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On Equations of Wiener–Poisson Type with Small Parameters

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Abstract

We provide a large deviation principle for some stochastic-jumps equations with periodic coefficients and highly oscillating drift. We employ the Legendre–Fenchel transform of the limit of normalized logarithm of an exponential moment and a uniform upper bound for the characteristic function of a Feller process.

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1 Introduction

Let $\varepsilon, \delta_\varepsilon > 0$. Our concern is a periodic stochastic differential equations (SDE) of Wiener–Poisson type in the whole space $\mathbb{R}^d$ satisfying

$$\begin{aligned} X_t^{x,\varepsilon,\delta_\varepsilon} =& x - \sqrt{\varepsilon}\int_0^t \sigma\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) dW_s + \frac{\varepsilon}{\delta_\varepsilon}\int_0^t b\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) ds + \int_0^t c\left(\frac{X_s^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) ds \\ &+ \int_0^t \int_{\mathbb{R}^d_*} k\left(\frac{X_{s-}^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, y\right)\left(\varepsilon N^{\varepsilon^{-1}}(dsdy) - \nu(dy)ds\right),\ x \in \mathbb{R}^d,\ t \geq 0, \end{aligned} \tag{1.1}$$

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as defined in a given complete Wiener–Poisson space $\left(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, W_t, N^{\varepsilon^{-1}}\right)$ in $\mathbb{R}^d \times \mathbb{R}^d_*$, $\mathbb{R}^d_* \equiv \mathbb{R}^d \backslash \{0\}$, with the Lévy measure ν, i.e., $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a filtered probability space with $\mathbb{F} := \{\mathcal{F}_t : \ t \geq 0\}$ being the $\mathbb{P}$-completion of the filtration $\mathcal{F}$, $\{W_t : \ t \geq 0\}$ is a d-dimensional standard Brownian motion with respect to $\mathcal{F}_t$ and $N^{\varepsilon^{-1}}$ is a Poisson random counting measure on $\mathbb{R}^d_*$ with intensity measure $\varepsilon^{-1}\nu$, independent of W and w.r.t. $\mathcal{F}_t$. One can refer to D. Applebaum [1] for a detailed study of the fundamental theory of stochastic integrals such as the Wiener–Lévy-type stochastic integrals and stochastic differential equations driven by pure jump processes. Our assumptions on the coefficients b, c, k and σ will be specified below.

The main purpose of this paper is to show that under suitable assumptions on the coefficients and the two parameters ε and δ_ε, the process $X_t^{x,\varepsilon,\delta_\varepsilon}$ (1.1) satisfies a large deviation principle (LDP) with good rate function. The family of equations (1.1) subject to ε (viscosity parameter) and δ_ε (homogenization parameter) is a classical problem which goes back to Paolo Baldi [2] at the end of 20'th century. In the case when $L_t^{\varepsilon,\delta_\varepsilon} \equiv 0$, the periodic SDE (1.1) with fast oscillating coefficients becomes one driven by small multiplicative noise and the similar issues was extensively investigated by Freidlin and Sowers [6]. It is well known that stochastic dynamical systems with Poisson jumps are more suitable for capturing sudden bursty fluctuations than classical diffusion models. However, there are still few results on LDP for stochastic evolution equations with jumps (see, for example [11,13,14]). The main difficulty in the study of SDE (1.1) is the presence of the jumps. As ε and δ_ε tend to zero, two well-known effects come into play. The effect of the viscosity parameter ε generates small excitations, and the other hand coefficients oscillate rapidly by means of the effect of the homogenization parameter δ_ε. In this paper, we consider the case that ε tends to zero sufficiently quickly compared to δ_ε. To do it, we should first treat δ_ε as fixed and carry out the calculations for slowly varying ε, then the theory of large deviation tells us how quickly $X^{x,\varepsilon,\delta_\varepsilon}$ tends to the deterministic dynamics given by actually setting ε to zero.

Definition 1.1. Let $X^{x,\varepsilon,\delta}$ be a $\mathbb{R}^d$-valued random variable and let $\mathbb{P}_{\varepsilon,\delta}$ denote its distribution on the Borel subsets of $\mathbb{R}^d$, that is, $\mathbb{P}_{\varepsilon,\delta}(A) = \mathbb{P}\left(X^{x,\varepsilon,\delta} \in A\right)$. The family $\left\{X^{x,\varepsilon,\delta}; \varepsilon, \delta \geq 0\right\}$ satisfies an LDP if there exists a lower semi-continuous function $I : \mathbb{R}^d \longrightarrow [0, +\infty]$ such that

- for each open set $O \subseteq \mathbb{R}^d$ $\liminf\limits_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left(X^{x,\varepsilon,\delta_\varepsilon} \in O\right) \geq - \inf\limits_{x \in O} I(x)$,
- for each closed set $F \subseteq \mathbb{R}^d$ $\limsup\limits_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left(X^{x,\varepsilon,\delta_\varepsilon} \in F\right) \leq - \inf\limits_{x \in F} I(x)$.

I is called the rate function for the LDP. A rate function is good if for each $a \in \mathbb{R}_+$, $\{x : I(x) \leq a\}$ is compact.

There are several *characterizations* for the large deviation principle (see, for example [3,5]). Our aim consists in computing the limit of $\varepsilon \log \mathbb{P}\left\{X^{x,\varepsilon,\delta_\varepsilon} \in A\right\}$ when ε and

δ_ε approach zero, where A is a Borel subset of $\mathcal{D}_x\left([0,T],\mathbb{R}^d\right)$, the set of càdlàd functions on $[0,T]$ with $\mathbb{R}^d$-values which take x at zero. In other word, we study the LDP for $X^{x,\varepsilon,\delta_\varepsilon}$ in a *pathwise* sense by very carefully using both characterizations: the change of measures (for the lower bound) and the derivative of the characteristic function of a Feller process (for the upper bound).

The rest of this paper is organized as follows. In Section 2 we set up some notation, make precisely our hypothesis and state the main result. In Section 3 we give the proofs of lower and upper bounds on LDP.

2 Main Results: Preliminaries and Formulation

2.1 Notation and Background

Denote expectation with respect to $\mathbb{P}$ by $\mathbb{E}$. We have already defined $\langle\cdot,\cdot\rangle$ as the standard Euclidean inner product on $\mathbb{R}^d$, and $|\cdot|$ as the associated norm. Let $C_p\left(\mathbb{R}^d,\mathbb{R}^d\right)$ be the collection of continuous mapping from $\mathbb{R}^d$ into $\mathbb{R}^d$ which are periodic of period 1 in each coordinate of the argument and let $\|\cdot\|_{C_p\left(\mathbb{R}^d,\mathbb{R}^d\right)}$ be the associated sup norm. Let $\mathcal{D}\left([0,T],\mathbb{R}^d\right)$ be the space of functions that map $[0,T]$ into $\mathbb{R}^d$, which are right continuous and having left hand limits. $\mathcal{D}\left([0,T],\mathbb{R}^d\right)$ is metricated by the Skorohod metric, with respect to which it is complete and separable (see, [4])). We will write $C_c^\infty\left(\mathbb{R}^d\right)$ for space of test functions.

The $\{\sigma_i:\ 1\le i\le d\}$ in (1.1) are assumed to be in $C_p\left(\mathbb{R}^d,\mathbb{R}^d\right)$, and we also assume that

$$\kappa := \inf\left\{\sum_{i=1}^{d}\langle\theta,\sigma_i(x)\rangle^2:\ x\in\mathbb{R}^d,\theta\in\mathbb{R}^d,|\theta|=1\right\}>0. \tag{2.1}$$

We assume that b,c in (1.1) are in $C_p\left(\mathbb{R}^d,\mathbb{R}^d\right)$.

We now turn our attention to the Poisson part. We first consider a Poisson random measure $N^{\varepsilon^{-1}}(.,.)$ on $[0,T)\times\mathbb{R}^d_*$ defined on the space probability $(\Omega,\mathcal{F},\mathbb{P})$, with Lévy measure $\varepsilon^{-1}\nu$ such that the standard integrability condition holds:

$$\int_{\mathbb{R}^d_*}\left(1\wedge|y|^2\right)\nu(dy)<+\infty. \tag{2.2}$$

In this paper we shall be interested in *Poisson process* of class (QL), namely a pure jump process whose counting measure has continuous compensator (see, [7]). More precisely, in light of the representation theorem of the Poisson jump ([7], Chap. II, Theorem 7.4), we shall assume that k is $C_p\left(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R}^d\right)$ with respect to first variable, integrable with respect to $dtdy$.

The considered stochastic process $X^{\varepsilon,\delta_\varepsilon}$ includes diffusion processes with jumps. Next let us write down its generator (an integro-differential operator) on twice continuously differentiable functions with compact support by

$$\mathcal{L}_{\varepsilon,\delta_\varepsilon}\phi\,(x) := \frac{\varepsilon}{2}\sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial^2\phi(x)}{\partial x_i\partial x_j} + \frac{\varepsilon}{\delta_\varepsilon}\sum_{i=1}^{d} b_i\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial\phi(x)}{\partial x_i} + \sum_{i=1}^{d} c_i\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial\phi(x)}{\partial x_i}$$
$$+\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\left[\phi\left(x+\varepsilon k\left(\frac{x}{\delta_\varepsilon},y\right)\right) - \phi(x) - \varepsilon\sum_{i=1}^{d} k_i\left(\frac{x}{\delta_\varepsilon},y\right)\frac{\partial\phi(x)}{\partial x_i}\right]\nu(dy), \quad (2.3)$$

where the matrix $a := (a_{ij})$ is factored as $a := \sigma\sigma^*$, and * denotes the transpose.

The following hypotheses are required:

H.1 (Main hypothesis) $\lim_{\varepsilon\to 0}\frac{\delta_\varepsilon}{\varepsilon} = +\infty$.

H.2 For any $\zeta := b, c, \sigma_i$ $(i := 1,\ldots,d)$ and k, there exists $C_i > 0$ $(i := 1, 2)$ such that

$$\begin{cases} \text{i) } \|\zeta(x') - \zeta(x)\| + \displaystyle\int_{\mathbb{R}^d_*}\|k(x',y) - k(x,y)\|\,dy \le C_1|x'-x|, \quad x', x \in \mathbb{R}^d; \\ \text{ii) } \|\zeta(x)\|^2 + \displaystyle\int_{\mathbb{R}^d_*}\|k(x,y)\|^2\,dy \le C_2\left(1+|x|^2\right),\ x \in \mathbb{R}^d. \end{cases}$$

The proof of the Proposition 3.1 (below) uses the *Girsanov's formula* (see, D. Applebaum [1] Chapter 5, Section 2). The proof of the Proposition 3.2 (below) uses an *analytic* approach and appeals to the *classical Hartman–Wintner condition* (see, for instance [12]). Such a condition plays a pivotal role in the sequel, because it allows us to show the existence of a transition density of a Feller process and to link it explicitly in terms of symbol q. A detailed exposition of the use of the symbol in the study of Markov processes can be found in [8–10]. It is well known that Feller generators are *variable coefficient* Lévy-type operators and once we fix x then $-q$ is the generator of a Lévy process. Notice that q is no longer the characteristic exponent of the Feller process $(Y_t)_{t\ge 0}$, that is, the formula $\mathbb{E}\left(e^{i\,\langle\xi, Y_t - x\rangle}\right) = e^{-tq(x,\xi)}$ is, in general, an inaccurate result. However, it is natural to expect that

$$\mathbb{E}\left(e^{i\,\langle\xi, Y_t - x\rangle}\right) \approx e^{-tq(x,\xi)}.$$

Before finishing this section, we point out that if the canonical process is Feller under $\mathbb{P}$, so that all requirements are satisfied, it is a *strong* Feller process, that is, its semigroup maps bounded measurable functions to continuous bounded functions. In this case, the symbol can also be written as

$$q(x,\xi) := -\lim_{t\to 0}\frac{\mathbb{E}\left(e^{i\langle Y_t - x,\xi\rangle}\right) - 1}{t}. \quad (2.4)$$

Hence, the symbol can be probabilistically interpreted as the derivative of the characteristic function of the process (defined entirely in analytic terms), i.e.,

$$\frac{\mathrm{d}}{\mathrm{dt}}\lambda_t(x,\xi)\Big|_{t=0} = -q(x,\xi) = e^{-i\langle x,\xi\rangle}\mathcal{L}e^{i\langle x,\xi\rangle}, \quad x,\xi \in \mathbb{R}^d, \quad (2.5)$$

where $\lambda_t(x,\xi) := e^{-i\langle x,\xi\rangle}\boldsymbol{T}_t e^{i\langle x,\xi\rangle}(x)$, and $(\boldsymbol{T}_t)_{t\ge 0}$ a semigroup on $C^\infty\left(\mathbb{R}^d\right)$ the space of continuous functions vanishing at infinity.

2.2 Main Results

Before proceeding, let us give some definitions.

($\boldsymbol{D.1}$) $V_L : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow [0, +\infty)$ is the energy function defined as

$$V_L(x,z) := V_L^1(x,z) + V_L^2(x,z)$$

where

$$V_L^1(x,z) := \inf_{\substack{\psi \in \mathcal{D}\left([0,L];\, \mathbb{R}^d\right) \\ \psi_0 = x,\ \psi_L = z}} \frac{1}{2} \int_0^L \left\| \dot{\psi}_s - c(\psi_s) - \int_{\mathbb{R}^d_*} k(\psi_s, y)\, \nu(dy) \right\|^2_{a^{-1}(\psi_s)} ds$$

$$V_L^2(x,z) := \inf_{\substack{\psi \in \mathcal{D}\left([0,L];\, \mathbb{R}^d\right) \\ \psi_0 = x,\ \psi_L = z}} \int_0^L \int_{\mathbb{R}^d_*} \varrho\left(\frac{|\dot{\psi}_s|}{\|k(\psi_s, y)\|} \right) ds\nu(dy),$$

with $\varrho(r) := r \log r - r + 1,\ r \in \mathbb{R}^*_+$, and $\|\theta\|_{a^{-1}} := \sqrt{\langle \theta, a^{-1}\theta \rangle}$ for all $\theta \in \mathbb{R}^d$.

($\boldsymbol{D.2}$) $\mathcal{J} : \mathbb{R}^d \longrightarrow [0\ ,\ \infty)$ the functional given by

$$\mathcal{J}(z) = \lim_{L \to +\infty} \frac{1}{L} V_L\big(0, Lz\big)$$

Now we state our main result.

Theorem 2.1. *Fix $x \in \mathbb{R}^d$, assume (**H.1**) and (**H.2**) hold. Then we have*

- *for each open set $G \subseteq \mathcal{D}\left([0,T], \mathbb{R}^d\right)$*

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left\{ X_T^{x,\varepsilon,\delta_\varepsilon} \in G \right\} \geq - \inf_{\substack{\varphi \in \mathcal{D}\left([0,T], \mathbb{R}^d\right) \\ \varphi(0) = x,\ \varphi(T) = z}} \int_0^T \mathcal{J}\left(\dot{\varphi}(s)\right) ds,$$

- *for each closed set $F \subseteq \mathcal{D}\left([0,T], \mathbb{R}^d\right)$*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left\{ X_T^{x,\varepsilon,\delta_\varepsilon} \in F \right\} \leq - \inf_{\substack{\varphi \in \mathcal{D}\left([0,T], \mathbb{R}^d\right) \\ \varphi(0) = x,\ \varphi(T) = z}} \int_0^T \mathcal{J}\left(\dot{\varphi}(s)\right) ds.$$

3 Large Deviation Principle

Before proceeding, we observe that the function $\mathcal{J}$ is convex, hence we can show that

$$\inf_{\substack{\varphi \in \mathcal{D}\left([0,T], \mathbb{R}^d\right) \\ \varphi(0) = x,\ \varphi(T) = z}} \int_0^T \mathcal{J}\left(\dot{\varphi}(s)\right) ds := T \mathcal{J}\left(\frac{z - x}{T} \right).$$

Next we are going to give the outline of the proof.

3.1 The Lower Bound

We start with the following *lower bound* in space.

Proposition 3.1. *Suppose the assumptions ($\boldsymbol{H.1}$) to ($\boldsymbol{H.2}$) hold. For each open subset $G \subseteq \mathbb{R}^d$ we have*

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left\{X_1^{x,\varepsilon,\delta_\varepsilon} \in G\right\} \geq - \inf_{z\in G} \mathcal{J}(z-x).$$

Proof. Let $\psi \in \mathcal{D}\left([0,1],\mathbb{R}^d\right)$ satisfying $\psi_0 = x$, $\psi_1 = z$. Let us set

$$\hat{X}_t^{\varepsilon,\delta_\varepsilon} := \frac{1}{\sqrt{\varepsilon}}\left(X_t^{x,\varepsilon,\delta_\varepsilon} - \psi(t)\right).$$

Now, fix $z \in G$, $\eta > 0$ and let $\delta'_\varepsilon > 0$ be small enough so that

$$\left\{z' \in \mathbb{R}^d : \|z' - z\| \leq \eta\delta'_\varepsilon\right\} \subseteq G.$$

Fix also $0 < \delta_\varepsilon < \delta'_\varepsilon$, then

$$\begin{aligned}
\mathbb{P}\left\{X_1^{x,\varepsilon,\delta_\varepsilon} \in G\right\} &\geq \mathbb{P}\left\{\left\|X_1^{x,\varepsilon\delta_\varepsilon} - z\right\| \leq \eta\delta_\varepsilon\right\} \\
&\geq \mathbb{P}\left\{\left\|X_1^{x,\varepsilon\delta_\varepsilon} - \psi\right\|_{\mathcal{D}\left([0,1],\mathbb{R}^d\right)} \leq \eta\delta_\varepsilon\right\} \\
&\geq \mathbb{P}\Big\{\underbrace{\|\hat{X}_1^{\varepsilon,\delta_\varepsilon}\|_{\mathcal{D}\left([0,1],\mathbb{R}^d\right)} \leq \frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\eta}_{:= A^\eta_{\varepsilon,\delta_\varepsilon}}\Big\}.
\end{aligned}$$

Next, we set

$$\begin{aligned}
\xi(t) := \Bigg[\dot{\psi}(t) - \frac{\varepsilon}{\delta_\varepsilon} b\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) - c\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right) \\
+ \int_{\mathbb{R}^d_*} k\left(\frac{X_{t^-}^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}, y\right)\nu(dy)\Bigg] \times \sigma^{-1}\left(\frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right),\ 0 \leq t \leq 1.
\end{aligned}$$

According to Girsanov's formula (see, D. Applebaum [1] Chapter 5, Section 2) and using a *dilation* by factor $\log(1+\phi)$, we build the Poissonian martingale (below) by taking

$$L(s,y) := \log\left(1+\phi(s,y)\right),\ H(s,y) := \log\left(1+\phi(s,y)\right),\ K(s,y) = k(.,y)(s).$$

Thus we define

$$\hat{N}_z^{\varepsilon^{-1}}(dtdy) := \frac{1}{\varepsilon}\log\left(1+\phi_\varepsilon(t,y)\right)\left(\varepsilon N^{\varepsilon^{-1}}(dtdy) - \nu(dy)\right)$$

$$- \frac{1}{\varepsilon} \log\left(1+\phi_\varepsilon(t,y)\right) \Bigg(\phi_\varepsilon(t,y) \mathbf{1}_{\{\|y\|<1\}} + \left(e^{k(z,y)} - 1\right) \mathbf{1}_{\{\|y\|\geq 1\}} \Bigg) \nu(dy),$$

for any $z \in \mathbb{R}^d$ and for any $\phi_\varepsilon \in \mathcal{H}^2(1,\nu)$ such that $\phi_\varepsilon > -1$.

Now, we introduce the new measure probability $\hat{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ defined as

$$\begin{aligned}\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} &:= e^{\left(-\frac{1}{2\varepsilon} \int_0^1 \|\xi(s)\|^2 ds - \frac{1}{\varepsilon} \int_0^1 \int_{\mathbb{R}^d_*} \left(\log\left(1+\phi_\varepsilon(s,y)\right) - \phi_\varepsilon(s,y) \right) \nu(dy) ds \right)} \\ &\times e^{\left(\frac{1}{\sqrt{\varepsilon}} \int_0^1 \xi(s) dW_s - \frac{1}{\varepsilon} \int_0^1 \int_{\mathbb{R}^d_*} \log\left(1+\phi_\varepsilon(s,y)\right) \left(\varepsilon N^{\varepsilon^{-1}}(dsdy) - \nu(dy)ds \right) \right)} \\ &\times e^{\left(\frac{1}{\varepsilon} \int_0^1 \int_{\mathbb{R}^d_*} \log\left(1+\phi_\varepsilon(s,y)\right) \left(e^{k\left(\frac{X^{x,\varepsilon,\delta_\varepsilon}_{s^-}}{\delta_\varepsilon}, y\right)} - 1 \right) \mathbf{1}_{\{\|y\|\geq 1\}} \nu(dy) ds \right)} \\ &\times e^{\left(-\frac{1}{\varepsilon} \int_0^1 \int_{\mathbb{R}^d_*} k\left(\frac{X^{x,\varepsilon,\delta_\varepsilon}_{s^-}}{\delta_\varepsilon}, y\right) \varepsilon N^{\varepsilon^{-1} \log(1+\phi_\varepsilon(s,y))}(dsdy) \right)}.\end{aligned}$$

Under $\hat{\mathbb{P}}$, $\hat{W}_t$ defined below is a Brownian motion

$$\hat{W}_t := W_t - \frac{1}{\sqrt{\varepsilon}} \int_0^t \xi(s) ds, \; 0 \leq t \leq 1.$$

Thereafter, one can see

$$\begin{aligned}\mathbb{P}\left(A^\eta_{\varepsilon,\delta_\varepsilon}\right) = \hat{\mathbb{E}} \Bigg\{ &\mathbf{1}_{A^\eta_{\varepsilon,\delta_\varepsilon}} e^{\left(\frac{1}{\sqrt{\varepsilon}} \int_0^1 \xi(s) dW_s - \int_0^1 \int_{\mathbb{R}^d_*} \hat{N}^{\varepsilon^{-1}}_{X/\delta_\varepsilon}(dsdy) - M^{\varepsilon,\phi_\varepsilon}_1 \right)} \\ &\times \mathbf{1}_{A^\eta_{\varepsilon,\delta_\varepsilon}} e^{\left(-\frac{1}{2\varepsilon} \int_0^1 \|\xi(s)\|^2 ds - \frac{1}{\varepsilon} \int_0^1 \int_{\mathbb{R}^d_*} \hat{\Phi}(s,y,\phi_\varepsilon) \nu(dy) ds \right)} \Bigg\}.\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}M^{\varepsilon,\phi_\varepsilon}_t &:= \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{R}^d_*} k\left(\frac{X^{x,\varepsilon,\delta_\varepsilon}_{s^-}}{\delta_\varepsilon}, y\right) \varepsilon N^{\varepsilon^{-1} \log(1+\phi_\varepsilon(s,y))}(dsdy), \\ \hat{\Phi}\left(t,y,\phi_\varepsilon\right) &:= \left(1+\phi_\varepsilon(t,y)\right) \log\left(1+\phi_\varepsilon(t,y)\right) - \phi_\varepsilon(t,y).\end{aligned}$$

Notice that

$$A^{\eta}_{\varepsilon,\delta_\varepsilon} \equiv \left\{ \sup_{0\le t\le 1} \left\| \frac{X_t^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} - \frac{\psi(t)}{\delta_\varepsilon} \right\| \le \eta \right\}$$

and on this set, we have

$$\tfrac{1}{2}\int_0^1 \|\xi(s)\|^2\, ds \le \hat{V}_\eta^{\varepsilon,\delta_\varepsilon}(1,\psi)$$

where

$$\hat{V}_\eta^{\varepsilon,\delta_\varepsilon}(t,\psi) := \sup_{\left\{\|\varpi\|_{\mathcal{D}([0,t];\mathbb{R}^d)}\le\eta\right\}} \tfrac{1}{2}\int_0^t \left\| \dot{\psi}_s - B^{\varepsilon,\delta_\varepsilon}\left(\frac{\psi_s}{\delta_\varepsilon} + \varpi(s)\right)\right\|^2_{a^{-1}\left(\frac{\psi_s}{\delta_\varepsilon}+\varpi(s)\right)} ds \tag{3.2}$$

and with

$$B^{\varepsilon,\delta_\varepsilon}(z) := \frac{\varepsilon}{\delta_\varepsilon} b(z) + c(z) - \int_{\mathbb{R}^d_*} k(z,y)\nu(dy).$$

For any $\varpi \in \mathcal{D}\left([0,1]\,;\mathbb{R}^d\right)$ with $\|\varpi\|_{\mathcal{D}\left([0,1];\mathbb{R}^d\right)} \le \eta$, as in [6] (Young's inequality), there exists constants κ_1 and $\kappa_2 > 0$ such that for all $\tilde{\eta} > 0$,

$$\begin{aligned}
&\frac{1}{2}\left\| \dot{\psi}_t - B^{\varepsilon,\delta_\varepsilon}\left(\frac{\psi_t}{\delta_\varepsilon} + \varpi(t)\right)\right\|^2_{a^{-1}\left(\frac{\psi_t}{\delta_\varepsilon}+\varpi(t)\right)} \\
&\qquad \le \frac{1}{2}(1+\kappa_1\eta)(1+\tilde{\eta}) \left\| \dot{\psi}_t - c\left(\frac{\psi(t)}{\delta_\varepsilon}\right) + k\left(\frac{\psi(t)}{\delta_\varepsilon}\right)\right\|^2_{a^{-1}\left(\frac{\psi_t}{\delta_\varepsilon}\right)} + \omega^{\eta,\tilde{\eta}}_{\varepsilon,\delta_\varepsilon},
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
\omega^{\eta,\tilde{\eta}}_{\varepsilon,\delta_\varepsilon} := \frac{1}{2}(1+\kappa_1\eta)\left(1+\tilde{\eta}^{-1}\right)\kappa_2 \sup_{\left\{\substack{z,z'\in\mathbb{R}^d\\ \|z-z'\|\le\eta}\right\}} &\left\{ \frac{\varepsilon}{\delta_\varepsilon}\|b(z)\| + \|c(z)-c(z')\| \right. \\
&\left. + \int_{\mathbb{R}^d_*} \|k(z,y)-k(z',y)\|\,\nu(dy) \right\}.
\end{aligned}$$

From (3.1) and the choice of ϕ_ε in the following way

$$\phi_\varepsilon(t,y) + 1 := \frac{|\dot{\psi}_t|}{\left\| k\left(\frac{\psi_t}{\delta_\varepsilon}, y\right)\right\|}, \quad \psi \in \mathcal{D}\left([0,1],\mathbb{R}^d\right) \text{ with } \psi_0 = x \text{ and } \psi_1 = z$$

we have

$$\mathbb{P}(A^\eta_{\varepsilon,\delta_\varepsilon}) \ge \exp\left(-\frac{\Psi_{\varepsilon,\eta}(1,\psi)}{\varepsilon}\right) \times \hat{\mathbb{E}}\mathbf{1}_{A^\eta_{\varepsilon,\delta_\varepsilon}} \left\{ \exp\left(-\tfrac{1}{\sqrt{\varepsilon}}\left|\Lambda_{\varepsilon,\phi_\varepsilon}(W,N)\right|\right)\right\}$$

where

$$\Psi_{\varepsilon,\eta}(1,\psi) := \hat{V}_{\eta}^{\varepsilon,\delta_\varepsilon}(1,\psi) + (1+\kappa_3\eta) \sup_{\substack{\psi\in\mathcal{D}\left([0,1];\ \mathbb{R}^d\right)\\ \psi_0=x,\ \psi_1=z}} \int_0^1 \int_{\mathbb{R}^d_*} \varrho\left(\frac{|\dot{\psi}_s|}{\left\|k\left(\frac{\psi_t}{\delta_\varepsilon},y\right)\right\|}\right) ds\nu(dy)$$

$$\Lambda_{\varepsilon,\phi_\varepsilon}(W,N) := \int_0^1 \xi(s)dW_s - \sqrt{\varepsilon}\int_0^1 \int_{\mathbb{R}^d_*} \hat{N}^{\varepsilon^{-1}}_{X/\delta_\varepsilon}(dsdy) - \sqrt{\varepsilon}M_1^{\varepsilon,\phi_\varepsilon}.$$

As a reminder $\int_0^t \int_{\mathbb{R}^d_*} \hat{N}^{\varepsilon^{-1}}_{X/\delta_\varepsilon}(dsdy)$ is a martingale under $\hat{\mathbb{P}}$. On the other hand, we have the following estimates:

(**E**$_1$) in the first place,

$$\begin{aligned}
&\hat{\mathbb{E}}\left(\exp\left\{\sqrt{\varepsilon}M_1^{\varepsilon,\phi_\varepsilon}\right\}\right)\\
&\leq \hat{\mathbb{E}}\Bigg[\exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^1 k\left(\frac{X_{s^-}^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},y\right)\varepsilon N^{\varepsilon^{-1}\log(1+\phi_\varepsilon(s,y))}(dsdy)\right.\\
&\left.-\frac{1}{\sqrt{\varepsilon}}\int_0^1\int_{\mathbb{R}^d_*}\log\left(1+\phi_\varepsilon(s,y)\right)\left[e^{k\left(\frac{X_{s^-}^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},y\right)}-1\right]\nu(dy)ds\right\}\\
&\times\exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^1\int_{\mathbb{R}^d_*}\log\left(1+\phi_\varepsilon(s,y)\right)\left[e^{k\left(\frac{X_{s^-}^{x,\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon},y\right)}-1\right]\nu(dy)ds\right\}\Bigg]\\
&\leq \exp\left\{\frac{1+\kappa_3\eta}{\sqrt{\varepsilon}}\sup_{\psi}\int_0^1\int_{\mathbb{R}^d_*}\log\left(\frac{|\dot{\psi}_s|}{\left\|k\left(\frac{\psi_s}{\delta_\varepsilon},y\right)\right\|}\right)\left\|k\left(\frac{\psi_s}{\delta_\varepsilon},y\right)\right\|\nu(dy)ds\right\};
\end{aligned}$$

(**E**$_2$) in the second place,

$$\begin{aligned}
&\hat{\mathbb{E}}\left(\exp\left\{\sqrt{\varepsilon}\int_0^1\int_{\mathbb{R}^d_*}\hat{N}^{\varepsilon^{-1}}_{X/\delta_\varepsilon}(dsdy)\right\}\right)\\
&\leq \hat{\mathbb{E}}\Bigg[\exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^1\int_{\mathbb{R}^d_*}\log\left(1+\phi_\varepsilon(s,y)\right)\left(\varepsilon N^{\varepsilon^{-1}}(dsdy)-\nu(dy)ds\right)\right.\\
&\qquad\left.+\frac{1}{\sqrt{\varepsilon}}\int_0^1\int_{\mathbb{R}^d_*}\Big(\log\left(1+\phi_\varepsilon(s,y)\right)-\phi_\varepsilon(s,y)\Big)\nu(dy)ds\right\}\\
&\times\exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^1\int_{\mathbb{R}^d_*}\Big(\log\left(1+\phi_\varepsilon(s,y)\right)-\phi_\varepsilon(s,y)\Big)\nu(dy)ds\right\}
\end{aligned}$$

$$\times \exp\left\{ -\frac{1}{\sqrt{\varepsilon}}\int_0^1\int_{\mathbb{R}^d_*}\phi_\varepsilon(s,y)\log\left(1+\phi_\varepsilon(s,y)\right)\nu(dy)ds\right\}$$
$$\times \exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^1\int_{\mathbb{R}^d_*}\log\left(1+\phi_\varepsilon(s,y)\right)\left[1-e^{k\left(\frac{X^{x,\varepsilon,\delta_\varepsilon}_{s^-}}{\delta_\varepsilon},y\right)}\right]\nu(dy)ds\right\}$$
$$\leq \exp\left\{ -\frac{1}{\sqrt{\varepsilon}}\sup_{\psi}\int_0^1\int_{\mathbb{R}^d_*}\varrho\left(\frac{|\dot{\psi}_s|}{\left\|k\left(\frac{\psi_s}{\delta_\varepsilon},y\right)\right\|}\right)ds\nu(dy)\right\}$$
$$\times \exp\left\{\frac{1+\kappa_3\eta}{\sqrt{\varepsilon}}\sup_{\psi}\int_0^1\int_{\mathbb{R}^d_*}\log\left(\frac{|\dot{\psi}_s|}{\left\|k\left(\frac{\psi_s}{\delta_\varepsilon},y\right)\right\|}\right)\left\|k\left(\frac{\psi_s}{\delta_\varepsilon},y\right)\right\|\nu(dy)ds\right\};$$

($\mathbf{E}_3$) in the third place,

$$\hat{\mathbb{E}}\left(\exp\left\{\int_0^1\xi(s)dW_s\right\}\right)$$
$$= \hat{\mathbb{E}}\left(\exp\left\{\int_0^1\xi(s)dW_s - \frac{1}{2}\int_0^1\|\xi(s)\|^2 ds + \frac{1}{2}\int_0^1\|\xi(s)\|^2 ds\right\}\right)$$
$$\leq \exp\left(\hat{V}^{\varepsilon,\delta_\varepsilon}_\eta(1,\psi)\right).$$

Girsanov's theorem tells us that $\hat{\mathbb{P}}(A^\eta_{\varepsilon,\delta_\varepsilon}) \longrightarrow 1$ when $\varepsilon,\delta_\varepsilon \to 0$. So, for $\varepsilon,\delta_\varepsilon > 0$ sufficiently small, $\hat{\mathbb{P}}(A^\eta_{\varepsilon,\delta_\varepsilon})$ is positive. Thus, we have

$$\mathbb{P}(A^\eta_{\varepsilon,\delta_\varepsilon}) \geq \exp\left(-\frac{\Psi_{\varepsilon,\eta}(1,\psi)}{\varepsilon}\right)\times\hat{\mathbb{P}}(A^\eta_{\varepsilon,\delta_\varepsilon})$$
$$\times\frac{\hat{\mathbb{E}}\mathbf{1}_{A^\eta_{\varepsilon,\delta_\varepsilon}}\left\{\exp -\frac{1}{\sqrt{\varepsilon}}\left|\Lambda_{\varepsilon,\phi_\varepsilon}(W,N)\right|\right\}}{\hat{\mathbb{P}}(A^\eta_{\varepsilon,\delta_\varepsilon})}.$$

Therefore,

$$\mathbb{P}(A^\eta_{\varepsilon,\delta_\varepsilon}) \geq \exp\left(-\frac{\Psi_{\varepsilon,\eta}(1,\psi)}{\varepsilon}\right)\times\hat{\mathbb{P}}(A^\eta_{\varepsilon,\delta_\varepsilon})$$
$$\times\underbrace{\exp\left(-\frac{1}{\sqrt{\varepsilon}}\frac{\left[\hat{\mathbb{E}}\mathbf{1}_{A^\eta_{\varepsilon,\delta_\varepsilon}}\left|\Lambda_{\varepsilon,\phi_\varepsilon}(W,N)\right|\right]}{\hat{\mathbb{P}}(A^\eta_{\varepsilon,\delta_\varepsilon})}\right)}_{\text{Jensen's inequality}}.$$

With due regard to estimates ($\mathbf{E}_i$), $i := 1,2,3$, we have

$$\mathbb{P}(A^\eta_{\varepsilon,\delta_\varepsilon}) \geq \exp\left(-\frac{\Psi_{\varepsilon,\eta}(1,\psi)}{\varepsilon}\right)$$

$$\underbrace{\times \exp\left(-\frac{K^{1/\kappa_4}}{\sqrt{\varepsilon}} \frac{\sqrt{2\Psi_{\varepsilon,\eta}(1,\psi)}}{\left(\tilde{\omega}_{\varepsilon,\delta_\varepsilon}\right)^{1/\kappa_4}} \right) \times \tilde{\omega}_{\varepsilon,\delta_\varepsilon}}_{\text{Burkholder-Davis-Gundy inequality}} \times \exp\left\{ \frac{1+\kappa_3\eta}{\sqrt{\varepsilon}} K \right\},$$

where, similarly as in [6, Lemma 4.4],

$$\tilde{\omega}_{\varepsilon,\delta_\varepsilon} := \left(\frac{2\min\left(1, \frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\kappa_5\right)}{\sqrt{2\pi e}} \right)^{\kappa_4}.$$

Now, we put everything together, rescale the integral on the right of (3.2), and vary ψ (over all $\psi \in \mathcal{D}\left([0,1], \mathbb{R}^d\right)$) such that $\psi_0 = x$ and $\psi_1 = z$. Then, by the remainder of ($\boldsymbol{D.1}$), we have

$$\begin{aligned}
\mathbb{P}(A^{\eta}_{\varepsilon,\delta_\varepsilon}) \geq{}& e^{\left\{ -\frac{1}{\varepsilon}(1+\kappa_1\eta)(1+\tilde{\eta})\delta_\varepsilon V^1_{1/\delta_\varepsilon}\left(\frac{x}{\delta_\varepsilon}, \frac{z}{\delta_\varepsilon}\right) + \frac{1}{\varepsilon}\omega^{\eta,\tilde{\eta}}_{\varepsilon,\delta_\varepsilon} \right\}} \\
&\times e^{\left\{ -\frac{1}{\varepsilon}(1+\kappa_3\eta)\delta_\varepsilon V^2_{1/\delta_\varepsilon}\left(\frac{x}{\delta_\varepsilon}, \frac{z}{\delta_\varepsilon}\right) \right\}} \times e^{\left\{ \frac{1+\kappa_3\eta}{\sqrt{\varepsilon}} K \right\}} \\
&\times e^{\left\{ -\frac{K^{1/\kappa_4}}{\sqrt{\varepsilon}} \frac{\sqrt{2(1+\kappa_1\eta)(1+\tilde{\eta})\delta_\varepsilon V^1_{1/\delta_\varepsilon}\left(\frac{x}{\delta_\varepsilon}, \frac{z}{\delta_\varepsilon}\right) + \omega^{\eta,\tilde{\eta}}_{\varepsilon,\delta_\varepsilon}}}{\left(\tilde{\omega}_{\varepsilon,\delta_\varepsilon}\right)^{1/\kappa_4}} \right\}} \\
&\times e^{\left\{ -\frac{K^{1/\kappa_4}}{\sqrt{\varepsilon}} \frac{\sqrt{2(1+\kappa_3\eta)\delta_\varepsilon V^2_{1/\delta_\varepsilon}\left(\frac{x}{\delta_\varepsilon}, \frac{z}{\delta_\varepsilon}\right)}}{\left(\tilde{\omega}_{\varepsilon,\delta_\varepsilon}\right)^{1/\kappa_4}} \right\}} \times \tilde{\omega}_{\varepsilon,\delta_\varepsilon}.
\end{aligned}$$

Thus, letting consecutively ε, δ_ε, η and then $\tilde{\eta}$ tend to zero in that order, we get

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left\{ X^{x,\varepsilon,\delta_\varepsilon}_1 \in G \right\} \geq -\inf_{z\in G} \mathcal{J}(z-x).$$

The proof is complete. □

3.2 The Upper Bound

We use $\left\{\hat{\boldsymbol{T}}^{\varepsilon,\delta_\varepsilon}_{t,s} : t < s\right\}$ to denote the semigroup on $C^\infty\left(\mathbb{R}^d\right)$ generated by the operator $\mathcal{L}_{\varepsilon,\delta_\varepsilon}$ with $C^\infty_c\left(\mathbb{R}^d\right) \subset \mathcal{D}\left(\mathcal{L}_{\varepsilon,\delta_\varepsilon}\right)$, and let $p^{\varepsilon,\delta_\varepsilon}(s-t,z,y)$ denote the heat kernel associated with the semigroup $\hat{\boldsymbol{T}}^{\varepsilon,\delta_\varepsilon}_{t,s}$ in the sense

$$\left(\hat{\boldsymbol{T}}^{\varepsilon,\delta_\varepsilon}_{t,s}\psi\right)(z) = \int_{\mathbb{R}^d} p^{\varepsilon,\delta_\varepsilon}(s-t,z,y)\psi(y)dy,\ t<s,\ z\in\mathbb{R}^d,\ \psi\in C^\infty_c\left(\mathbb{R}^d\right). \quad (3.4)$$

Thus, for any $A \in \mathcal{B}\left(\mathbb{R}^d\right)$ the σ-algebra of all Borel subsets of $\mathbb{R}^d$,

$$\mathbb{P}\left(X_t^{x,\varepsilon,\delta_\varepsilon} \in A\right) := \int_{A/\delta_\varepsilon} p^{\varepsilon,\delta_\varepsilon}\left(\left(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon}\right)^2 t, z, \frac{x}{\delta_\varepsilon}\right) dz,\ t \geq 0. \tag{3.5}$$

With our requirements, it is well known that $X^{x,\varepsilon,\delta_\varepsilon}$ is a strong Feller process. Then let us define its symbol. Before continuing, let us set

$$\hat{\mathcal{L}}^1_{\varepsilon,\delta_\varepsilon} := \frac{1}{2}\sum_{i,j=1}^d a_{ij}\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial}{\partial x_i} + \frac{\delta_\varepsilon}{\varepsilon}\sum_{i=1}^d c_i\left(\frac{x}{\delta_\varepsilon}\right)\frac{\partial}{\partial x_i}$$
$$\hat{\mathcal{L}}^2_{\varepsilon,\delta_\varepsilon} := \int_{\mathbb{R}^d_*}\left\{\sum_{l=1}^d\left(ik_l\left(\frac{x}{\delta_\varepsilon}, y\right)\xi_l + \frac{\varepsilon}{\delta_\varepsilon}k_l\left(\frac{x}{\delta_\varepsilon}, y\right)\frac{\partial}{\partial x_l}\right) + \left(e^{i\left\langle k\left(\frac{x}{\delta_\varepsilon}, y\right), \xi\right\rangle} - 1\right)\right\}\nu(dy)$$
$$\hat{\mathcal{L}}^3_{\varepsilon,\delta_\varepsilon} := \frac{\varepsilon}{\delta_\varepsilon}\sum_{l,r=1}^d a_{l,r}\left(\frac{x}{\delta_\varepsilon}\right)\xi_l\frac{\partial}{\partial x_l} + i\sum_{l=1}^d\left(\frac{\varepsilon}{\delta_\varepsilon}b_l + c_l\right)\left(\frac{x}{\delta_\varepsilon}\right)\xi_l - \frac{1}{2}\sum_{l,r=1}^d \xi_l^* a_{l,r}\left(\frac{x}{\delta_\varepsilon}\right)\xi_r.$$

Next, we have

$$\begin{aligned}-q^{\varepsilon,\delta_\varepsilon}\left(\frac{x}{\delta_\varepsilon}, \xi\right) &= \varepsilon \exp\left(-i\left\langle\frac{x}{\delta_\varepsilon}, \xi\right\rangle\right)\mathcal{L}_{\varepsilon,\delta_\varepsilon}\exp\left(i\left\langle\frac{x}{\delta_\varepsilon}, \xi\right\rangle\right)\\ &= \left(\frac{\varepsilon}{\delta_\varepsilon}\right)^2\hat{\mathcal{L}}^1_{\varepsilon,\delta_\varepsilon} + \hat{\mathcal{L}}^2_{\varepsilon,\delta_\varepsilon} + \hat{\mathcal{L}}^3_{\varepsilon,\delta_\varepsilon}\end{aligned}$$

Heuristically if $\frac{x}{\delta_\varepsilon} \longrightarrow z$ when $\varepsilon \longrightarrow 0$, it can be seen that this operator converges to $-q$ defined as

$$\begin{aligned}-q(z,\xi) = &-\frac{1}{2}\sum_{l,r=1}^d \xi_l^* a_{l,r}(z)\,\xi_r + i\sum_{r=1}^d\left\{c_i(z)\,\xi_r + \int_{\mathbb{R}^d_*} k_r(z,y)\,\xi_r\nu(dy)\right\}\\ &+ \int_{\mathbb{R}^d_*}\left(e^{i\langle k(z,y),\xi\rangle} - 1\right)\nu(dy).\end{aligned}$$

Now we have the following *upper bound* in space.

Proposition 3.2. *Suppose the assumptions (**H**.1) to (**H**.2) hold true. for each closed subset* $F \subseteq \mathbb{R}^d$

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left\{X_1^{x,\varepsilon,\delta_\varepsilon} \in F\right\} \leq -\inf_{z\in F}\mathcal{J}(z-x).$$

Proof. Before proceeding, let $\hat{Q}_1$ be the quadratic form defined as $\hat{Q}_1(v) := \langle v, av\rangle$ and let $\hat{Q}_1^*$ be the conjugate quadratic form of $\hat{Q}_1$ defined as

$$\hat{Q}_1^*(v) := \sup_{t\in\mathbb{R}^d}\Big\{2\,\langle t,v\rangle - \hat{Q}_1(t)\Big\}.$$

It is well known that if the inverse of the matrix a exists then

$$\hat{Q}_1^*(v) := \left\langle v, a^{-1}v\right\rangle.$$

Next, fix $\theta\in\mathbb{R}^d$ and let $\hat{Q}_2^*(v,\theta) = \sup_{t\in\mathbb{R}^d}\Big\{\langle t,v\rangle - \Big[\exp\left(\langle\theta,t\rangle\right) - 1\Big]\Big\}$. Then, it is easy to see that

$$\hat{Q}_2^*(v,\theta) = \hat{Q}_2\left(\|v\|\times\|\theta\|^{-1}\right), \quad \text{with } \hat{Q}_2(x) := x\log x - x + 1.$$

Now, fix $x, z\in\mathbb{R}^d$ and $t>0$. Thanks to [12] we have

$$\begin{aligned}\sup_{x,z\in\mathbb{R}^d} p^{\varepsilon,\delta_\varepsilon}(t,x,z) &\le \int \exp\left(-\frac{1}{16}\inf_{\substack{\psi\in\mathcal{D}([0,t];\mathbb{R}^d)\\ \psi_0=x,\ \psi_t=z}}\int_0^t \mathrm{Re}\,q^{\varepsilon,\delta_\varepsilon}\left(\frac{\psi_s}{\delta_\varepsilon},\xi\right)ds\right)d\xi\\
&\le \int \exp\left(\frac{1}{2}\inf_{\substack{\psi\in\mathcal{D}([0,t];\,\mathbb{R}^d)\\ \psi_0=x,\ \psi_t=z}}\int_0^t \left\langle\xi, a\left(\frac{\psi_s}{\delta_\varepsilon}\right)\xi\right\rangle ds\right.\\
&\quad\left.+\inf_{\substack{\psi\in\mathcal{D}([0,t];\,\mathbb{R}^d)\\ \psi_0=x,\ \psi_t=z}}\int_0^t\int_{\mathbb{R}^d_*}\mathrm{Re}\left\{1-e^{\left(i\left\langle k\left(\frac{\psi_s}{\delta_\varepsilon},y\right),\xi\right\rangle\right)}\right\}ds\,dy\right)+o(1).\end{aligned}$$

Thus

$$\begin{aligned}&\sup_{x,z\in\mathbb{R}^d} p^{\varepsilon,\delta_\varepsilon}(t,x,z)\\
&\le \exp\left(-\frac{1}{2}\inf_{\substack{\psi\in\mathcal{D}([0,t];\,\mathbb{R}^d)\\ \psi_0=x,\ \psi_t=z}}\int_0^t \hat{Q}_1^*\left(\dot{\psi}_s - \frac{\delta_\varepsilon}{\varepsilon}\left\{c+\int_{\mathbb{R}^d_*}k(.,y)\nu(dy)\right\}\left(\frac{\psi_s}{\delta_\varepsilon}\right)\right)ds\right)\\
&\times\exp\left(\inf_{\substack{\psi\in\mathcal{D}([0,t];\,\mathbb{R}^d)\\ \psi_0=x,\ \psi_t=z}}\sup_{\xi\in\mathbb{R}^d}\int_0^t\left\langle\dot{\psi}_s - \frac{\delta_\varepsilon}{\varepsilon}\left\{c+\int_{\mathbb{R}^d_*}k(.,y)\nu(dy)\right\}\left(\frac{\psi_s}{\delta_\varepsilon}\right),\xi\right\rangle ds\right)\\
&\times\int\exp\left(\inf_{\substack{\psi\in\mathcal{D}([0,t];\,\mathbb{R}^d)\\ \psi_0=x,\ \psi_t=z}}\int_0^t\int_{\mathbb{R}^d_*}\mathrm{Re}\left\{1-e^{\left(i\left\langle k\left(\frac{\psi_s}{\delta_\varepsilon},y\right),\xi\right\rangle\right)}\right\}ds\,dy\right)d\xi+o(1).\end{aligned}$$

Thereupon, we have

$$\begin{aligned}
&\sup_{x,z\in\mathbb{R}^d} p^{\varepsilon,\delta_\varepsilon}(t,x,z) \\
&\leq \exp\left(-\frac{1}{2}\inf_{\substack{\psi\in\mathcal{D}\left([0,t];\,\mathbb{R}^d\right)\\ \psi_0=x,\ \psi_t=z}}\int_0^t \hat{Q}_1^*\left(\dot{\psi}_s-\frac{\delta_\varepsilon}{\varepsilon}\left\{c+\int_{\mathbb{R}^d_*}k(.,y)\nu(dy)\right\}\left(\frac{\psi_s}{\delta_\varepsilon}\right)\right)ds\right) \\
&\times K\exp\left(-\inf_{\substack{\psi\in\mathcal{D}\left([0,t];\,\mathbb{R}^d\right)\\ \psi_0=x,\ \psi_t=z}}\left\{\int_0^t \hat{Q}_2^*\left(\dot{\psi}_s,\int_{\mathbb{R}^d_*}k\left(\frac{\psi_s}{\delta_\varepsilon},y\right)dy\right)ds\right.\right. \\
&\qquad\left.\left. +\frac{\delta_\varepsilon}{\varepsilon}\sup_{\xi\in\mathbb{R}^d}\int_0^t\left\langle\left\{c+\int_{\mathbb{R}^d_*}k(.,y)\nu(dy)\right\}\left(\frac{\psi_s}{\delta_\varepsilon}\right),\xi\right\rangle ds\right]\right\}\right)+o(1).
\end{aligned}$$

It follows

$$\begin{aligned}
&\sup_{x,z\in\mathbb{R}^d} p^{\varepsilon,\delta_\varepsilon}\left(t,\frac{x}{\delta_\varepsilon},\frac{z}{\delta_\varepsilon}\right) \\
&\leq \exp\left(-\frac{1}{2}\inf_{\substack{\psi\in\mathcal{D}\left([0,t];\,\mathbb{R}^d\right)\\ \psi_0=\frac{x}{\delta_\varepsilon},\ \psi_t=\frac{z}{\delta_\varepsilon}}}\int_0^t \hat{Q}_1^*\left(\frac{\dot{\psi}_s}{\delta_\varepsilon}-\frac{\delta_\varepsilon}{\varepsilon}\left\{c+\int_{\mathbb{R}^d_*}k(.,y)\nu(dy)\right\}\left(\frac{\psi_s}{\delta_\varepsilon}\right)\right)ds\right) \\
&\times K\exp\left(-\inf_{\substack{\psi\in\mathcal{D}\left([0,t];\,\mathbb{R}^d\right)\\ \psi_0=\frac{x}{\delta_\varepsilon},\ \psi_t=\frac{z}{\delta_\varepsilon}}}\left\{\int_0^t \hat{Q}_2^*\left(\frac{\dot{\psi}_s}{\delta_\varepsilon},\int_{\mathbb{R}^d_*}k\left(\frac{\psi_s}{\delta_\varepsilon},y\right)dy\right)ds\right.\right. \\
&\qquad\left.\left. +\frac{\delta_\varepsilon}{\varepsilon}\sup_{\xi\in\mathbb{R}^d}\int_0^t\left\langle\left\{c+\int_{\mathbb{R}^d_*}k(.,y)\nu(dy)\right\}\left(\frac{\psi_s}{\delta_\varepsilon}\right),\xi\right\rangle ds\right]\right\}\right)+o(1).
\end{aligned} \tag{3.6}$$

From (3.5), we then have

$$\begin{aligned}
\mathbb{P}\left(X_1^{x,\varepsilon,\delta_\varepsilon}\in F\right) &= \int_{F/\delta_\varepsilon} p^{\varepsilon,\delta_\varepsilon}\left(\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2,\frac{x}{\delta_\varepsilon},z\right)dz \\
&= \underbrace{\delta_\varepsilon^{-d}\int_F p^{\varepsilon,\delta_\varepsilon}\left(\left(\sqrt{\varepsilon}/\delta_\varepsilon\right)^2,\frac{x}{\delta_\varepsilon},\frac{z}{\delta_\varepsilon}\right)dz}_{\text{by scaling property}}.
\end{aligned}$$

By (3.6), we deduce

$$\begin{aligned}
&\varepsilon\log\mathbb{P}\left(X_1^{x,\varepsilon,\delta_\varepsilon}\in F\right) \\
&\leq -\frac{\varepsilon}{2}\inf_{\substack{\psi\in\mathcal{D}\left(\left[0,\frac{\varepsilon}{\delta_\varepsilon^2}\right];\mathbb{R}^d\right)\\ \psi_0=\frac{x}{\delta_\varepsilon},\ \psi_{\frac{\varepsilon}{\delta_\varepsilon^2}}=\frac{z}{\delta_\varepsilon}}}\int_0^{\frac{\varepsilon}{\delta_\varepsilon^2}}\hat{Q}_1^*\left(\frac{\dot{\psi}_s}{\delta_\varepsilon}-\frac{\delta_\varepsilon}{\varepsilon}\left\{c+\int_{\mathbb{R}^d_*}k(.,y)\nu(dy)\right\}\left(\frac{\psi_s}{\delta_\varepsilon}\right)\right)ds
\end{aligned}$$

$$
\begin{aligned}
& -\varepsilon \inf_{\substack{\psi\in\mathcal{D}\left(\left[0,\frac{\varepsilon}{\delta_\varepsilon^2}\right];\mathbb{R}^d\right)\\ \psi_0=\frac{x}{\delta_\varepsilon},\ \psi_{\frac{\varepsilon}{\delta_\varepsilon^2}}=\frac{z}{\delta_\varepsilon}}} \left\{ \int_0^{\frac{\varepsilon}{\delta_\varepsilon^2}} \hat{Q}_2^*\left(\frac{\dot\psi_s}{\delta_\varepsilon}, \int_{\mathbb{R}^d_*} k\left(\frac{\psi_s}{\delta_\varepsilon}, y\right) dy\right) ds \right.\\
&\qquad\qquad \left. + \frac{\delta_\varepsilon}{\varepsilon}\sup_{\xi\in\mathbb{R}^d} \int_0^{\frac{\varepsilon}{\delta_\varepsilon^2}} \left\langle \left\{ c + \int_{\mathbb{R}^d_*} k(.,y)\nu(dy)\right\}\left(\frac{\psi_s}{\delta_\varepsilon}\right), \xi\right\rangle ds\right]\right\} + o(1)\\
&\le -\frac{\delta_\varepsilon}{2} \inf_{\substack{\psi\in\mathcal{D}\left(\left[0,\frac{1}{\delta_\varepsilon}\right];\mathbb{R}^d\right)\\ \psi_0=\frac{x}{\delta_\varepsilon},\ \psi_{\frac{1}{\delta_\varepsilon}}=\frac{z}{\delta_\varepsilon}}} \int_0^{\frac{1}{\delta_\varepsilon}} \hat{Q}_1^*\left(\dot\psi_s - \left\{c + \int_{\mathbb{R}^d_*} k(.,y)\nu(dy)\right\}(\psi_s)\right) ds\\
&\quad - \inf_{\substack{\psi\in\mathcal{D}\left(\left[0,\frac{\varepsilon}{\delta_\varepsilon}\right];\mathbb{R}^d\right)\\ \psi_0=\frac{x}{\delta_\varepsilon},\ \psi_{\frac{1}{\delta_\varepsilon}}=\frac{z}{\delta_\varepsilon}}} \left\{ \delta_\varepsilon \int_0^{\frac{1}{\delta_\varepsilon}} \hat{Q}_2^*\left(\dot\psi_s, \int_{\mathbb{R}^d_*} k\left(\psi_s, y\right) dy\right) ds \right.\\
&\qquad\qquad \left. + \varepsilon\sup_{\xi\in\mathbb{R}^d} \int_0^{\frac{1}{\delta_\varepsilon}} \left\langle \left\{ c + \int_{\mathbb{R}^d_*} k(.,y)\nu(dy)\right\}\left(\psi_s\right), \xi\right\rangle ds\right]\right\} + o(1).
\end{aligned}
$$

Therefore, the claim follows, i.e.,

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(X_1^{x,\varepsilon,\delta_\varepsilon} \in F\right) &\le -\lim_{\varepsilon\to 0} \delta_\varepsilon V^1_{1/\delta_\varepsilon}\left(\frac{x}{\delta_\varepsilon}, \frac{z}{\delta_\varepsilon}\right) - \lim_{\varepsilon\to 0} \delta_\varepsilon V^2_{1/\delta_\varepsilon}\left(\frac{x}{\delta_\varepsilon}, \frac{z}{\delta_\varepsilon}\right)\\
&\le -\inf_{z \inf F} \mathcal{J}(z-x).
\end{aligned}
$$

The proof is complete. □

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