Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 15, Number 2, pp. 113–124 (2020)
http://campus.mst.edu/adsa

Initial Value Problems for Fractional Differential Equations of Riemann–Liouville Type

Muhammad N. Islam
University of Dayton
Department of Mathematics
Dayton, OH 45469 USA
misl am1@udayton.edu

Jeffrey T. Neugebauer
Eastern Kentucky University
Department of Mathematics and Statistics
jeffrey.neugebauer@eku.edu

Abstract

Consider fractional initial value problem

$$D^q_{0^+}x(t) = -f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x_0 \neq 0,$$

where $f : (0, \infty)\times\mathbb{R} \to \mathbb{R}$ is continuous, and $D^q_{0^+}$ denotes the Riemann–Liouville differential operator of order $q \in (0, 1)$. By studying an equivalent Volterra integral equation, we show the existence of a continuous solution on $(0, T]$ for some $T > 0$. We then show for a special case when $f(t, x) = x + g(t, x)$ that if a continuous solution exists on $(0, \infty)$, then it is absolutely integrable on the same interval.

AMS Subject Classifications: 34A08, 34A12, 45D05, 45E10, 45G05.
Keywords: Fractional differential equation, Volterra integral equation, existence of solutions, uniqueness of solutions, absolute integrability.

1 Introduction

It is known that many real world problems can be modeled by the fractional initial value problem of Riemann–Liouville type:

$$D^q_{0^+}x(t) = -f(t, x(t)), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x_0 \neq 0. \tag{1.1}$$

Received April 6, 2020; Accepted June 24, 2020
Communicated by Martin Bohner

For various results on (1.1) in terms of theory and applications, we refer the interested reader to [4,6–8,10,11]. We also refer to [1–3] for more recent studies on the existence of solutions of the initial value problem (1.1).

In the present paper, we study (1.1) where $f : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, and $D^q_{0^+}$ denotes the Riemann–Liouville differential operator of order $q \in (0,1)$ defined by

$$D^q_{0^+}x(t) = \frac{1}{\Gamma(1-q)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-q}x(s)\mathrm{d}s.$$

Here $\Gamma : (0,\infty) \to \mathbb{R}$ is Euler's Gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t.$$

Under certain conditions, it is known that the initial value problem (1.1) is equivalent to the Volterra integral equation

$$x(t) = x_0 t^{q-1} - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x(s))\mathrm{d}s. \tag{1.2}$$

By equivalent, we mean that x is a solution of (1.1) if and only if x is a solution of (1.2). Some results of equivalence can be found in [1], where equivalence is shown on an interval $(0,T]$. In Section 4, we will assume that (1.1) and (1.2) are equivalent on the interval $(0,\infty)$. In Section 2, an example where (1.1) and (1.2) are equivalent on the interval $(0,\infty)$ is given.

We start with some preliminaries and a motivating example in Section 2. In Section 3, we show the existence of a continuous solution of (1.2) on $(0,T]$ for some $T > 0$. Then in Section 4, we obtain a result showing that if a continuous solution of (1.2) exists on $(0,\infty)$, then it is absolutely integrable on the same interval, i.e., if x is a continuous solution of (1.2) on $(0,\infty)$, then $\int_0^\infty |x(t)|\mathrm{d}t < \infty$. Absolute integrability of solutions when they exist is an important issue. We feel that our work of Section 4 in which we have obtained such a result is an important contribution to the research in Riemann–Liouville type equations. Although we consider a special case where $f(t,x) = x + g(t,x)$, we hope that our work will motivate researchers to pursue more general cases.

2 A Motivating Example

Example 2.1. Consider the fractional differential equation

$$D^{1/2}_{0^+}x(t) = -\frac{\sqrt{\pi}}{2}\left(\sqrt{t}x(t)\right)^{3/2}, \tag{2.1}$$

satisfying the initial condition

$$\lim_{t\to 0^+}\frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{-1/2}x(s)\mathrm{d}s = \sqrt{\pi}. \tag{2.2}$$

In [1], initial condition (2.2) is shown to be equivalent to an initial condition of the form

$$\lim_{t\to 0^+} t^{-1/2}x(t) = 1.$$

It is shown in [1] that the function

$$x(t) = \frac{1}{\sqrt{t}(1+t)}$$

satisfies (2.1), (2.2) on the interval $(0,\infty)$ and also satisfies the integral equation

$$x(t) = \frac{1}{\sqrt{t}} - \frac{1}{2}\int_0^t (t-s)^{-1/2}(\sqrt{s}x(s))^{3/2}\mathrm{d}s$$

on $(0,\infty)$. Also, notice

$$\begin{aligned}\int_0^\infty x(t)\mathrm{d}t &= \int_0^\infty \frac{1}{\sqrt{t}(1+t)}\\ &= 2\int_0^\infty \frac{1}{1+u^2}\mathrm{d}u\\ &= 2\arctan u|_0^\infty\\ &= \pi.\end{aligned}$$

So x is absolutely integrable on $(0,\infty)$.

Motivated by this example, in Section 3, we give conditions when (1.2) has a unique solution on $(0,T^*]$ for some $T^*>0$, where Theorem 3.2 gives equivalence of (1.1) and (1.2). In Section 4, we assume (1.1) and (1.2) are equivalent on $(0,\infty)$. When (1.1) has a solution x on $(0,\infty)$, we give sufficient conditions that imply x is absolutely integrable on $(0,\infty)$.

3 Existence of Solutions

In this section, we present some results on the existence of a continuous solution of (1.1) without showing detailed proofs. These results can be derived from the results in [2].

Definition 3.1. For a given $q \in (0,1)$, a function $\varphi : (0,T] \to \mathbb{R}$ is said to be a solution of (1.2) if φ is continuous, φ satisfies (1.2) on $(0,T]$, and $t^{1-q}\varphi$ is continuous on $[0,T)$ with $\lim_{t\to 0^+} t^{1-q}\varphi(t) = x_0$.

The following theorem given in [1] establishes some conditions under which (1.1) and (1.2) are equivalent.

Theorem 3.2. *Let* $q \in (0,1)$ *and* $x_0 \neq 0$. *Let* $f(t,x)$ *be a function that is continuous on the set* $B = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T,\ x \in I\}$, *where* $I \subset \mathbb{R}$ *is an unbounded interval. Suppose* $x : (0,T] \to I$ *is continuous and both* $x(t)$ *and* $f(t,x(t))$ *are absolutely integrable on* $(0,T]$. *Then* $x(t)$ *satisfies* (1.1) *on* $(0,T]$ *if and only if* $x(t)$ *satisfies* (1.2) *on* $(0,T]$.

We assume $f(t,x)$ satisfies the following local Lipschitz condition.

(A1) For each $T > 0$, there exists a $k = k(T) > 0$ such that

$$|f(t,x) - f(t,y)| \leq k|x-y|,$$

for all $x, y \in \mathbb{R}$, $0 < t \leq T$.

Notice that (A1) implies

$$|f(t,x)| \leq k|x| + |f(t,0)|.$$

Define $f_0(t) = f(t,0)$. We also assume $f_0 \in X$.

For a fixed $T > 0$ and for $g(t) = t^{q-1}$, let X be the space of all continuous functions $\varphi : (0,T] \to \mathbb{R}$ with

$$|\varphi|_g = \sup_{0<t\leq T} \frac{|\varphi(t)|}{g(t)} < \infty.$$

It is shown in [2, Theorem 2.2] that $(X, |\cdot|_g)$ is a Banach space.

Lemma 3.3. *If* $\varphi \in X$, *then* φ *is absolutely integrable on* $(0,T]$.

The proof of the following lemma can be found in [1, Lemma 4.6]

Lemma 3.4. *Suppose* $\varphi : (0,T] \to \mathbb{R}$ *is a continuous and absolutely integrable function on* $(0,T]$. *Then*

$$h(t) := \int_0^t (t-s)^{q-1}\varphi(s)\mathrm{d}s$$

is continuous and absolutely integrable on $(0,T]$.

Define a mapping P on X as follows. For $\varphi \in X$,

$$(P\varphi)(t) := x_0 t^{q-1} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,\varphi(s))\mathrm{d}s. \tag{3.1}$$

Define

$$b_\varphi(t) := \int_0^t (t-s)^{q-1} f(s,\varphi(s))\mathrm{d}s.$$

Since $f_0 \in X$, (A1) and Lemma 3.4 imply $b_\varphi \in X$. This implies $P\varphi \in X$. So $P : X \to X$.

Theorem 3.5. *Suppose assumption (A1) holds and suppose* $f_0 \in X$*. Then there exists a* $T^* > 0$ *such that* (1.1) *has a unique continuous solution on* $(0, T^*]$.

Proof. Let $\varphi, \psi \in X$. Then, by (A1),

$$\begin{aligned}
|(P\varphi)(t) - (P\psi)(t)| &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,\varphi(s)) - f(s,\psi(s))| \mathrm{d}s \\
&\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} k |\varphi(s) - \psi(s)| \mathrm{d}s \\
&\le \frac{1}{\Gamma(q)} k |\varphi - \psi|_g \int_0^t (t-s)^{q-1} s^{q-1} \mathrm{d}s \\
&= \frac{1}{\Gamma(q)} k |\varphi - \psi|_g t^{2q-1} \frac{\Gamma^2(q)}{\Gamma(2q)}.
\end{aligned}$$

Therefore

$$\frac{|(P\varphi)(t) - (P\psi)(t)|}{t^{q-1}} \le k|\varphi - \psi|_g \frac{\Gamma(q)}{\Gamma(2q)} t^q.$$

Let $T^* > 0$ be such that

$$k \frac{\Gamma(q)}{\Gamma(2q)} t^q \le k^* < 1$$

for $0 < t \le T^*$. Thus, we have

$$|(P\varphi) - (P\psi)|_g \le k^* |\varphi - \psi|_g.$$

Since $k^* < 1$, the mapping $P : X \to X$ is a contraction for $0 < t \le T^*$. Therefore there exists a unique continuous $\varphi \in X$ such that $P\varphi = \varphi$. Since φ is continuous and both $\varphi(t)$ and $f(t, \varphi(t))$ are absolutely integrable on $(0, T^*]$, (1.1) and (1.2) are equivalent. Thus $\varphi(t)$ is a unique continuous solution of (1.1) on $(0, T^*]$. □

4 Absolute Integrability of Solutions

In this section, we assume (1.1) and (1.2) are equivalent on $(0, \infty)$. Let

$$C(t-s) = \frac{1}{\Gamma(q)} (t-s)^{q-1}. \tag{4.1}$$

Then, for $f(t,x) = x + g(t,x)$, the integral equation (1.2) becomes

$$x(t) = x_0 t^{q-1} - \int_0^t C(t-s)[x(s) + g(s, x(s))] \mathrm{d}s. \tag{4.2}$$

We assume $|g(t,x)| \le h(t)$ for $t << 1$ and all $x \in \mathbb{R}$, with

$$\lim_{t \to 0^+} t^{1-q} \int_0^t C(t-s) h(s) \mathrm{d}s = 0. \tag{4.3}$$

First, we present some known results regarding (4.2) and the associated resolvent equation (see [9, pp. 189-193]). A function x is a solution of (4.2) if and only if x satisfies

$$x(t) = y(t) - \int_0^t R(t-s)g(s, x(s))\mathrm{d}s, \tag{4.4}$$

where the function y is given by

$$y(t) = x_0 t^{q-1} - \int_0^t R(t-s)x_0 s^{q-1}\mathrm{d}s, \tag{4.5}$$

and the function R, known as the resolvent kernel of C, is the solution of the resolvent equation

$$R(t) = C(t) - \int_0^t C(t-s)R(s)\mathrm{d}s. \tag{4.6}$$

The function C defined in (4.1) is completely monotone on $(0,\infty)$. Thus by [9, Theorem 6.2], the associated resolvent kernel R satisfies, for $t > 0$,

$$0 \le R(t) \le C(t), \quad R(t) \to 0 \text{ as } t \to \infty \tag{4.7}$$

and that

$$C \notin L^1(0,\infty) \text{ implies } \int_0^\infty R(t)\mathrm{d}t = 1. \tag{4.8}$$

If x satisfies (4.4), it must satisfy the condition $\lim_{t\to 0^+} t^{1-q}x(t) = x_0$. To see this, notice that (4.3) implies

$$\begin{aligned}\lim_{t\to 0^+} t^{1-q}\left|\int_0^t R(t-s)g(s,x(s))\mathrm{d}s\right| &\le \lim_{t\to 0^+} t^{1-q}\int_0^t C(t-s)h(s)\mathrm{d}s \\ &= 0.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{t\to 0^+} t^{1-q}\int_0^t R(t-s)x_0 s^{q-1}\mathrm{d}s &\le \lim_{t\to 0^+} t^{1-q}\int_0^t \frac{1}{\Gamma(q)}(t-s)^{q-1}x_0 s^{q-1}\mathrm{d}s \\ &= \lim_{t\to 0^+} \frac{\Gamma(q)}{\Gamma(2q)} x_0 t^q \\ &= 0.\end{aligned}$$

So

$$\lim_{t\to 0^+} t^{1-q}x(t) = \lim_{t\to 0^+} t^{1-q}y(t) = x_0.$$

Thus if x satisfies (4.4), then x is a solution of (1.1).

Suppose there exists a continuous solution $x(t)$ of (4.2) on $(0,\infty)$. In this section, we show that $x(t)$ is absolutely integrable on $(0,\infty)$.

Multiplying both sides of (4.6) by $x_0\Gamma(q)$ gives

$$\begin{aligned} x_0\Gamma(q)R(t) &= x_0\Gamma(q)C(t) - x_0\Gamma(q)\int_0^t C(t-s)R(s)\mathrm{d}s \\ &= x_0\Gamma(q)\frac{1}{\Gamma(q)}t^{q-1} - x_0\Gamma(q)\int_0^t \frac{1}{\Gamma(q)}(t-s)^{q-1}R(s)\mathrm{d}s \\ &= x_0t^{q-1} - x_0\int_0^t (t-s)^{q-1}R(s)\mathrm{d}s \\ &= x_0t^{q-1} - \int_0^t R(t-s)x_0s^{q-1}\mathrm{d}s \\ &= y(t), \end{aligned}$$

the last equality coming from (4.5). Therefore $y(t)$ is a constant multiple of $R(t)$. Since $R(t)$ is continuous, so is $y(t)$, $0 < t < \infty$. Also, by (4.7) and (4.8), it is clear that

$$\int_0^\infty y(t)\mathrm{d}t < \infty.$$

Remark 4.1. If $g(t,x) \equiv 0$, then (1.1) becomes the linear initial value problem

$$D_{0^+}^q x(t) = -x(t), \quad \lim_{t\to 0^+} t^{1-q}x(t) = x_0 \neq 0. \tag{4.9}$$

Here

$$x(t) = y(t) = x_0t^{q-1} - \int_0^t R(t-s)x_0s^{q-1}\mathrm{d}s.$$

Since $y(t)$ is a constant multiple of $R(t)$, this also implies $x(t)$ is a constant multiple of $R(t)$. Then (4.7) and (4.8) imply that x is absolutely integrable on $(0,\infty)$.

For a specific example, consider (4.9) with $q = \frac{1}{2}$ and $x_0 = \frac{1}{\sqrt{\pi}}$. In [11, p. 138], it is shown that the solution of the differential equation is given by

$$x(t) = C\left(\frac{1}{\sqrt{\pi t}} - e^t\operatorname{erfc}(\sqrt{t})\right),$$

for $t > 0$. The initial condition gives $C = 1$. Notice

$$\int_0^\infty \left(\frac{1}{\sqrt{\pi t}} - e^t\operatorname{erfc}(\sqrt{t})\right)\mathrm{d}t = 1,$$

so x is absolutely integrable on $(0,\infty)$. In this case, $x(t) = R(t)$, and so

$$R(t) = \frac{1}{\sqrt{\pi t}} - e^t\operatorname{erfc}(\sqrt{t}),$$

which was shown in [5].

Assume g satisfies a global Lipschitz condition

$$|g(t,x) - g(t,y)| \leq k|x-y| \text{ for all } t \in (0,\infty),\ x,y \in \mathbb{R}.$$

This condition implies that

$$|g(t,x)| \leq k|x| + |g(t,0)| \text{ for all } t \in (0,\infty),\ x \in \mathbb{R}. \tag{4.10}$$

Theorem 4.2. *Suppose for* $t << 1$, $|g(t,x)| \leq h(t)$ *for all* $x \in \mathbb{R}$, *where* h *satisfies* (4.3). *Suppose* g *satisfies* (4.10) *with* $k < 1$,

$$\int_0^\infty |g(t,0)|\mathrm{d}t < \infty,$$

and

$$\lim_{t\to 0^+} \int_0^t |g(s,0)|\mathrm{d}s = 0.$$

If there exists a solution x *of equation* (4.4) *on* $(0,\infty)$, *then* x *is absolutely integrable on* $(0,\infty)$.

Proof. Define, for the solution x and for $t > 0$,

$$V(t) = \int_0^t \int_{t-s}^\infty R(u)\mathrm{d}u[k|x(s)| + |g(s,0)|]\mathrm{d}s. \tag{4.11}$$

Now

$$\int_{t-s}^\infty R(u)\mathrm{d}u \leq \int_0^\infty R(u)\mathrm{d}u \leq 1.$$

So

$$\int_0^t \int_{t-s}^\infty R(u)\mathrm{d}u[k|x(s)| + |g(s,0)|]\mathrm{d}s \leq \int_0^t [k|x(s)| + |g(s,0)|]\mathrm{d}s.$$

Since $\lim_{t\to 0^+} t^{1-q}x(t) = x_0$, there exists a $T > 0$ such that

$$\frac{|x_0|}{2}t^{q-1} \leq |x(t)| \leq \frac{3|x_0|}{2}t^{q-1}, \quad t \in (0,1).$$

So

$$\begin{aligned} 0 &\leq V(t) \\ &\leq \int_0^t [k|x(s)| + |g(s,0)|]\mathrm{d}s \\ &\leq \int_0^t k\frac{3|x_0|}{2}s^{q-1}\mathrm{d}s + \int_0^t |g(s,0)|\mathrm{d}s \\ &\leq k\frac{3|x_0|t^q}{2q} + \int_0^t |g(s,0)|\mathrm{d}s. \end{aligned}$$

Since $\lim_{t\to 0^+} \int_0^t |g(s,0)|\mathrm{d}s = 0$, it follows that

$$\lim_{t\to 0^+} V(t) = 0.$$

So V can be defined on $[0,\infty)$ so that $V(0)=0$.

Next,

$$V'(t) = \int_0^\infty R(u)\mathrm{d}u[k|x(t)| + |g(t,0)|] - \int_0^t R(t-s)[k|x(s)| + |g(s,0)|]\mathrm{d}s. \tag{4.12}$$

Since $\int_0^\infty R(u)\mathrm{d}u = 1$, (4.12) implies that

$$V'(t) = [k|x(t)| + |g(t,0)|] - \int_0^t R(t-s)[k|x(s)| + |g(s,0)|]\mathrm{d}s. \tag{4.13}$$

Now, from (4.4),

$$\begin{aligned} |x(t)| &\le |y(t)| + \int_0^t R(t-s)|g(s,x(s))|\mathrm{d}s \\ &\le |y(t)| + \int_0^t R(t-s)[k|x(s)| + |g(s,0)|]\mathrm{d}s. \end{aligned}$$

Therefore

$$-\int_0^t R(t-s)[k|x(s)| + |g(s,0)|]\mathrm{d}s \le |y(t)| - |x(t)|.$$

So (4.13) gives

$$\begin{aligned} V'(t) &\le k|x(t)| + |g(t,0)| + |y(t)| - |x(t)| \\ &= (k-1)|x(t)| + |y(t)| + |g(t,0)|. \end{aligned}$$

Integrating from 0 to t yields

$$V(t) - V(0) \le (k-1)\int_0^t |x(s)|\mathrm{d}s + \int_0^t |y(s)|\mathrm{d}s + \int_0^t |g(s,0)|\mathrm{d}s.$$

Since $V'(t) \ge 0$ and $V(0)=0$, the previous inequality implies that

$$(1-k)\int_0^t |x(s)|\mathrm{d}s \le \int_0^t |y(s)|\mathrm{d}s + \int_0^t |g(s,0)|\mathrm{d}s. \tag{4.14}$$

Since

$$\int_0^\infty |y(t)|\mathrm{d}t < \infty$$

and

$$\int_0^\infty |g(t,0)|\mathrm{d}t < \infty,$$

it follows from (4.14) that

$$\int_0^\infty |x(t)|\mathrm{d}t < \infty,$$

proving x is absolutely integrable on $(0,\infty)$. □

Example 4.3. Consider the fractional differential equation

$$D^q_{0^+}x(t) = \begin{cases} -x + t^{q-1} - \sin x, & 0 \le t \le 1, \\ -x + t^{q-2} - \sin x, & 1 \le t, \end{cases} \qquad \lim_{t\to 0^+} t^{1-q}x(t) = x_0 \neq 0. \tag{4.15}$$

Here

$$f(t,x) = \begin{cases} x - t^{q-1} + \sin x, & 0 \le t \le 1, \\ x - t^{q-2} + \sin x, & 1 \le t, \end{cases}$$

and

$$g(t,x) = \begin{cases} -t^{q-1} + \sin x, & 0 \le t \le 1, \\ -t^{q-2} + \sin x, & 1 \le t. \end{cases}$$

Now

$$|f(t,x) - f(t,y)| \le |x-y| + |\sin x - \sin y| \le 2|x-y|,$$

so (A1) holds. Set $f(t,0) = f_0(t)$. Then

$$\frac{|f_0(t)|}{t^{q-1}} = \begin{cases} 1, & 0 \le t \le 1, \\ t^{-1}, & 1 \le t, \end{cases}$$

So $|f_0|_g = 1$ and $f_0 \in X$. Therefore Theorem 3.5 gives the existence of a $T^* > 0$ such that (4.15) has a unique solution x on $(0, T^*]$.

Notice for small t, $|g(t,x)| \le t^{q-1} + 1$. So

$$t^{1-q}\int_0^t \frac{1}{\Gamma(q)}(t-s)^{q-1}(s^{q-1}+1)\mathrm{d}s = \frac{\Gamma(q)}{\Gamma(2q)}t^q + \frac{1}{\Gamma(q+1)}t,$$

implying

$$\lim_{t\to 0^+} t^{1-q}\int_0^t \frac{1}{\Gamma(q)}(t-s)^{q-1}(s^{q-1}+1)\mathrm{d}s = 0.$$

The Lipschitz condition holds since

$$|g(t,x) - g(t,y)| = |\sin x - \sin y| \le |x-y|.$$

Now,

$$\int_0^\infty |g(t,0)|\mathrm{d}t = \int_0^1 t^{q-1}\mathrm{d}t + \int_1^\infty t^{q-2}\mathrm{d}t = \frac{1}{q} + \frac{1}{1-q} < \infty.$$

Finally, for small t,

$$\int_0^t |g(s,0)|\mathrm{d}s = \frac{t^q}{q}.$$

Hence

$$\lim_{t\to 0^+} \int_0^t |g(s,0)|\mathrm{d}s = 0.$$

Therefore, if there exists a solution x^* of (4.15) that can be extended to $(0,\infty)$, Theorem 4.2 guarantees that x^* is absolutely integrable on $(0,\infty)$.

References

[1] L. C. Becker, T. A. Burton, and I. K. Purnaras. Complementary equations: a fractional differential equation and a Volterra integral equation. *Electron. J. Qual. Theory Differ. Equ.*, 2015(12):1–24, 2015.

[2] L. C. Becker, T. A. Burton, and I. K. Purnaras. An inversion of a fractional differential equation and fixed points. *Nonlinear Dyn. Syst. Theory*, 15(3):242–271, 2015.

[3] L. C. Becker, T. A. Burton, and I. K. Purnaras. Integral and fractional equations, positive solutions, and Schaefer's fixed point theorem. *Opuscula Math.*, 36(4):431–458, 2016.

[4] Kai Diethelm. *The analysis of fractional differential equations*, volume 2004 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010. An application-oriented exposition using differential operators of Caputo type.

[5] Muhammad N. Islam. Bounded, asymptotically stable, and L^1 solutions of Caputo fractional differential equations. *Opuscula Math.*, 35(2):181–190, 2015.

[6] Anatoly A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo. *Theory and applications of fractional differential equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006.

[7] Vangipuram Lakshmikantham, Srinivasa Leela, and J Vasundhara Devi. *Theory of fractional dynamic systems*. Cambridge Scientific Publishers, 2009.

[8] Kenneth S. Miller and Bertram Ross. *An introduction to the fractional calculus and fractional differential equations*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1993.

[9] Richard K. Miller. *Nonlinear Volterra integral equations*. W. A. Benjamin, Inc., Menlo Park, Calif., 1971. Mathematics Lecture Note Series.

[10] Keith Oldham and Jerome Spanier. *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, volume 111. Elsevier, 1974.

[11] Igor Podlubny. *Fractional differential equations*, volume 198 of *Mathematics in Science and Engineering*. Academic Press, Inc., San Diego, CA, 1999. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.