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On Hyers–Ulam and Hyers–Ulam–Rassias Stability of a Second Order Linear Dynamic Equation on Time Scales

Alaa E. Hamza
Department of Mathematics, College of Science
University of Jeddah, Jeddah, 21589 Saudi Arabia
& Department of Mathematics, Faculty of Science
Cairo University, Cairo, Egypt
hamzaaeg2003@yahoo.com

Maryam Alghamdi and **Alaa Aljehani**
Department of Mathematics, College of Science
University of Jeddah, Jeddah, 21589 Saudi Arabia
maaalghamdi4@gmail.com
a.a418@hotmail.com

Abstract

In this paper, we investigate new sufficient conditions for Hyers–Ulam and Hyers–Ulam–Rassias stability of second- order linear dynamic equations on time scales of the form

$$\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t) = 0, \quad t \in [a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T},$$

where $\mathbb{T}$ is a time scale and f is rd–continuous from $[a, b]_{\mathbb{T}}$ to a Banach space X. Our results depend on creating an equivalent integral equation and using the fixed point theorem.

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1 Preliminaries and Introduction

In 1940, Ulam presented the following problem related to the stability of functional equations: “give conditions in order for a linear mapping near an approximately linear

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mapping to exist". See [19]. The case of approximately additive mappings was solved by Hyers [7] who proved that the Cauchy equation is stable in Banach spaces. Since then, this type of stability founded by Ulam and Hyers, famed for Hyers–Ulam stability. Recently, there has been hundreds papers appeared concerning Hyers–Ulam stability due to its applications in control theory and numerical analysis etc. In 1978, Rassias [14] extended Hyers–Ulam stability concept and called it Hyers–Ulam–Rassias stability. For more details, we refer the reader to the monograph of Jung [8].

In 1998, Ger and Alsina [5] were first authors who investigated the Hyers–Ulam stability of differential equations. This result has been generalized by Miura, Takahasi and Choda [9], by Miura [10], and by Takahasi et al. and Miura et al. [11, 12]. Popa proved the Hyers–Ulam stability of linear recurrence with constant coefficients [13]. Many articles, dealing with Hyers–Ulam stability, were edited by Rassias [15]. Wang, Zhou and Sun introduced the Hyers–Ulam stability of linear differential equations of first order [20]. In 2012 Anderson, Gates and Heuer [1] extended the work of Li and Shen [16, 17] to prove the Hyers–Ulam stability of the scalar second-order linear non-homogeneous dynamic equation on bounded time scales. They obtained their results via a related Riccati dynamic equation. Also in 2012 András and Mészáros studied the Hyers– Ulam stability of some linear and nonlinear dynamic equations and integral equations on time scales based on the theory of Picard operators [2]. Hamza and Yassen extended the work of Douglas, Gates and Heuer, and investigated Hyers–Ulam stability of abstract second-order linear dynamic equations on unbounded time scales [6]. In 2017, Shen established Ulam stability of first-order linear dynamic equations and its adjoint equation on time scales by using the integrating factor method [18]. Recently there has been a great interest in studying stability of dynamic equations on time scales.

In this paper, we investigate new sufficient conditions for Hyers–Ulam and Hyers–Ulam–Rassias stability of second-order linear dynamic equations on time scales of the form

$$\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t) = 0, \quad t \in [a, b]_{\mathbb{T}} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $p \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ and $f \in C_{rd}([a, b]_{\mathbb{T}}, X)$. Here, $[a, b]_{\mathbb{T}}$ is the time scale

$$[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}.$$

Our results depend basically on finding an equivalent integral equation to equation (1.1). The main result of the paper is that a sufficient condition for equation (1.1) to have Hyers–Ulam stability is the existence of a unique solution ψ satisfying the initial conditions $\psi^{\Delta^i}(a) = a_i, i = 0, 1$ for any initial values $a_0, a_1 \in X$.

For the terminology and notations used here, we refer the reader to the very interesting monographs of Bohner and Peterson [3] and [4]. We start the paper by introducing some of the basic definitions and notations of the calculus of time scales.

Definition 1.1. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.

Definition 1.2. Let $\mathbb{T}$ be a time scale. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$.

Definition 1.3. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

From now on, X denotes a Banach space with a norm $\|.\|$.

Definition 1.4. **(1)** For a function $f : \mathbb{T} \to X$, $f^\sigma(t)$ is understood to mean $f(\sigma(t))$.

(2) A function $f : \mathbb{T} \to X$ is said to be right-dense continuous or rd-continuous provided f is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points t in $\mathbb{T}$.
The set of all rd-continuous functions $f : \mathbb{T} \to X$ will be denoted by $C_{rd}(\mathbb{T}, X)$.

(3) Assume $f : \mathbb{T} \to X$, and let $t \in \mathbb{T}^k$. The delta-derivative of f at t, denoted $f^\Delta(t)$, is defined to be the element of X with the property that given any $\epsilon > 0$, there is a neighborhood $\mathbf{U}$ of t such that

$$\big\|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big\| \le \epsilon\big|\sigma(t) - s\big|, \ \forall\, s \in \mathbf{U}.$$

If $f^\Delta(t)$ exists we say f is delta-differentiable at t, and we say $f^\Delta : \mathbb{T}^k \to X$ is the delta–derivative of f on $\mathbb{T}^k$. For the notion $\mathbb{T}^k$, see [3] page 2. We denote by

$$f^{\Delta\sigma} = (f^\Delta)^\sigma \ \text{ and } f^{\sigma\Delta} = (f^\sigma)^\Delta.$$

Throughout the rest of the article, we denote by

$$C^2_{rd}([a, b]_\mathbb{T}, X) = \left\{ f : [a, b]_\mathbb{T} \to X | f^\Delta, f^{\Delta^2} \text{ exist and rd-continuous} \right\}.$$

As usual for a bounded function f from $\mathbb{T}$ to X, we denote by

$$\|f\|_\infty = \sup_{t \in \mathbb{T}} \|f(t)\|.$$

2 Main Results

In this section, assume that $p \in C_{rd}([a, b]_\mathbb{T}, \mathbb{R})$ and $f \in C_{rd}([a, b]_\mathbb{T}, X)$. We investigate Hyers–Ulam and Hyers–Ulam–Rassias stability of equation (1.1). First we recall the concept of Hyers–Ulam and Hyers–Ulam–Rassias stability. See [8].

Definition 2.1. We say that equation (1.1) has Hyers–Ulam stability if for any $\epsilon > 0$ and any $\psi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ satisfies

$$\|\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)\| < \epsilon, \quad t \in [a,b]_{\mathbb{T}},$$

there exists a solution $\phi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ of equation (1.1) such that

$$\|\psi(t) - \phi(t)\| < L\epsilon, \quad t \in [a,b]_{\mathbb{T}},$$

for some $L > 0$.

Definition 2.2. Let $\mathcal{C}$ be a family of positive rd-continuous functions on $[a,b]_{\mathbb{T}}$. We say that equation (1.1) has Hyers–Ulam–Rassias stability of type $\mathcal{C}$, if for any $\omega \in \mathcal{C}$ and any $\psi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ that satisfies

$$\|\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)\| < \omega(t), \quad t \in [a,b]_{\mathbb{T}},$$

there exists a solution $\phi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ of equation (1.1) such that

$$\|\psi(t) - \phi(t)\| < L\omega(t), \quad t \in [a,b]_{\mathbb{T}},$$

for some $L > 0$.

We need the following lemma in proving our results.

Lemma 2.3. *ψ is a solution of equation* (1.1) *if and only if ψ satisfies the integral equation*

$$\psi(t) = a_0 + a_1(t-a) + \int_a^t (s - t + \mu(s))(p(s)\psi(s) - f(s))\Delta s. \qquad (2.1)$$

for some constants $a_0, a_1 \in X$.

Proof. Assume that ψ satisfies the integral equation (2.1). We denote by

$$M(t) = \int_a^t (s - t + \mu(s))(p(s)\psi(s) - f(s))\Delta s.$$

By [3, Theorem 1.117], we conclude that

$$M^{\Delta}(t) = -\int_a^t (p(s)\psi(s) - f(s))\Delta s,$$

and

$$M^{\Delta^2}(t) = -p(t)\psi(t) + f(t).$$

This implies that $\psi^{\Delta^2}(t) = -p(t)\psi(t) + f(t)$.

To prove the other direction, assume ψ is a solution of equation (1.1). We denote by

$$g(t) = p(t)\psi(t) - f(t),$$

$$G(t) = \int_a^t g(s)\,\Delta s,$$

and

$$L(t) = \int_a^t G(s)\,\Delta s.$$

Simple calculations show, by integrating two times both sides of (1.1), that

$$\psi(t) = a_0 + a_1(t-a) - L(t).$$

Here $a_i = \psi^{\Delta^i}(a), i = 0, 1.$ It is readily seen that, $M(t) = -L(t)$ for every t. Indeed, we have

$$\begin{aligned} L^{\Delta}(t) &= G(t) \\ &= \int_a^t g(s)\Delta s \\ &= -M^{\Delta}(t). \end{aligned}$$

Consequently, $M(t) = -L(t) + C, t \in [a, \infty) \cap \mathbb{T}$. We have $C = M(a) + L(a) = 0$. Therefore ψ, satisfies equation (2.1). □

Corollary 2.4. *For any two elements* $a_0, a_1 \in X$*, equation* (1.1) *has at most one solution satisfying* $\psi^{\Delta^i}(a) = a_i, i = 0, 1.$

Proof. Assume that ψ_1 and ψ_2 are solutions of equation (1.1) which satisfy same initial conditions. Then both of them satisfy equation (2.1). This implies that

$$\|\psi_1(t) - \psi_2(t)\| \le \int_a^t |s - t + \mu(s)||p(s)|\|\psi_1(s) - \psi_2(s)\|\Delta s.$$

By the Grönwall inequality [3], $\psi_1 = \psi_2$. □

Remark 2.5. It is well known that if equation (1.1) is regressive, that is $1+\mu^2(t)p(t) \neq 0$ for all $t \in [a, b]_{\mathbb{T}}$, then it has a unique solution x that satisfies the initial conditions $x^{\Delta^i}(a) = a_i, i = 0, 1$ for every $a_0, a_1 \in X$. See [3].

Another sufficient condition for the existence of a unique solution of equation (1.1) can be stated in the following theorem.

Theorem 2.6. *If there is* $\alpha \in (0, 1)$ *such that*

$$\int_a^t |p(s)|\,\Delta s \le \frac{\alpha}{b - a + \|\mu\|_\infty}, t \in [a, b]_{\mathbb{T}},$$

then equation (1.1) *has a unique solution* ψ *that satisfies the initial conditions*

$$\psi^{\Delta^i}(a) = a_i, i = 0, 1$$

for any $a_0, a_1 \in X$.

Proof. Fix $a_0, a_1 \in X$. Define the operator $T : C_{rd}([a,b]_{\mathbb{T}}, X) \to C_{rd}([a,b]_{\mathbb{T}}, X)$ by

$$T\psi(t) = a_0 + a_1(t-a) + \int_a^t (s - t + \mu(s))(p(s)\psi(s) - f(s))\Delta s.$$

For $\phi, \psi \in C_{rd}([a,b]_{\mathbb{T}}, X)$, we have

$$\begin{aligned} \|T\psi(t) - T\phi(t)\| &\leq (b - a + \|\mu\|_\infty)\|\psi - \phi\|_\infty \int_a^t |p(s)|\,\Delta s \\ &\leq \alpha\|\psi - \phi\|_\infty, \ \ t \in [a,b]_{\mathbb{T}}. \end{aligned}$$

This implies that T is a contraction. Therefore T has a unique fixed point ψ which is the solution of the integral equation (2.1) satisfying the initial conditions. □

The next theorem indicates the equivalence between the existence of a solution of the scalar homogeneous second order equation and the existence of a solution z of the corresponding Riccati equation. See also [3].

Theorem 2.7. *Assume that Riccati equation*

$$z^\Delta(1 - \mu(t)z(t)) - z^2(t) = p(t), \ \ t \in [a,b]_{\mathbb{T}} \tag{2.2}$$

associated with the scalar equation

$$\psi^{\Delta^2}(t) + p(t)\psi(t) = 0, \ \ t \in [a,b]_{\mathbb{T}}, \tag{2.3}$$

has a solution z *that satisfies* $1 - \mu(t)z(t) \neq 0, t \in [a,b]_{\mathbb{T}}$. *Then equation* (2.3) *has a solution. Conversely, if equation* (2.3) *has a solution with no zeros, then equation* (2.2) *has a solution.*

Proof. Let z be a solution of (2.2) that satisfies $1 - \mu(t)z(t) \neq 0, t \in [a,b]_{\mathbb{T}}$. The function

$$x(t) = e_{-z}(t, a)$$

is a solution of the dynamic equation

$$x^\Delta(t) = -z(t)x(t).$$

Now, for $t \in [a,b]_{\mathbb{T}}$, we have

$$\begin{aligned} x^{\Delta\Delta}(t) + p(t)x(t) &= z^2(t)x(t) - z^{\Delta}(t)x(\sigma(t)) + p(t)x(t) \\ &= z^2(t)x(t) - z^{\Delta}(t)(x(t) + \mu(t)x^{\Delta}(t)) + p(t)x(t) \\ &= z^2(t)x(t) - z^{\Delta}(t)(x(t) - \mu(t)z(t)x(t)) + p(t)x(t) \\ &= x(t)\left(z^2(t) - z^{\Delta}(t)(1-\mu(t)z(t)) + p(t)\right) \\ &= 0. \end{aligned}$$

Conversely, assume that x is a scalar solution of equation (2.3) with no zeros. Define z by

$$z(t) = -\frac{x^{\Delta}(t)}{x(t)}, t \in [a,b]_{\mathbb{T}}.$$

Then

$$z^{\Delta}(t) = -\frac{x(t)x^{\Delta\Delta}(t) - (x^{\Delta}(t))^2}{x(t)x^{\sigma}(t)}.$$

Simple calculations show that

$$\begin{aligned} z^{\Delta}(t)(1-\mu(t)z(t)) - z^2(t) &= -\frac{x^{\Delta\Delta}(t)}{x(t)} \\ &= p(t), t \in [a,b]_{\mathbb{T}}. \end{aligned}$$

□

In the following result we establish a new sufficient condition for the existence of a unique solution x of the scalar equation (2.3) that satisfies the initial conditions $x^{\Delta^i}(a) = a_i, i = 0, 1$ for any $a_0, a_1 \in \mathbb{R}$.

Theorem 2.8. *Assume that Riccati equation* (2.2) *has a solution z that satisfies $1 - \mu(t)z(t) \neq 0, t \in [a,b]_{\mathbb{T}}$. Then equation* (2.3) *has a unique solution x that satisfies the initial conditions $x^{\Delta^i}(a) = a_i, i = 0, 1$ for any $a_0, a_1 \in \mathbb{R}$.*

Proof. In view of Theorem 2.7, assume that $x(t) = e_{-z}(t,a)$ is a solution of equation (2.3). We investigate another solution y of the form

$$y(t) = u(t)x(t),$$

where u is a scalar function which will be chosen such that $\{x, y\}$ is a fundamental set for equation (2.3). We have

$$y^{\Delta}(t) = u(t)x^{\Delta}(t) + u^{\Delta}(t)x^{\sigma}(t),$$

and consequently

$$\begin{aligned} y^{\Delta^2}(t) + p(t)y(t) &= u(t)x^{\Delta^2}(t) + u^{\Delta}(t)x^{\Delta\sigma}(t) \\ &\quad + u^{\Delta}(t)x^{\sigma\Delta}(t) + u^{\Delta^2}(t)x^{\sigma^2}(t) + p(t)u(t)x(t) \\ &= u^{\Delta^2}(t)x^{\sigma^2}(t) + u^{\Delta}(t)x^{\Delta\sigma}(t) + u^{\Delta}(t)x^{\sigma\Delta}(t). \end{aligned}$$

Thus y is a solution of equation (2.3) if and only if u satisfies the following equation

$$u^{\Delta^2}(t)e^{\sigma^2}_{-z}(t,a) + u^{\Delta}(t)(e^{\Delta\sigma}_{-z}(t,a) + e^{\sigma\Delta}_{-z}(t,a)) = 0.$$

The previous equation yields

$$v^{\Delta}(t) + q(t)v(t) = 0,$$

whose solution is given by

$$v(t) = e_{-q}(t,a),$$

where $q(t) = \dfrac{e^{\Delta\sigma}_{-z}(t,a) + e^{\sigma\Delta}_{-z}(t,a)}{e^{\sigma^2}_{-z}(t,a)}$ and $v = u^{\Delta}$. Hence $u(t) = \displaystyle\int_a^t e_{-q}(s,a)\Delta s$. The Wronskian of x and $y = ux$ is given by

$$\begin{aligned} W(x,y)(t) &= e_{-z}(t,a)(u(t)e^{\Delta}_{-z}(t,a) + u^{\Delta}(t)e^{\sigma}_{-z}(t,a)) - e_{-z}(t,a)u(t)e^{\Delta}_{-z}(t,a) \\ &= e_{-z}(t,a)e^{\sigma}_{-z}(t,a)u^{\Delta}(t) \\ &= e_{-z}(t,a)e^{\sigma}_{-z}(t,a)e_{-q}(t,a). \end{aligned}$$

It follows that the Wronskian is nonzero and consequently $\{x,y\}$ is a fundamental set of equation (2.3). Then for any $a_0, a_1 \in \mathbb{R}$, there exist $c_1, c_2 \in \mathbb{R}$ such that the solution $\psi(t) = c_1 x(t) + c_2 y(t)$ of equation (2.3) satisfies the initial conditions $\psi^{\Delta^i}(a) = a_i, i = 0, 1$. □

The following result establishes a new sufficient condition for the Hyers–Ulam stability of equation (1.1).

Theorem 2.9. *Assume that for any $a_0, a_1 \in X$ equation* (1.1) *has a unique solution ϕ that satisfies $\phi^{\Delta^i}(a) = a_i, i = 0, 1$. Then equation* (1.1) *has Hyers–Ulam stability.*

Proof. Let $\epsilon > 0$ and $\psi \in C^2_{rd}([a,b]_{\mathbb{T}}, X)$ satisfies

$$\|\psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)\| < \epsilon, \;\; t \in [a,b]_{\mathbb{T}}.$$

Set $h(t) = \psi^{\Delta^2}(t) + p(t)\psi(t) - f(t)$. Then ψ satisfies the equation

$$\psi^{\Delta^2}(t) + p(t)\psi(t) = f(t) + h(t), \;\; t \in [a,b]_{\mathbb{T}}.$$

Let $a_i = \psi^{\Delta^i}(a), i = 0, 1$. Hence ψ satisfies

$$\psi(t) = a_0 + a_1(t-a) + \int_a^t [s - t + \mu(s)][p(s)\psi(s) - f(s) - h(s)]\Delta s.$$

There exists a unique solution ϕ of equation (1.1) satisfying

$$\phi^{\Delta^i}(a) = a_i, i = 0, 1.$$

Equivalently, ϕ satisfies the integral equation

$$\phi(t) = a_0 + a_1(t-a) + \int_a^t [s-t+\mu(s)][p(s)\phi(s) - f(s)]\Delta s.$$

We have

$$\begin{aligned}\|\psi(t)-\phi(t)\| &\le \int_a^t |s-t+\mu(s)||h(s)|\Delta s \\ &\quad + \int_a^t |s-t+\mu(s)|\|p(s)(\psi(s)-\phi(s))\|\,\Delta s \\ &\le (b-a)(b-a+\|\mu\|_\infty)\epsilon \\ &\quad + (b-a+\|\mu\|_\infty)\|p\|_\infty \int_a^t \|\psi(s)-\phi(s)\|\,\Delta s.\end{aligned}$$

This inequality yields

$$\|\psi(t)-\phi(t)\| \le \frac{(b-a)K}{\|p\|_\infty}\epsilon + K\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s,$$

where

$$K = (b-a+\|\mu\|_\infty)\|p\|_\infty,$$

and

$$M = \sup\{e_K(t,a) : t \in [a,b]_{\mathbb{T}}\}.$$

By the Grönwall inequality [3], we deduce that

$$\|\psi(t)-\phi(t)\| \le \frac{(b-a)KM}{\|p\|_\infty}\,\epsilon, \quad t \in [a,b]_{\mathbb{T}}.$$

Therefore, equation (1.1) is Hyers–Ulam stable. □

Since a regressive equation has a unique solution satisfying any initial conditions, [3], we get the following result

Theorem 2.10. *If equation* (1.1) *is regressive, then it has Hyers–Ulam stability.*

We combine theorems 2.6 and 2.9, to obtain a new sufficient condition for Hyers–Ulam stability of equation (1.1).

Theorem 2.11. *If there is* $\alpha \in (0,1)$ *such that*

$$\int_a^t |p(s)|\,\Delta s \le \frac{\alpha}{b-a+\|\mu\|_\infty}, \quad t \in [a,b]_{\mathbb{T}},$$

then equation (1.1) *has Hyers–Ulam stability.*

We combine theorems 2.8 and 2.9 to obtain another sufficient condition for the Hyers–Ulam stability of the scalar equation (2.3).

Theorem 2.12. *The scalar equation* (2.3) *has Hyers–Ulam stability if the corresponding Riccati equation* (2.2) *has a solution* z *that satisfies* $1-\mu(t)z(t)\neq 0, t\in[a,b]_{\mathbb{T}}$.

The following results are concerning with Hyers–Ulam–Rassias stability. Throughout the rest of the paper, we denote by

$$\mathcal{M}=\left\{\omega\in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R}):\omega \text{ is positive, } \int_a^t \omega^2(s)\,\Delta s\le\omega^2(t)\right\}.$$

Theorem 2.13. *Assume that for any* $a_0,a_1\in X$ *equation* (1.1) *has a unique solution* ϕ *that satisfies* $\phi^{\Delta^i}(a)=a_i, i=0,1.$ *Then equation* (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}$.

Proof. Let $\omega\in\mathcal{M}$ and $\psi\in C^2_{rd}([a,b]_{\mathbb{T}},X)$ satisfies

$$\|\psi^{\Delta^2}(t)+p(t)\psi(t)-f(t)\|<\omega(t),\ \ t\in[a,b]_{\mathbb{T}}.$$

Set $h(t)=\psi^{\Delta^2}(t)+p(t)\psi(t)-f(t)$. Then ψ satisfies the equation

$$\psi^{\Delta^2}(t)+p(t)\psi(t)=f(t)+h(t),\ \ t\in[a,b]_{\mathbb{T}}.$$

Hence it satisfies the integral equation

$$\psi(t)=a_0+a_1(t-a)+\int_a^t[s-t+\mu(s)][p(s)\psi(s)-f(s)-h(s)]\Delta s,$$

where $a_i=\psi^{\Delta^i}(a), i=0,1$. There exists a unique solution ϕ of equation (1.1). Equivalently, ϕ satisfies the integral equation

$$\phi(t)=a_0+a_1(t-a)+\int_a^t[s-t+\mu(s)][p(s)\phi(s)-f(s)]\Delta s.$$

By the Hölder inequality [3], we have

$$\begin{aligned}\|\psi(t)-\phi(t)\|&\leqslant\int_a^t|s-t+\mu(s)|\|h(s)\|\Delta s\\&\quad+\int_a^t|s-t+\mu(s)||p(s)|\|\psi(s)-\phi(s)\|\Delta s\\&\leqslant(b-a+\|\mu\|_\infty)\int_a^t\omega(s)\Delta s\\&\quad+(b-a+\|\mu\|_\infty)\|p\|_\infty\int_a^t\|\psi(s)-\phi(s)\|\Delta s\\&\leqslant\sqrt{b-a}(b-a+\|\mu\|_\infty)\omega(t)\\&\quad+(b-a+\|\mu\|_\infty)\|p\|_\infty\int_a^t\|\psi(s)-\phi(s)\|\Delta s.\end{aligned}$$

We denote by

$$K = (b - a + \|\mu\|_\infty)\|p\|_\infty,$$

and

$$L = \sup_{t\in[a,b]_{\mathbb{T}}} \left(\int_a^b e_K^2(t,\sigma(s))\Delta s \right)^{\frac{1}{2}}.$$

This implies that

$$\|\psi(t) - \phi(t)\| \le \frac{\sqrt{b-a}K}{\|p\|_\infty}\omega(t) + K\int_a^t \|\psi(s) - \phi(s)\|\Delta s.$$

By the Grönwall inequality [3], we deduce that

$$\|\psi(t) - \phi(t)\| \le \frac{\sqrt{b-a}K}{\|p\|_\infty}\,\omega(t) + \frac{\sqrt{b-a}K^2}{\|p\|_\infty}\int_a^t e_K(t,\sigma(s))\omega(s)\Delta s,\ \ t \in [a,b]_{\mathbb{T}}.$$

Again by the Hölder inequality [3], it follows that

$$\|\psi(t) - \phi(t)\| \le \frac{\sqrt{b-a}K}{\|p\|_\infty}\,\omega(t) + \frac{\sqrt{b-a}K^2L}{\|p\|_\infty}\left(\int_a^t \omega^2(s)\Delta s\right)^{\frac{1}{2}},\ \ t \in [a,b]_{\mathbb{T}}.$$

This implies that

$$\|\psi(t) - \phi(t)\| \le \frac{\sqrt{b-a}K}{\|p\|_\infty}\,(1 + KL)\ \omega(t),\ \ t \in [a,b]_{\mathbb{T}}.$$

Therefore, equation (1.1) is Hyers–Ulam–Rassias stable of type $\mathcal{M}$. □

Theorem 2.14. *If equation* (1.1) *is regressive, then it has Hyers–Ulam–Rassias stability of type* $\mathcal{M}$.

We combine theorem 2.6 and Theorem 2.13, to obtain a new sufficient condition for Hyers–Ulam–Rassias stability of equation (1.1).

Theorem 2.15. *If there is* $\alpha \in (0,1)$ *such that*

$$\int_a^t |p(s)|\,\Delta s \le \frac{\alpha}{b - a + \|\mu\|_\infty},\ \ t \in [a,b]_{\mathbb{T}},$$

then equation (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}$.

We combine theorems 2.8 and 2.13 to get another sufficient condition for Hyers–Ulam–Rassias stability of the scalar equation (2.3).

Theorem 2.16. *The scalar equation* (2.3) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}$ *if the corresponding Riccati equation* (2.2) *has a solution* z *that satisfies* $1 - \mu(t)z(t) \neq 0, t \in [a,b]_{\mathbb{T}}$.

Remark 2.17. Theorems 2.13–2.16 hold if we replace $\mathcal{M}$ by

$$\mathcal{K} = \left\{ \omega \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}) : \omega \text{ is positive, } \int_a^t \omega(s)\,\Delta s \le \omega(t) \right\}.$$

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