{r \neq s = 0}^{d-2} \frac{d-r-s}{(d-1)^2}\alpha_s; \\ \frac{\partial^2 \Lambda_\mu(P_{\mathbb{R}})}{\partial \alpha_r^2} &= \frac{\partial^2 \Lambda_\mu(P_{\mathbb{C}})}{\partial \alpha_r^2} = \frac{d-2r+1}{2(d-1)^2}. \end{aligned}$$

The proof is complete. □

The observations in [14] by the authors, are then simple corollaries to specific cases when $d = 2$ and $d = 3$. We complete this paper with these following corollaries.

Corollary 6.1. *For a monic centred hyperbolic quadratic polynomial*

$$R_2(z) \;=\; z^2 + \alpha + i\beta \quad \textit{with} \;\; \alpha^2 + \beta^2 < 1,$$

we have

$$\Lambda_{\mu_{\vec{p}}}(R_2) \;\to\; -\log 2 \;+\; \alpha \;+\; \frac{3}{2}\left(\alpha^2 - \beta^2\right), \;\; \textit{as } p_1 \uparrow 1 \textit{ or as } p_2 \uparrow 1.$$

Corollary 6.2. *For a monic centred hyperbolic cubic polynomial*

$$R_3(z) \;=\; z^3 + (\alpha_1 + i\beta_1)\, z + (\alpha_0 + i\beta_0) \quad \textit{with} \;\; \alpha_r^2 + \beta_r^2 < 1 \;\textit{for } r = 1, 0,$$

we have

$$\begin{aligned} \Lambda_{\mu_{\vec{p}}}(R_3) \;\to\; & -\log 3 \;+\; \frac{1}{2}(\alpha_1 + \alpha_0) \;+\; \frac{1}{2}\left(\alpha_0^2 - \beta_0^2\right) \;+\; \frac{1}{4}\left(\alpha_1^2 - \beta_1^2\right) \\ & + \frac{3}{4}(\alpha_1\alpha_0 - \beta_1\beta_0) \qquad \textit{as } p_1 \uparrow 1,\; p_2 \uparrow 1 \textit{ or as } p_3 \uparrow 1. \end{aligned}$$

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