Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 15, Number 2, pp. 79–97 (2020)
http://campus.mst.edu/adsa

Existence of Positive Solutions for Discrete Fractional Boundary Value Problems

Buse Eralp and **Fatma Serap Topal**
Department of Mathematics, Ege University
35100 Bornova, Izmir, Turkey
buseeeralp@gmail.com
f.serap.topal@ege.edu.tr

Abstract

In this paper, by using the Schauder fixed point theorem and the Banach fixed point theorem, we obtain the existence of positive solutions for the following higher order discrete fractional boundary value problem

$$-\nabla^{\mu}_{a*}x(t) = g(t, x(t-(N-1))), \quad t \in \mathbb{N}^{b-1}_{a+1}$$
$$\alpha x(a-i) - \beta \nabla x(a-(i-1)) = 0, \quad i = 1, 2, 3, ..., N-1$$
$$\gamma x(b) + \delta \nabla x(b) = 0$$

where $\mu > 1,\ N := \lceil \mu \rceil,\ \alpha, \gamma, \beta, \delta > 0$ such that $\dfrac{\alpha}{\beta} > N-1,\ g : \mathbb{N}^{b-1}_{a+1} \longrightarrow \mathbb{R}$ and x are defined on $\mathbb{N}_{a-N+1}$. First, establishing some useful inequalities satisfied by the Green's function associated with the above boundary value problem, we will give the sufficient conditions to ensure the existence of positive solutions for this problem. Also, we establish Lyapunov-type inequalities for this problem. Our results extend some recent works in the literature.

AMS Subject Classifications: 39A10, 39A12, 39A70.
Keywords: Discrete fractional calculus, Caputo nabla fractional difference, positive solutions, fixed point theorem.

1 Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration. Basic backgrounds in fractional calculus can be found in [12]. In recent years, fractional differential equations have been receiving of great interest. It is caused both by

Received March 13, 2020; Accepted September 3, 2020
Communicated by Mehmet Ünal

the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry and engineering. Discrete fractional calculus consists of studying fractional derivatives of functions defined on a discrete domain. Very recently, there appeared a number of papers on the discrete fractional calculus, which has helped to build up some of the basic theory of this area. In particular, several recent papers by Atici and Eloe [9] as well as other recent papers by the present authors have addressed some basic theory of both discrete fractional initial value problems and discrete fractional boundary value problems (DFBVPs). More specifically, Atici and Eloe [9] have already analyzed a transform method in discrete fractional calculus. Goodrich [10] considered a discrete right-focal fractional boundary value problem. All of the fundamental background in discrete fractional calculus can be found in [11] which is written by Goodrich and Peterson. Also Baoguo, Erbe and Peterson have studied on nabla and delta fractional differences in [18, 19] and they have a lot of works on this field. Other recent works have considered DFBVPs with a variety of boundary conditions, see [13,14] and the references therein.

Our aim is, using the fixed point theorems, to obtain the existence of positive solutions for the following $(N-1,1)$-conjugate DFBVP (1.1)–(1.2), as shown

$$-\nabla^{\mu}_{a*}x(t) = g(t, x(t-(N-1))), \quad t \in \mathbb{N}_{a+1}^{b-1} \tag{1.1}$$

$$\begin{gathered} \alpha x(a-i) - \beta\nabla x(a-(i-1)) = 0, \quad i = 1,2,3,...,N-1 \\ \gamma x(b) + \delta\nabla x(b) = 0 \end{gathered} \tag{1.2}$$

in the context of discrete nabla fractional calculus, where $\mu > 1,\ N := \lceil \mu \rceil,\ \alpha, \gamma, \beta, \delta > 0$ such that $\dfrac{\alpha}{\beta} > N-1$.

We organize the rest of this paper as follows. In the preliminaries, we will present some definitions and background results. For sake of convenience, we will also state the fixed point theorems. Considering the Green's function for this conjugate DFBVP, we will give some properties and bounds of this Green's function in Section 3. Also in this section, using these bounds, we will establish Lyapunov-type inequalities and give existence results for $(N-1,1)$-conjugate DFBVP (1.1)–(1.2).

2 Preliminaries

In this section, we collect some basic definitions and lemmas for manipulating discrete fractional operators.

For any real number β, let $N_\beta = \{\beta, \beta+1, \beta+2, ...\}$ and we define $t^{\overline{k}} = \dfrac{\Gamma(t+k)}{\Gamma(t)}$ for any $t, k \in \mathbb{R}$. If $n \in \mathbb{N}$ then $t^{\overline{n}} := t(t+1)...(t+n-1)$.

Remark 2.1 (See [12]). Let n and N be nonnegative integers. Then

$$\frac{\Gamma(-n)}{\Gamma(-N)} = (-1)^{N-n}\frac{N!}{n!}.$$

Also, if t is a nonpositive integer and $t+r$ is not a nonpositive integer, then

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)} = 0.$$

Theorem 2.2. *The following equality hold*

$$\nabla(t+a)^{\overline{n}} = n(t+a)^{\overline{n-1}}, \quad t \in \mathbb{R}$$

for values of $n \in \mathbb{N}$ *and* $a \in \mathbb{R}$.

Definition 2.3. We define the nabla Taylor monomials, $H_n(t,a),\ n \in \mathbb{N}_0$ by $H_0(t,a) = 1,\ t \in \mathbb{N}_a$ and

$$H_n(t,a) = \frac{(t-a)^{\overline{n}}}{n!},\ t \in \mathbb{N}_{a-n+1}$$

for $n \in \mathbb{N}_1$.

Definition 2.4. Let $\mu \neq -1, -2, \ldots$ we define μ-th order nabla fractional Taylor monomial, $H_\mu(t,a)$, by

$$H_\mu(t,a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} \tag{2.1}$$

whenever the right hand side of the equation (2.1) makes sense.

Definition 2.5. Let $f : \mathbb{N}_{a+1} \longrightarrow \mathbb{R}$ be given and $\mu \in \mathbb{R}^+$, then

$$\nabla_a^{-\mu} f(t) := \int_a^t H_{\mu-1}(t, \rho(s)) f(s) \nabla s$$

for $t \in \mathbb{N}_a$, where by convention $\nabla_a^{\mu}(a) = 0$.

Definition 2.6. Let $f : \mathbb{N}_{a-N+1} \longrightarrow \mathbb{R},\ \mu \in \mathbb{R}^+$ and $N-1 < \mu \leq N$. Then we define μ-th nabla fractional difference $\nabla_a^{\mu} f(t)$ by

$$\nabla_a^{\mu} f(t) := \nabla^N \nabla_a^{-(N-\mu)} f(t).$$

Definition 2.7. Assume $f : \mathbb{N}_{a-N+1} \longrightarrow \mathbb{R},\ \mu \in \mathbb{R}^+$ and $N-1 < \mu \leq N$. Then we define μ-th Caputo nabla fractional difference of f is defined by

$$\nabla_{a^*}^{\mu} f(t) := \nabla_a^{-(N-\mu)} \nabla^N f(t),\ t \in \mathbb{N}_{a+1}$$

where $N := \lceil \mu \rceil$.

Definition 2.8. $\rho : \mathbb{N}_{a+1} \longrightarrow \mathbb{N}_a$ for the Nabla transformation defined as $\rho(t) = t-1$ the operator is called the backward jump operator.

The following propositions will be used in the theorem that follows stating the bounds of the Green's function.

Proposition 2.9 (See [15]). *Let f, g be real-valued functions on a set S, such that $f(t), g(t) \geq 0$ for all $t \in S$. Moreover, assume there exists $s_0, s_1 \in S$ where $\max_{s \in S} f(s) = f(s_0)$ and $\max_{s \in S} g(s) = g(s_1)$, i.e. f and g attain their maximum in S. Then for each fixed $t \in S$,*

$$\begin{aligned} |f(t) - g(t)| &\leq \max\{f(t), g(t)\} \\ &\leq \max\{\max_{t \in S} f(t), \max_{t \in S} g(t)\} \end{aligned}$$

Proposition 2.10. *Let $g(\tau, s) := (\tau - \rho(s))^{\overline{a}}$ where $a > 0,\ s \in \mathbb{N}_{a+1}^{b}$, and $\tau \in \mathbb{N}_{s-1}^{b}$.*
(i) $g(\tau, s) \geq 0$,
(ii) g is decreasing function of s,
(iii) g is increasing function of τ.

Proof. Using [11, Theorem 3.5], we get

$$\nabla_s g(\tau, s) = \nabla_s (\tau - \rho(s))^{\overline{a}} = \nabla_s [(\tau + 1) - s]^{\overline{a}} = -a(\tau + 1 - \rho(s))^{\overline{a-1}}.$$

So,

$$\nabla_s g(\tau, s) = -a(\tau + 1 - \rho(s))^{\overline{a-1}}. \tag{2.2}$$

Since $a > 0$, we know that $-a < 0$. Now

$$(\tau + 1 - \rho(s))^{\overline{a-1}} = \frac{\Gamma(\tau + 1 - \rho(s) + a - 1)}{\Gamma(\tau + 1 - \rho(s))} = \frac{\Gamma(\tau - s + 1 + a)}{\Gamma(\tau - s + 2)}$$

Since $\tau \in \mathbb{N}_{s-1}^{b},\ \tau - s + 1 \geq 0$, and hence $\tau - s + 2 > 0$. Also, $\tau - s + 1 + a > 0$ since $a > 0$. Thus $\Gamma(\tau - s + 1 + a) > 0$ and $\Gamma(\tau - s + 2) > 0$ and it follows that $(\tau + 1 - \rho(s))^{\overline{a-1}} > 0$. Thus we have (2.2) that $\nabla_s g(\tau, s) < 0$, and it follows that g is a decreasing function of s. This shows (ii).

Now since g is a decreasing function of s, to show (i), it will suffice to show that $g(\tau, b) \geq 0$ for each $\tau \in \mathbb{N}_{s-1}^{b}$, we have either $\tau = b - 1$ or $\tau = b$. If $\tau = b - 1$, we get

$$g(b - 1, b) = (b - 1 - b + 1)^{\overline{a}} = 0^{\overline{a}} = \frac{\Gamma(a)}{\Gamma(0)} = 0,$$

by Remark 2.1 for $a > 0$. If $\tau = b$, we have

$$g(b, b) = (b - \rho(b))^{\overline{a}} = (b - b + 1)^{\overline{a}} = 1^{\overline{a}} = \Gamma(1 + a),$$

where the last inequality follows since $1 + a > 0$. Hence $g(\tau, s) \geq 0$ for each $\tau \in \mathbb{N}_{a+1}^{b}$. Next, consider

$$\nabla_\tau g(\tau, s) = \nabla_\tau (\tau - \rho(s))^{\overline{a}} = a(\tau - \rho(s))^{\overline{a-1}} = a\frac{\Gamma(\tau - \rho(s) + a - 1)}{\Gamma(\tau - \rho(s))}.$$

First consider the case $\tau \in \mathbb{N}_{s-1}^{b}$. Then $\tau-\rho(s)+a-1=\tau-s+a>0$ and $\tau-\rho(s)>0$. Since $\Gamma(\tau-\rho(s)+a-1)>0$ and $\Gamma(\tau-\rho(s))>0$, then $\nabla_\tau g(\tau,s)>0$, so g is an increasing function of τ. Next if $\tau=s-1$, we get

$$\begin{aligned}\nabla_\tau g(\tau,s) &= a(\tau-\rho(s))^{\overline{a-1}} = a0^{\overline{a-1}} = \frac{a\Gamma(a-1)}{\Gamma(0)}\\ &= \begin{cases} 0, & a-1 \notin \{0,-1,-2,...\}\\ (-1)^{0-(1-a)}\dfrac{0!}{(1-a)!}, & otherwise, \end{cases}\end{aligned}$$

by Remark 2.1, since $\Gamma(z) \longrightarrow \infty$ as $z \longrightarrow 0$. Using $a>0$, we have $a-1>-1$, so the only case to consider for $(-1)^{0-(1-a)}\dfrac{0!}{(1-a)!}$ is $a-1=0$. In this case,

$$(-1)^{0-(1-a)}\frac{0!}{(1-a)!} = (-1)^{0-0}\frac{0!}{0!} = 1 > 0.$$

Hence in this case we also have $\nabla_\tau g(\tau,s)>0$, so g is an increasing function of τ for $\tau \in \mathbb{N}_{s-1}^{b}$, which shows (iii). $\square$

Theorem 2.11. *Let $a,\mu \in \mathbb{R},\ \mu>1,\ N:=\lceil\mu\rceil$ and $g:\mathbb{N}_{a+1}\longrightarrow\mathbb{R}$. Then a general solution to the nabla Caputo fractional equation*

$$-\nabla_{a^*}^{\mu}x(t)=g(t), \quad t\in\mathbb{N}_{a+1} \tag{2.3}$$

is given by

$$x(t)=\sum_{k=0}^{N-1}\frac{A_k}{k!}(t-a)^{\overline{k}}-\nabla_a^{-\mu}g(t) \tag{2.4}$$

for $t \in \mathbb{N}_{a-N+1}$.

Proof. First, we will show that $x(t)$ given by (2.4), is a solution to (2.3) on $\mathbb{N}_{a-N+1}$. For $t\in\mathbb{N}_{a-N+1}$, we consider

$$\begin{aligned}-\nabla_{a^*}^{\mu}x(t) &= -\nabla_{a^*}^{\mu}\left[\sum_{k=0}^{N-1}c_k(t-a)^{\overline{k}}-\nabla_a^{-\mu}g(t)\right]\\ &= \sum_{k=0}^{N-1}-c_k\nabla_{a^*}^{\mu}\left[(t-a)^{\overline{k}}\right]+\nabla_{a^*}^{\mu}\nabla_a^{-\mu}g(t)\end{aligned}$$

where we have made use of the linearity of $-\nabla_{a^*}^{\mu}$. Now,

$$\begin{aligned}\nabla_{a^*}^{\mu}\left[(t-a)^{\overline{k}}\right] &= \nabla_a^{-(N-\mu)}\nabla^N(t-a)^{\overline{k}}, \quad k\in\{0,1,...,N-1\}\\ &= \nabla_a^{-(N-\mu)}\nabla^{N-k-1}\nabla^{k+1}(t-a)^{\overline{k}}\end{aligned}$$

$$\begin{aligned}
&= \nabla_a^{-(N-\mu)} \nabla^{N-k-1} \left[k(k-1)...2\right] \nabla^2 (t-a) \\
&= \nabla_a^{-(N-\mu)} \nabla^{N-k-1} k! \nabla(1) \\
&= \nabla_a^{-(N-\mu)} \nabla^{N-k-1} k!\ 0 \\
&= 0,
\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$. Hence

$$\sum_{k=0}^{N-1} -c_k \nabla_{a^*}^{\mu} \left[(t-a)^{\overline{k}}\right] + \nabla_{a^*}^{\mu} \nabla_a^{-\mu} g(t) = g(t),$$

which shows that $x(t)$ is a solution to (2.3). Next, we will show that any solution, $y(t)$, of (2.3) is of the form (2.4). We will show that we can express y in the form (2.4) for fixed given constants $c_k \in \mathbb{R}$.

First, define $A_k := \nabla^k y(a),\ for\ k \in \mathbb{N}^{N-1}$. Then note that $y(t)$ is a solution to the IVP

$$\begin{gathered}
-\nabla_{a^*}^{\mu} y(t) = g(t), \quad t \in \mathbb{N}_{a+1} \\
\nabla^k y(a) = A_k.
\end{gathered} \tag{2.5}$$

Using [2, Theorem 3.12], the unique solution $y(t)$ to the IVP (2.5) is

$$y(t) = \sum_{k=0}^{N-1} H_k(t,a) A_k - \nabla_a^{-\mu} g(t) = \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{k}}}{k!} A_k - \nabla_a^{-\mu} g(t).$$

Then, taking $c_k := \dfrac{A_k}{k!}$, we shown that y is of the form (2.4). □

The following fixed point theorems are fundamental and important to the proof of our main results.

Theorem 2.12 (Schauder–Tychonov Fixed Point Theorem)**.** *Let X be a Banach space. Assume that K is a closed, bounded, convex subset of X. If $T : K \longrightarrow K$ is compact, then T has a fixed point in K.*

Theorem 2.13 (Banach Fixed Point Theorem)**.** *Let T be a contraction mapping from a closed subset K of a Banach space X into K. Then there exists a unique fixed point x in K such that $T(x) = x$.*

3 Main Results

In this section, we prove the existence of positive solutions of the $(N-1,1)$-conjugate DFBVP (1.1)–(1.2) by using Theorem 2.3 and Theorem 2.4. To prove the main results, we will employ following theorems.

Theorem 3.1. *Consider the* $(N-1,1)$*-conjugate DFBVP* (1.1)–(1.2) *with* $\mu > 1,\ N := \lceil \mu \rceil$ *and* $g : \mathbb{N}_{a+1} \longrightarrow \mathbb{R}$*. Then* $x : \mathbb{N}_{a-N+1} \longrightarrow \mathbb{R}$ *is a solution of the* $(N-1,1)$*-conjugate DFBVP* (1.1)–(1.2) *if and only if* $x(t)$ *satisfies the integral equation*

$$x(t) = \int_a^b G_\mu(t,s) g(s) \nabla s, \tag{3.1}$$

for $t \in \mathbb{N}_{a-N+1}^b$*, where* $G_\mu(t,s)$ *is the Green's function for the homogeneous equation*

$$-\nabla_a^\mu x(t) = 0, \quad t \in \mathbb{N}_{a+1}$$

with the boundary conditions (1.2) *and is given by*

$$G_\mu(t,s) := \begin{cases} u(t,s), & t \le \rho(s) \\ v(t,s), & \rho(s) \le t \end{cases} \tag{3.2}$$

where

$$u(t,s) = \frac{\displaystyle\sum_{i=0}^{N-1} \frac{(t-a)^{\overline{i}}}{i!(\alpha - i\beta)} \Big(\gamma H_{\mu-1}\big(b,\rho(s)\big) + \delta H_{\mu-2}\big(b,\rho(s)\big) \Big)}{\displaystyle\gamma \sum_{k=0}^{N-1} \frac{(b-a)^{\overline{k}}}{k!(\alpha - k\beta)} + \delta \sum_{k=0}^{N-2} \frac{(b-a)^{\overline{k}}}{k!(\alpha - (k+1)\beta)}}$$

and

$$v(t,s) = u(t,s) - \frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)} = u(t,s) + H_{\mu-1}(t,\rho(s)).$$

Note that $G_\mu : \mathbb{N}_{a-N+1}^b \longrightarrow \mathbb{R}$.

Proof. Assume $x(t)$ is a solution of the $(N-1,1)$-conjugate DFBVP (1.1)–(1.2). By Theorem 2.11, we have

$$x(t) = \sum_{k=0}^{N-1} c_k (t-a)^{\overline{k}} - \nabla_a^{-\mu} g(t). \tag{3.3}$$

First, for each $i \in \mathbb{N}_1^{N-1}$, we have

$$\begin{aligned} \alpha x(a-i) + \beta \nabla x(a-(i-1)) =& \alpha \sum_{k=0}^{N-1} c_k (a-i-a)^{\overline{k}} - \alpha \nabla_a^{-\mu} g(a-i) \\ & - \beta \nabla \left(\sum_{k=0}^{N-1} c_k (t-a)^{\overline{k}} - \nabla_a^{-\mu} g(t) \right)_{t=a-(i-1)} \\ =& \alpha \nabla_a^{-\mu} g(a-i) + \alpha \sum_{i=0}^{N-1} c_k (-i)^{\overline{k}} \end{aligned}$$

$$-\beta\sum_{k=1}^{N-1}c_k k(1-i)^{\overline{k-1}}+\beta\nabla\nabla_a^{-\mu}g(a-i+1).$$

For $t\in\mathbb{N}_{a-N+2}^{b-1}$ we know that $-\nabla_{a^*}^{\mu}x(t)=g(t)$. So we have $\alpha\nabla_a^{-\mu}g(a-i)-\beta\nabla\nabla_a^{-\mu}g(a-i+1)=\alpha x(a-i)-\beta\nabla x(a-i+1)$ for $i=1,2,...,N-1$. Using the first boundary value condition we get $\alpha\nabla_a^{-\mu}g(a-i)-\beta\nabla\nabla_a^{-\mu}g(a-i+1)=0$. So

$$\begin{aligned}\alpha x(a-i)+\beta\nabla x(a-(i-1))=&\alpha\sum_{k=0}^{N-1}c_k(-i)^{\overline{k}}-\beta\sum_{k=1}^{N-1}c_k k(1-i)^{\overline{k-1}}\\=&\alpha\sum_{k=0}^{N-1}c_k\frac{\Gamma(k-i)}{\Gamma(-i)}-\beta\sum_{k=1}^{N-1}c_k k\frac{\Gamma(k-i)}{\Gamma(1-i)}\\=&\alpha\sum_{k=0}^{N-1}c_k\frac{\Gamma(k-i)}{\Gamma(-i)}-\beta\sum_{k=0}^{N-2}(k+1)c_{k+1}\frac{\Gamma(k+1-i)}{\Gamma(-i)}.\end{aligned}$$

By Remark 2.1, we have

$$\frac{\Gamma(k-i)}{\Gamma(-i)}=\begin{cases}(-1)^{i-(i-k)}\dfrac{i!}{(i-k)!}, & k\le i\\ 0, & k>i\end{cases}$$

and so we get

$$\begin{aligned}\alpha x(a-i)+\beta\nabla x(a-(i-1))=&\alpha\sum_{k=0}^{N-1}c_k(-1)^k\frac{i!}{(i-k)!}\\&-\beta\sum_{k=0}^{N-2}(k+1)c_{k+1}(-1)^{k+1}\frac{i!}{(i-k-1)!}.\end{aligned}$$

Then from the boundary value conditions (1.2), we have for each $i\in\mathbb{N}_1^{N-1}$,

$$\alpha\sum_{k=0}^{i}c_k(-1)^k\frac{i!}{(i-k)!}-\beta\sum_{k=0}^{i-1}(k+1)c_{k+1}(-1)^{k+1}\frac{i!}{(i-k-1)!}=0. \tag{3.4}$$

So we claim that from (3.4), we obtain $c_n=(n+1)\dfrac{\alpha-(n+1)\beta}{\alpha-n\beta}c_{n+1}$ for $n\in\{0,1,...,N-2\}$. We will show the claim holds by strong introduction on n for $n\in\mathbb{N}_0^{N-1}$.

For the case base, we let $i=1$ in (3.4) to get

$$\alpha\sum_{k=0}^{1}c_k(-1)^k\frac{1!}{(1-k)!}-\beta\sum_{k=0}^{0}(k+1)c_{k+1}(-1)^{k+1}\frac{0!}{(1-k-1)!}=0$$

so $c_0 = \dfrac{\alpha-\beta}{\alpha}c_1$. Hence the claim holds for $n=0$.

Next, assume for the inductive hypothesis that the claim holds for $0,1,...,n-1$, for $n \in \mathbb{N}_0^{N-2}$. We will show that it follows that $c_n = (n+1)\dfrac{\alpha-(n+1)\beta}{\alpha-n\beta}c_{n+1}$ for $n \in \mathbb{N}_0^{N-1}$. First we will note that from the strong inductive hypothesis, we have that

$$\begin{aligned} c_0 &= \frac{\alpha-\beta}{\alpha}c_1, \\ c_1 &= 2\frac{\alpha-2\beta}{\alpha-\beta}c_2, \\ &\vdots \\ c_{n-1} &= n\frac{\alpha-n\beta}{\alpha-(n-1)\beta}c_n. \end{aligned}$$

From this, we get that $c_0 = n!\dfrac{\alpha-n\beta}{\alpha}c_n$, $c_1 = 2.3.4\cdots n\dfrac{\alpha-n\beta}{\alpha-\beta}c_n$ and in general $c_k = \dfrac{n!(\alpha-n\beta)}{k!(\alpha-k\beta)}c_n$ for all $k \in \{0,1,...,n\}$.

Now we let $i = n+1$ where $n+1 \in \mathbb{N}_1^{N-1}$ in (3.4) to get

$$\alpha\sum_{k=0}^{n+1} c_k(-1)^k\frac{(n+1)!}{(n+1-k)!} - \beta\sum_{k=0}^{n}(k+1)c_{k+1}(-1)^{k+1}\frac{(n+1)!}{(n-k)!} = 0.$$

This means that

$$\begin{aligned} &\alpha\sum_{k=0}^{n} c_k(-1)^k\frac{(n+1)!}{(n+1-k)!} + \alpha c_{n+1}(-1)^{n+1}(n+1)! \\ &\qquad - \beta\sum_{k=0}^{n-1}(k+1)c_{k+1}(-1)^{k+1}\frac{(n+1)!}{(n-k)!} - \beta(n+1)c_{n+1}(-1)^{n+1}(n+1)! = 0. \end{aligned}$$

Replacing c_k with $\dfrac{n!(\alpha-n\beta)}{k!(\alpha-k\beta)}c_n$, this gives

$$\begin{aligned} &\alpha\sum_{k=0}^{n}\frac{n!(\alpha-n\beta)}{k!(\alpha-k\beta)}c_n(-1)^k\frac{(n+1)!}{(n+1-k)!} + \alpha c_{n+1}(-1)^{n+1}(n+1)! \\ &\qquad - \beta\sum_{k=0}^{n-1}\frac{(n+1)!(\alpha-n\beta)}{(k+1)!(\alpha-(k+1)\beta)}c_n(k+1)(-1)^{k+1}\frac{(n+1)!}{(n-k)!} \\ &\qquad\qquad - \beta(n+1)c_{n+1}(-1)^{n+1}(n+1)! = 0. \end{aligned}$$

Since $\binom{n+1}{k} := \dfrac{(n+1)!}{k!(n+1-k)!}$, we get

$$\begin{aligned}\alpha\sum_{k=0}^{n}\frac{n!(\alpha-n\beta)}{\alpha-k\beta}c_n(-1)^k\binom{n+1}{k}+\alpha c_{n+1}(-1)^{n+1}(n+1)!\\ -\beta\sum_{k=0}^{n-1}\frac{(n+1)!(\alpha-n\beta)}{\alpha-(k+1)\beta}c_n(k+1)(-1)^{k+1}\binom{n+1}{k+1}\\ -\beta(n+1)c_{n+1}(-1)^{n+1}(n+1)!=0.\end{aligned}$$

We have now established that $c_n=(n+1)\dfrac{\alpha-(n+1)\beta}{\alpha-n\beta}c_{n+1}$ for $n=0,1,\cdots,N-2$. Then it follows that

$$c_k=\frac{(N-1)!(\alpha-(N-1)\beta)}{k!(\alpha-k\beta)}c_{N-1}\quad for\ k=0,\cdots,N-1. \tag{3.5}$$

Next, from the boundary condition (1.2) and (3.3) we get

$$\begin{aligned}\gamma x(b)+\delta\nabla x(b)=&\gamma\sum_{k=0}^{N-1}\frac{(N-1)!(\alpha-(N-1)\beta)}{k!(\alpha-k\beta)}c_{N-1}(b-a)^{\overline{k}}-\gamma\nabla_a^{-\mu}g(b)\\ &+\delta\sum_{k=1}^{N-1}\frac{(N-1)!(\alpha-(N-1)\beta)}{k!(\alpha-k\beta)}c_{N-1}k(b-a)^{\overline{k-1}}-\delta\nabla\nabla_a^{-\mu}g(b)\\ =&0.\end{aligned}$$

Now from (3.5) we have

$$c_i=\frac{\gamma\nabla^{-\mu}g(b)+\delta\nabla^{1-\mu}g(b)}{i!(\alpha-i\beta)\left(\gamma\displaystyle\sum_{k=0}^{N-1}\frac{(b-a)^{\overline{k}}}{k!(\alpha-k\beta)}+\delta\sum_{k=0}^{N-2}\frac{(k+1)(b-a)^{\overline{k}}}{(k+1)!(\alpha-(k+1)\beta)}\right)}, \tag{3.6}$$

for $i=0,1,\cdots,N-1$. Hence from (3.3), we have

$$\begin{aligned}x(t)=&\sum_{i=0}^{N-1}\frac{\left(\gamma\nabla^{-\mu}g(b)+\delta\nabla^{1-\mu}g(b)\right)(t-a)^{\overline{i}}}{i!(\alpha-i\beta)\left(\gamma\displaystyle\sum_{k=0}^{N-1}\frac{(b-a)^{\overline{k}}}{k!(\alpha-k\beta)}+\delta\sum_{k=0}^{N-2}\frac{(k+1)(b-a)^{\overline{k}}}{(k+1)!(\alpha-(k+1)\beta)}\right)}\\ &-\nabla^{-\mu}g(t)\\ =&-\int_a^t H_{\mu-1}(t,\rho(s))g(s)\nabla s\end{aligned}$$

$$+\sum_{i=0}^{N-1}\frac{\left(\gamma\nabla^{-\mu}g(b)+\delta\nabla^{1-\mu}g(b)\right)(t-a)^{\overline{i}}}{i!(\alpha-i\beta)\left(\gamma\sum_{k=0}^{N-1}\frac{(b-a)^{\overline{k}}}{k!(\alpha-k\beta)}+\delta\sum_{k=0}^{N-2}\frac{(k+1)(b-a)^{\overline{k}}}{(k+1)!(\alpha-(k+1)\beta)}\right)}$$

$$=-\int_a^t H_{\mu-1}(t,\rho(s))g(s)\nabla s$$

$$+\frac{\left(\gamma\nabla^{-\mu}g(b)+\delta\nabla^{1-\mu}g(b)\right)\sum_{i=0}^{N-1}\frac{(t-a)^{\overline{i}}}{i!(\alpha-i\beta)}}{\gamma\sum_{k=0}^{N-1}\frac{(b-a)^{\overline{k}}}{k!(\alpha-k\beta)}+\delta\sum_{k=0}^{N-2}\frac{(k+1)(b-a)^{\overline{k}}}{(k+1)!(\alpha-(k+1)\beta)}}$$

$$=-\int_a^t H_{\mu-1}(t,\rho(s))g(s)\nabla s$$

$$+\frac{\sum_{i=0}^{N-1}\frac{(t-a)^{\overline{i}}}{i!(\alpha-i\beta)}\int_a^b(\gamma H_{\mu-1}(b,\rho(s))+\delta H_{\mu-2}(b,\rho(s)))g(s)\nabla s}{\gamma\sum_{k=0}^{N-1}\frac{(b-a)^{\overline{k}}}{k!(\alpha-k\beta)}+\delta\sum_{k=0}^{N-2}\frac{(k+1)(b-a)^{\overline{k}}}{(k+1)!(\alpha-(k+1)\beta)}}.$$

Thus for $s\in\mathbb{N}_{a+1}^{t}$ and $t\in\mathbb{N}_{s}^{b}$, we have

$$G_\mu(t,s)=v(t,s):=u(t,s)-H_{\mu-1}(t,\rho(s))$$

$$=\frac{\sum_{i=0}^{N-1}\frac{(t-a)^{\overline{i}}}{i!(\alpha-i\beta)}\left(\gamma H_{\mu-1}(b,\rho(s))+\delta H_{\mu-2}(b,\rho(s))\right)}{\gamma\sum_{k=0}^{N-1}\frac{(b-a)^{\overline{k}}}{k!(\alpha-k\beta)}+\delta\sum_{k=0}^{N-2}\frac{(b-a)^{\overline{k}}}{k!(\alpha-(k+1)\beta)}}-\frac{(t-\rho(s))^{\mu-1}}{\Gamma(\mu)}.$$

Also, for $s\in\mathbb{N}_{t+1}^{b}$ and $t\in\mathbb{N}_{a}^{s-1}$ or $s\in\mathbb{N}_{a+1}^{b}$ and $t\in\mathbb{N}_{a-N+1}^{a}$, we get

$$G_\mu(t,s)=u(t,s):=\frac{\sum_{i=0}^{N-1}\frac{(t-a)^{\overline{i}}}{i!(\alpha-i\beta)}\left(\gamma H_{\mu-1}(b,\rho(s))+\delta H_{\mu-2}(b,\rho(s))\right)}{\gamma\sum_{k=0}^{N-1}\frac{(b-a)^{\overline{k}}}{k!(\alpha-k\beta)}+\delta\sum_{k=0}^{N-2}\frac{(b-a)^{\overline{k}}}{k!(\alpha-(k+1)\beta)}}.$$

Noting that $u(t,s)=v(t,s)$ when $t=\rho(s)$, we can also write the Green's function as $v(t,s)$ for $\rho(s)\le t$ and $u(t,s)$ for $t\le\rho(s)$ for $(t,s)\in\mathbb{N}_{a-N+1}\times\mathbb{N}_{a+1}^{b}$.

Thus, we have $x(t)=\int_a^b G_\mu(t,s)g(s)\nabla s$, where $G_\mu(t,s)$ is given by (3.2) are determined by the boundary value conditions (1.1) and (1.2) is shown by the above steps, it follows that $\int_a^b G_\mu(t,s)g(s)\nabla s$ is a solution to the $(N-1,1)$-conjugate DFBVP (1.1)–(1.2). $\square$

The next theorem proves that the Green's function for the $(N-1,1)$- conjugate boundary value problem is bounded.

Theorem 3.2. *Let G_μ be defined as* (3.2). *Then*

$$|G_\mu(t,s)| \leq DH_{\mu-1}(b,a), \tag{3.7}$$

where $D := \dfrac{(\gamma+\delta)\sum_{i=0}^{N-1}\dfrac{(b-a)^{\overline{i}}}{i!(\alpha-i\beta)}}{d}$ *with*
$d := \gamma\sum_{k=0}^{N-1}\dfrac{(b-a)^{\overline{k}}}{k!(\alpha-k\beta)} + \delta\sum_{k=0}^{N-2}\dfrac{(b-a)^{\overline{k}}}{k!(\alpha-(k+1)\beta)}$ *for* $(t,s)\in\mathbb{N}^b_{a-N+1}\times\mathbb{N}^b_{a+1}$.

Proof. First, consider $(t,s)\in\mathbb{N}^b_s\times\mathbb{N}^t_{a+1}$. Then note that from (3.2), $|G_\mu(t,s)|$ is equal to

$$\left|\frac{\gamma(b-\rho(s))^{\overline{\mu-1}}+\delta(\mu-1)(b-\rho(s))^{\overline{\mu-2}}}{d\Gamma(\mu)}\sum_{i=0}^{N-1}\frac{(t-a)^{\overline{i}}}{i!(\alpha-i\beta)}-\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}\right|.$$

Now by Proposition 2.10, $(t-\rho(s))^{\overline{\mu-1}}\geq 0,\ (t-\rho(s))^{\overline{\mu-2}}\geq 0$ for $t\in\mathbb{N}^b_s$ and also $(b-\rho(s))^{\mu-1}\geq 0,\ (b-\rho(s))^{\mu-2}\geq 0$. Note that for each $i\in\mathbb{N}_0^{N-1}$, $(t-a)^{\overline{i}} = \dfrac{\Gamma(t-a+i)}{\Gamma(t-a)}\geq 0$ since $(t,s)\in\mathbb{N}^b_s\times\mathbb{N}^t_{a+1}$ implies $t\geq a+1$.

Since $d>0$ and Proposition 2.9, we have

$$\begin{aligned}|G_\mu(t,s)| \leq \max\Bigg\{&\max_{t\in\mathbb{N}^b_a}\left[\frac{\gamma(b-\rho(s))^{\overline{\mu-1}}+\delta(\mu-1)(b-\rho(s))^{\overline{\mu-2}}}{d\Gamma(\mu)}\sum_{i=0}^{N-1}\frac{(t-a)^{\overline{i}}}{i!(\alpha-i\beta)}\right],\\ &\max_{t\in\mathbb{N}^b_a}\left[\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}\right]\Bigg\}.\end{aligned}$$

Since $\dfrac{(b-\rho(s))^{\overline{\mu-1}}}{\mu-1} > (b-\rho(s))^{\overline{\mu-2}}$, we have

$$\begin{aligned}&|G_\mu(t,s)|\\ &\leq \max\left\{\max_{t\in\mathbb{N}^b_a}\left[\frac{(\gamma+\delta)(b-\rho(s))^{\mu-1}}{d\Gamma(\mu)}\sum_{i=0}^{N-1}\frac{(t-a)^{\overline{i}}}{i!(\alpha-i\beta)}\right],\max_{t\in\mathbb{N}^b_a}\left[\frac{(t-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu)}\right]\right\}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(b-\rho(s))^{\mu-1}}{\Gamma(\mu)} \max \left\{ \frac{(\gamma+\delta) \displaystyle\sum_{i=0}^{N-1} \frac{(b-a)^{\overline{i}}}{i!(\alpha - i\beta)}}{d}, 1 \right\} \\
&= H_{\mu-1}(b,\rho(s)) \max\{D, 1\}.
\end{aligned}
$$

Thus, we get

$$|G_\mu(t,s)| \leq DH_{\mu-1}(b,\rho(s)) \leq DH_{\mu-1}(b,a),$$

since $D > 1$, using $\dfrac{\alpha}{\beta} > N-1$.

Next, consider $(t,s) \in \mathbb{N}_{a-N+1}^{s-1} \times \mathbb{N}_{\max\{a+1,t+1\}}^{b}$. Then from (3.2), we get

$$
\begin{aligned}
|G_\mu(t,s)| &= \left| \frac{\gamma(b-\rho(s))^{\overline{\mu-1}} + \delta(\mu-1)(b-\rho(s))^{\overline{\mu-2}}}{d\Gamma(\mu)} \sum_{i=0}^{N-1} \frac{(t-a)^{\overline{i}}}{i!(\alpha-i\beta)} \right| \\
&\leq \frac{1}{\Gamma(\mu)} \left| \frac{\gamma(b-a)^{\overline{\mu-1}} + \delta(b-a)^{\overline{\mu-1}}}{d} \sum_{i=0}^{N-1} \frac{(t-a)^{\overline{i}}}{i!(\alpha-i\beta)} \right| \\
&\leq D\frac{(b-a)^{\overline{\mu-1}}}{\Gamma(\mu)} \\
&= DH_{\mu-1}(b,a).
\end{aligned}
$$

Hence we have $|G_\mu(t,s)| \leq DH_{\mu-1}(b,a)$ for all $(t,s) \in \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b}$. □

Corollary 3.3. *The Green's function,* $G_\mu(t,s) : \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b} \longrightarrow \mathbb{R}$ *for the* $(N-1,1)$*-conjugate DFBVP* (1.1)–(1.2) *satisfies the inequality*

$$\int_a^b |G_\mu(t,s)| \nabla s \leq DH_{\mu-1}(b,a)(b-a).$$

Theorem 3.4. *Assume that* g *is continuous on* $[a-N+1,b] \times \mathbb{R}$. *If* $M > 0$ *and* $DH_{\mu-1}(b,a)(b-a) \leq \dfrac{M}{Q}$ *where* $Q > 0$ *satisfies*

$$Q \geq \max\{|g(t,x)| : t \in [a-N+1,b],\ |x| \leq M\},$$

then the $(N-1,1)$*-conjugate DFBVP* (1.1)–(1.2) *has a solution.*

Proof. Define C to be the Banach space of all continuous functions on $[a-N+1,b]$ equipped with the norm $\|.\|$ defined by

$$\|x\| := \max_{t\in[a-N+1,b]} |x(t)| \quad \text{for all } x \in C.$$

Let

$$K := \{x \in C : \|x\| \leq M\}.$$

It can be shown that K is a closed, bounded and convex subset of C. Define $T : K \longrightarrow C$ by

$$Tx(t) := \int_a^b G_\mu(t,s)g(s, x(s-N+1))\nabla s$$

for $t \in [a-N+1, b]$. It can be shown that $T : K \longrightarrow C$ is continuous. Let $x \in K$ and consider

$$\begin{aligned}
|Tx(t)| &= \left|\int_a^b G_\mu(t,s)g(s, x(s-N+1))\nabla s\right| \\
&\leq \int_a^b |G_\mu(t,s)g(s, x(s-N+1))| \nabla s \\
&\leq Q \int_a^b |G_\mu(t,s)| \nabla s \\
&\leq QDH_{\mu-1}(b,a)(b-a) \\
&\leq M,
\end{aligned}$$

for all $t \in [a-N+1, b]$. But this implies that $\|Tx\| \leq M$. Hence $T : K \longrightarrow K$. Using the Arzela–Ascoli theorem (see [16, Theorem 4.44]), it can be shown that $T : K \longrightarrow K$ is a compact operator. Hence T has a fixed point x in K by the Schauder-Tychonov theorem. This implies that x is a solution of the $(N-1, 1)$ conjugate boundary value problem (1.1) and (1.2). □

Corollary 3.5. *If g is continuous and bounded on $[a-N+1, b] \times \mathbb{R}$, then the $(N-1,1)$-conjugate DFBVP* (1.1)–(1.2) *has a solution.*

Theorem 3.6. *Assume that $g : [a-N+1, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to the second constant with $l > 0$; that is*

$$|g(t,x) - g(t,y)| \leq l\,|x-y|$$

for all $(t,x), (t,y) \in [a-N+1, b] \times \mathbb{R}$. If

$$DH_{\mu-1}(b,a)(b-a) < \frac{1}{l} \tag{3.8}$$

then the DFBVP (1.1)–(1.2) *has a unique solution.*

Proof. Let $x, y \in K$. Then

$$|Tx(t) - Ty(t)| \leq \int_a^b |G_\mu(t,s)|\,|g(s, x(t)) - g(s, y(t))|\,\nabla s$$

$$\begin{aligned}&\leq l \int_a^b |G_\mu(t,s)| \, |x(s-N+1) - y(s-N+1)| \, \nabla s \\ &\leq l \int_a^b |G_\mu(t,s)| \, \|x-y\| \, \nabla s \\ &\leq lDH_{\mu-1}(b,a)(b-a)| \, \|x-y\| \, ,\end{aligned}$$

which on taking the norm for $[a-N+1,b]$ yields

$$\|Tx - Ty\| < lDH_{\mu-1}(b,a)(b-a) \, \|x-y\| \, .$$

In view of the assumption (3.8) that the operator T is a contraction on K. Thus, by the contraction mapping principle (Banach fixed point theorem), we conclude that $(N-1,1)$-conjugate DFBVP (1.1)–(1.2) has a unique solution on $[a+1,b]$. $\square$

In this part, we will give Lyapunov inequalities for an $(N-1,1)$ conjugate boundary value problem. The following theorem gives a necessary condition for a boundary value problem with $(N-1,1)$ conjugate boundary conditions to have a nontrivial solution. This means that from contrapositive statement of the theorem, we obtain a sufficient condition for the DFBVP to have only the trivial solution.

Theorem 3.7. *Let $\mu > 1,\ N := \lceil \mu \rceil$, $q : \mathbb{N}_a^b \longrightarrow \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $|f(x)| \leq x$. Consider the conjugate boundary value problem*

$$-\nabla_{a^*}^{\mu} x(t) + q(t) f(x(t-N+1)) = 0, \ \ t \in \mathbb{N}_{a-N+2}^{b-1}, \tag{3.9}$$

$$\begin{aligned}\alpha x(a-i) - \beta \nabla x(a-(i-1)) = 0, \ \ i = 1, \cdots, N-1, \\ \gamma x(b) + \delta \nabla x(b) = 0.\end{aligned} \tag{3.10}$$

If DFBVP (3.9) *and* (3.10) *has a nontrivial solution $x : \mathbb{N}_{a-N+1}^b \longrightarrow \mathbb{R}$, then*

$$\int_a^b |q(t)| \, \nabla t \geq \frac{1}{DH_{\mu-1}(b,a)}.$$

Proof. From Theorem 3.1, we have that a solution $x : \mathbb{N}_{a-N+1}^b \longrightarrow \mathbb{R}$ for the DFBVP (3.9)–(3.10) satisfies the integral equation

$$x(t) = \int_a^b G_\mu(t,s) q(s) f(x(s-N+1)) \nabla s, \quad t \in \mathbb{N}_{a-N+1}^b. \tag{3.11}$$

Assume that x is nontrivial and let $L := \max_{t \in \mathbb{N}_{a-N+1}^b} |x(t)|$. Then,

$$|x(t)| \leq \int_a^b |G_\mu(t,s)| \, |q(s)| \, |x(s-N+1)| \, \nabla s, \ \ t \in \mathbb{N}_{a-N+1}^b.$$

Then in particular of $t = t_0$ such that $|x(t_0)| = L$, we have

$$|x(t_0)| \leq \int_a^b |G_\mu(t_0, s)| \, |q(s)| \, |x(s-N+1)| \, \nabla s.$$

Since $|x(s-N+1)| \leq L \; for \; s-N+1 \in \mathbb{N}_{a-N+1}^b$, we have

$$|x(t_0)| \leq \int_a^b |G_\mu(t_0, s)| \, |q(s)| \, L \nabla s.$$

Hence we have

$$L \leq \int_a^b |G_\mu(t_0, s)| \, |q(s)| \, L \nabla s,$$

i.e., by Theorem 3.2

$$1 \leq \int_a^b |G_\mu(t_0, s)||q(s)| \nabla s \leq \int_a^b DH_{\mu-1}(b, a) \, |q(s)| \, \nabla s.$$

Thus we have

$$\int_a^b |q(s)| \, \nabla s \geq \frac{1}{DH_{\mu-1}(b, a)}.$$

□

The following examples illustrates an application of Theorem 3.7 and Theorem 3.4.

Example 3.8. Let $\mu = \dfrac{27}{10}, \; N = 3, \; q(t) \equiv \dfrac{1}{t}$ for $q : \mathbb{N}_3^5 \longrightarrow \mathbb{R}$ and $f(x) = sin^2(x)$ for $f : \mathbb{R} \longrightarrow \mathbb{R}$. We consider the following boundary value problem

$$\nabla_{2^*}^{\frac{27}{10}} x(t) + \frac{1}{t} sin^2(x(t-2)) = 0, \; t \in \mathbb{N}_3^5, \tag{3.12}$$

$$\begin{gathered} \frac{2}{3} x(2-i) - \frac{1}{6} \nabla x(2-(i-1)) = 0, \;\; i = 1, 2 \\ 8x(6) + 7\nabla x(6) = 0. \end{gathered} \tag{3.13}$$

We must show that

$$\int_2^6 |q(t)| \, \nabla t \geq \frac{1}{DH_{\mu-1}(6, 2)}.$$

We can easily calculate

$$\int_2^6 \left| \frac{1}{t} \right| \nabla t = \frac{57}{60} \cong 0.95,$$

and $D = \dfrac{818}{534} \cong 1.53$, so we get

$$\frac{1}{DH_{\mu-1}(6,2)} = \frac{1}{102} \cong 0.01.$$

Since really $0.95 \geq 0.01$, we have the condition $\displaystyle\int_a^b |q(t)|\, \nabla t \geq \frac{1}{DH_{\mu-1}(b,a)}$.

Then by Theorem 3.7, we see that the boundary value problem (3.12)-(3.13) has a nontrivial solution $x : \mathbb{N}_0^6 \longrightarrow \mathbb{R}$.

Example 3.9. Let $\mu = \dfrac{30}{9}$, $N = 4$ and $g(t,x) = 2 + \dfrac{1}{4.10^4} \arctan x$. We consider the following boundary value problem

$$\nabla_{1^*}^{\frac{30}{9}} x(t) + 2 + \frac{1}{4.10^4} \arctan x(t-3) = 0,\ t \in \mathbb{N}_2^9, \tag{3.14}$$

$$\begin{aligned} \frac{1}{4}x(1-i) - \frac{1}{20}\nabla x(1-(i-1)) = 0, \quad i = 1,2,3 \\ 9x(10) + 10\nabla x(10) = 0. \end{aligned} \tag{3.15}$$

We must show that there exist a constant l such that

$$|g(t,x) - g(t,y)| \leq l\,|x-y|$$

satisfying

$$DH_{\mu-1}(b,a)(b-a) < \frac{1}{l}.$$

We can see that the function $g(t,x) = 2 + \dfrac{1}{4.10^4} \arctan x$ is satisfy a uniform Lipschitz condition with $l = \dfrac{1}{4.10^4}$.

Also we can easily calculate $d = 23141, D \cong 8,7$ and $H_{\frac{21}{9}}(10,1) \cong 45$, so we get the condition

$$8,7.45.9 < 4.10^4.$$

Then by Theorem 3.4, we see that the boundary value problem (3.14)-(3.15) has a unique solution $x : \mathbb{N}_{-2}^{10} \longrightarrow \mathbb{R}$.

Acknowledgements

e are very grateful to the reviewers for their useful comments that led to improvement of this manuscript.

References

[1] Nieto JJ, Pimentel J, Positive solutions of a fractional thermostat model, *Bound. Value Probl.* 2013 (5) (2013), 11 pp.

[2] Ding XL, Jiang YL, Waveform relaxation methods for fractional functional differential equations, *Fract. Calc. Appl. Anal.* 16 2013, no.3, 573-594.

[3] Miller KS, Ross B, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York (1993)

[4] Nisar KS, Baleanu D, Qurashi MMA: Fractional calculus and application of generalized Struve function. *Springer Plus* 5(1) (2016), 1-13.

[5] Oldham KB, Spanier J, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Dover, New York (2002)

[6] Virginia K, The special functions of fractional calculus as generalized fractional calculus operators of some basic functions, *Comput. Math. Appl.* 59(3) (2010), 1128-1141.

[7] Guy J, On the derivative chain-rules in fractional calculus via fractional difference and their application to systems modelling, *Open Phys.* 11(6) (2016), 617-633.

[8] Pilipović, S, Fractional calculus with applications in mechanics: vibrations and diffusion processes. Drug Dev. Ind. Pharm. (2014)

[9] Atici FM, Eloe PW, A transform method in discrete fractional calculus, *Int. J. Difference Equ.* 2 (2007), 165-176.

[10] Goodrich CS, Solutions to a discrete right-focal fractional boundary value problem, *Int. J. Difference Equ.* 5(2) (2010), 195-216.

[11] Goodrich CS, Peterson A, Discrete Fractional Calculus, Springer (2015)

[12] Oldham K, Spanier J, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Dover Publications, Mineola, New York (2002)

[13] Ibrahim RW, Jalab HA, Existence of a class of fractional difference equations with two point boundary value problem, *Adv. Differ. Equ.* 269 (2015), 12 pp.

[14] Lv W, Existence of solutions for discrete fractional boundary value problems with a p-Laplacian operator, *Adv. Differ. Equ.* 2012(1) (2012), 163, 10 pp.

[15] Areeba I, Peterson A, Green's Functions and Lyapunov Inequalities For Nabla Caputo Boundary Value Problems, PhD. (2018)

[16] G. Folland, Real Analysis: Modern Techniques and Their Apllications, John Wiley and Sons, Inc., New York, Second Edition (1999)

[17] Singh V, Pandey DN, Controllability of multi-term time-fractional differential systems with state-dependent delay, Elsevier (2017)

[18] Baoguo J, Erbe L, Peterson A, Two monotonicity results for nabla and delta fractional differences, Arch. Math. (Basel), 104, 589-597 (2015)

[19] Baoguo J, Erbe L, Peterson A, Convexity for nabla and delta fractional differences, J. Differ. Equai Appl., 21, 360-373 (2015)