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Caputo Type Modification of Erdélyi–Kober Fractional Differential Inclusions with Retarded and Advanced Arguments

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Abstract

In this article, we establish the existence of solutions for a class of problems for Caputo type modification of the Erdélyi–Kober fractional differential inclusions with retarded and advanced arguments. We illustrate our results by an example in the last section.

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1 Introduction

Fractional differential equations and inclusions are valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, numerous applications have been addressed in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. For examples and details on the progress of fractional calculus, we refer the reader to the monographs [1–3, 17, 20, 25, 27, 30], and the references therein.

In [4, 6, 10, 12, 15], the authors studied the existence and uniqueness of solutions for boundary value problems of Hadamard-type fractional functional differential equations and inclusions involving both retarded and advanced arguments. In [24] the authors provide some properties of Caputo-type modification of the Erdélyi–Kober fractional

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derivative. More details on the Erdélyi–Kober fractional integral and fractional derivative are given in [7,20–22,28].

In the present work, we investigate the existence of solutions for a class of problem for nonlinear Caputo type modification of the Erdélyi–Kober fractional differential inclusions involving both retarded and advanced arguments given by

$$ {}^{\rho}_{c}D^{\alpha}_{a+}y(t) \in F(t, y^t), \text{ for } t \in I := [a, T], \ 1 < \alpha \leq 2, \tag{1.1}$$

$$ y(t) = \phi(t), \ t \in [a - r, a], \ r > 0 \tag{1.2}$$

$$ y(t) = \psi(t), \ t \in [T, T + \beta], \ \beta > 0, \tag{1.3}$$

where ${}^{\rho}_{c}D^{\alpha}_{a+}$ is the Caputo type modification of the Erdélyi–Kober fractional derivative, $F : I \times C([-r, \beta], \mathbb{R}) \to \mathcal{P}(\mathbb{R})$ is a given function, $\phi \in C([a - r, a], \mathbb{R})$ with $\phi(a) = 0$ and $\psi \in C([T, T + \beta], \mathbb{R})$ with $\psi(T) = 0$. We denote by y^t the element of $C([-r, \beta])$ defined by

$$ y^t(s) = y(t + s) : s \in [-r, \beta]. $$

This paper initiates the study of differential inclusions involving the Erdélyi–Kober fractional derivative, which include the Hadamard fractional derivative as special case.

2 Preliminaries

In this part, we present notations and definitions we will use throughout this paper. By $C([-r, \beta], \mathbb{R})$ we denote the Banach space of all continuous functions from $[-r, \beta]$ into $\mathbb{R}$ equipped with the norm

$$ \|y\|_{[-r,\beta]} = \sup\{|y(t)| : -r \leq t \leq \beta\}, $$

and $C([a, T], \mathbb{R})$ is the Banach space endowed with the norm

$$ \|y\|_{[a,T]} = \sup\{|y(t)| : a \leq t \leq T\}. $$

By $L^1(I, \mathbb{R})$ we denote the Banach space of functions $y : I \longrightarrow \mathbb{R}$ which are Lebesgue integrable with norm

$$ \|y\|_{L^1} = \int_a^T |y(t)| dt. $$

Also, let $E_1 = C([a - r, a], \mathbb{R})$, $E_2 = C([T, T + \beta], \mathbb{R})$, and

$$ AC^1(I) := \{w : I \longrightarrow \mathbb{R} : w' \in AC(I)\}, $$

where

$$ w'(t) = t\frac{d}{dt}w(t), \ \ t \in I, $$

$AC(I,\mathbb{R})$ is the space of absolutely continuous functions on I,

$$\mathcal{C} = \{y : [a-r, T+\beta] \longmapsto \mathbb{R} : y \mid_{[a-r,a]} \in C([a-r,a]), y \mid_{[a,T]} \in AC^1([a,T]),$$

and

$$y \mid_{[T,T+\beta]} \in C([T,T+\beta])\}$$

are the spaces endowed, respectively, with the norms

$$\|y\|_{[a-r,a]} = \sup\{|y(t)| : a-r \le t \le a\},$$

$$\|y\|_{[T,T+\beta]} = \sup\{|y(t)| : T \le t \le T+\beta\},$$

$$\|y\|_{\mathcal{C}} = \sup\{|y(t)| : a-r \le t \le T+\beta\}.$$

Consider the space $X_c^p(a,b)$, $(c \in \mathbb{R},\ 1 \le p \le \infty)$ of those complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \ \ (1 \le p < \infty, c \in \mathbb{R}).$$

In particular, where $c = \dfrac{1}{p}$ the space $X_c^p(a,b)$ coincides $L^p(a,b)$ space, i.e.,

$$X_{\frac{1}{p}}^p(a,b) = L^p(a,b).$$

Definition 2.1 (Erdélyi–Kober fractional integral [19, 21, 22]). Let $\alpha \in \mathbb{R}, c \in \mathbb{R}$ and $g \in X_c^p(a,b)$, the Erdélyi–Kober fractional integral of order α is defined by

$$({}^\rho I_{a^+}^\alpha g)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} (t^\rho - s^\rho)^{\alpha-1} g(s)ds, \qquad t > a, \rho > 0 \tag{2.1}$$

where Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,\ \alpha > 0.$$

Definition 2.2 (See [18]). The generalized fractional derivative, corresponding to the fractional integral (2.1), is defined, for $0 \le a < t$, by

$$\begin{aligned} {}^\rho D_{a^+}^\alpha g(t) &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-n+\alpha}} g(s)ds \\ &= \delta_\rho^n ({}^\rho I_{a^+}^{n-\alpha} g)(t), \end{aligned}$$

where $\delta_\rho^n = \left(t^{1-\rho}\dfrac{d}{dt}\right)^n$.

Definition 2.3 (See [18, 24])**.** The Caputo-type generalized fractional derivative ${}^{\rho}_{c}D^{\alpha}_{a+}$ is defined via the above generalized fractional derivative as follows

$$({}^{\rho}_{c}D^{\alpha}_{a+}g)(t) = \left({}^{\rho}D^{\alpha}_{a+}\left[g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(s-a)^k\right]\right). \tag{2.2}$$

Lemma 2.4 (See [18])**.** *Let $\alpha, \rho \in \mathbb{R}^+$, then*

$$({}^{\rho}I^{\alpha}_{a+}{}^{\rho}_{c}D^{\alpha}_{a+}g)(t) = g(t) - \sum_{k=0}^{n-1} c_k \left(\frac{t^\rho - a^\rho}{\rho}\right)^k, \tag{2.3}$$

for some $c_k \in \mathbb{R}$, $n = [\alpha] + 1$.

Let $(E, \|\cdot\|)$ be a Banach space. We define the following subsets of $\mathcal{P}(E)$:

$P_{cl}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is closed}\}$,

$P_b(E) = \{Y \in \mathcal{P}(E) : Y \text{ is bounded}\}$,

$P_{cp}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is compact}\}$,

$P_{cv}(E) = \{Y \in \mathcal{P}(E) : Y \text{ is convex}\}$,

$P_{cp,cv}(E) = P_{cp}(E) \cap P_{cv}(E)$.

Definition 2.5. A multivalued map $G : E \to \mathcal{P}(E)$ is said to be convex (closed) valued if $G(x)$ is convex (closed) for all $x \in E$. A multivalued map G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in E for all $B \in P_b(E)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}$ exists).

Definition 2.6. A multivalued map $G : E \to \mathcal{P}(E)$ is called upper semi-continuous (u.s.c.) on E if for each $x_0 \in E$, the set $G(x_0)$ is a nonempty closed subset of E, and for each open set N of E containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(\mathbb{R})$.

Definition 2.7. Let $G : X \to \mathcal{P}(E)$ be completely continuous with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in E$ such that $x \in G(x)$.

We denote by $FixG$ the fixed point set of the multivalued operator G.

Definition 2.8. A multivalued map $G : J \to P_{cl}(E)$ is said to be measurable if for every $y \in E$, the function

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Lemma 2.9 (See [26]). *Let G be a completely continuous multivalued map with nonempty compact values, then G is u.s.c. if and only if G has a closed graph.*

Definition 2.10. A multivalued map $F : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(1) $t \to F(t,u)$ is measurable for each $u \in \mathbb{R}$

(2) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in I$.

F is said to be L^1-Carathéodory if $(1), (2)$ and the following condition holds:

(3) For each $q > 0$, there exists $\varphi_q \in L^1(I, \mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{|v| \ : v \in F(t,u)\} \leq \varphi_q \text{ for all } |u| \leq q \text{ and for a.e. } t \in I.$$

For each $y \in C(I)$, define the set of selections of F by

$$S_{F\circ y} = \{v \in L^1(I) : v(t) \in F(t, y(t)) \text{ a.e. } t \in I\}.$$

Let $(\mathbb{R}, d)$ be a metric space induced from the normed space $(\,|\cdot|)$. The function H_d : $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$$

is known as the Hausdorff–Pompeiu metric. For more details on multivalued maps see the books of Hu and Papageorgiou [26].

Lemma 2.11 (See [23]). *Let I be a compact real interval. Let F be a Carathéodory multivalued map and let Θ be a linear continuous map from $L^1(I) \to C(I)$. Then the operator*

$$\Theta \circ S_{F\circ u} : C(I) \to \mathcal{P}_{cv,cp}(C(I)), \quad u \mapsto (\Theta \circ S_{F\circ u})(u) = \Theta(S_{F\circ u})$$

is a closed graph operator in $C(I) \times C(I)$.

Lemma 2.12 (Nonlinear alternative for Kakutani maps [16]). *Let E be a Banach space, Ca closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that N : $U \longrightarrow \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) N has a fixed point in U, or

(ii) there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda N(u)$.

Lemma 2.13 (Covitz and Nadler [13]). *Let (X,d) be a complete metric space. If N : $X \longrightarrow \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3 Existence of Solutions

Lemma 3.1. *Let* $1 < \alpha \le 2$, $\phi \in C([a-r,a],\mathbb{R})$ *with* $\phi(a) = 0$, $\psi \in C([T,T+\beta],\mathbb{R})$ *with* $\psi(T) = 0$ *and* $h : I \to \mathbb{R}$ *be a integrable function. Then the linear problem*

$$ {}^{\rho}_{c}D^{\alpha}_{a^+}y(t) = h(t), \text{ for a.e. } t \in I := [a,T],\ 1 < \alpha \le 2, \tag{3.1}$$

$$ y(t) = \phi(t),\ t \in [a-r,a],\ r > 0 \tag{3.2}$$

$$ y(t) = \psi(t),\ t \in [T,T+\beta],\ \beta > 0, \tag{3.3}$$

has a unique solution, which is given by

$$ y(t) = \begin{cases} \phi(t), \text{ if } & t \in [a-r,a], \\ \\ -\displaystyle\int_a^T G(t,s)h(s)ds, \text{ if } & t \in I \\ \\ \psi(t), \text{ if } & t \in [T,T+\beta], \end{cases} \tag{3.4}$$

where $G(t,s)$ *is given by*

$$ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)} - s^{\rho-1}(t^\rho - s^\rho)^{\alpha-1}, & a \le s \le t \le T, \\ \\ \dfrac{(t^\rho - a^\rho)(T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho - a^\rho)}, & a \le t \le s \le T. \end{cases} \tag{3.5}$$

Here $G(t,s)$ *is called the Green function of the boundary value problem* (3.1)–(3.3).

Proof. From (2.3), we have

$$ y(t) = c_0 + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right) + {}^{\rho}I^{\alpha}_{a^+}h(s), \quad c_0, c_1 \in \mathbb{R}, \tag{3.6}$$

therefore

$$ y(a) = c_0 = 0, $$

$$ y(T) = c_1\left(\frac{T^\rho - a^\rho}{\rho}\right) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds, $$

and

$$ c_1 = -\frac{\rho^{2-\alpha}}{(T^\rho - a^\rho)\Gamma(\alpha)}\int_a^T (T^\rho - s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds. $$

Substitute the value of c_0 and c_1 into equation (3.6), we get (3.4).

$$y(t) = \begin{cases} \phi(t), \text{ if } \ t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G(t,s)h(s)ds, \text{ if } \ t \in I \\ \\ \psi(t), \text{ if } \ t \in [T, T+\beta], \end{cases}$$

where G is defined by (3.5), the proof is complete. □

Lemma 3.2. *Let $F : I \times C[-r, \beta] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ be a Carathéodory multivalued map. A function $y \in \mathcal{C}$ is a solution for the inclusion problem* (1.1)–(1.3) *if and only if y satisfies the integral equation*

$$y(t) = \begin{cases} \phi(t), \ \textit{if} \ \ t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G(t,s)h(s)ds, \ \textit{if} \ \ t \in I \\ \\ \psi(t), \ \textit{if} \ \ t \in [T, T+\beta], \end{cases}$$

where $h \in L^1(I)$ with

$$h(t) = F(t, y^t) \ \ a.e. \ \ t \in I,$$

$$\widetilde{G} = \sup\left\{ \int_a^T |G(t,s)|ds, t \in I \right\}.$$

The following hypotheses will be used in the sequel:

(H_1) The multivalued map $F : I \times C([-r, \beta]) \to \mathcal{P}_{cp,c}(\mathbb{R})$ is Carathéodory.

(H_2) There exist $p \in L^\infty(I, \mathbb{R}_+)$, and $\Omega : [0, \infty) \longrightarrow (0, \infty)$ continuous and nondecreasing such that

$$\|F(t,u)\|_{\mathcal{P}} = \sup\{\|v\|_C : v(t) \in F(t,u)\} \le p(t)\Omega(\|v\|_{[-r,\beta]}),$$

for a.e. $t \in I$, and each $u \in C([-r, \beta])$.

(H_3) there exists $l \in L^1(I, \mathbb{R})$, such that

$$H_d(F(t, u^t), F(t, \bar{u^t})) \le l(t)\|u - \bar{u}\|_{[-r,\beta]}$$

for every $u, \bar{u} \in C([-r, \beta])$, and

$$d(0, F(0, u^t)) \le l(t) \ \ a.e. \ \ t \in I.$$

(H_4) There exists a number $K_1 > 0$ such that

$$\frac{K_1}{\widetilde{G}p^*\Omega(K_1)} > 1. \tag{3.7}$$

Set

$$p^* = ess\sup_{t\in I} p(t).$$

Now, we state and prove our existence result for problem (1.1)–(1.3) based on a nonlinear alternative for Kakutani maps.

Theorem 3.3. *Assume that* $(H_1), (H_2)$ *and* (H_4) *hold. Then the problem* (1.1)–(1.3) *has at least one solution.*

Proof. Let the operator $N : \mathcal{C} \longmapsto \mathcal{P}(\mathcal{C})$ defined by

$$(Ny)(t) = \left\{ \begin{array}{c} h : I \longrightarrow \mathcal{C} : \\ \\ h(t) = \begin{cases} \phi(t), \text{ if } \ t \in [a-r, a], \\ \\ -\displaystyle\int_a^T G(t,s)v(s)ds, \text{ if } \ t \in I \\ \\ \psi(t), \text{ if } \ t \in [T, T+\beta], \end{cases} \end{array} \right\} \tag{3.8}$$

where

$$v \in S_{F\circ y} = \{v : \Omega \longrightarrow L^1(I) : v(t) \in F(t, y^t) \text{ a.e. } t \in I\}.$$

By Lemma 3.2 it is clear that the fixed points of N are solutions (1.1)–(1.3).

Step 1. $N(u)$ *is convex for each* $u \in C(I)$. Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v_1, v_2 \in S_{F\circ u}$ such that for each $t \in I$ we have

$$h_i(t) = \int_a^T G(t,s)v_i(s)ds; \ i = 1, 2.$$

Let $0 \le \lambda \le 1$. Then, for each $t \in I$, we have

$$(\lambda h_1 + (1-\lambda)h_2)(t) = \int_a^T G(t,s)(\lambda v_1(s) + (1-\lambda)v_2(s))ds.$$

Since $S_{F\circ u}$ is convex (because F has convex values), we have $\lambda h_1 + (1-\lambda)h_2 \in N(u)$.

Let the constant R be such that:

$$R \ge \max\left\{p^*\Omega(R)\widetilde{G}, \|\phi\|_{[a-r,a]}, \|\psi\|_{[T,T+\beta]}\right\}, \tag{3.9}$$

and define

$$D_R = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \le R\}.$$

It is clear that D_R is a bounded, closed and convex subset of $\mathcal{C}$.

Step 2. $N(D_R) \subset D_R$. Let $u \in D_R$. Then for each $h \in N(u)$, there exists $v \in S_{F\circ u}$ such that

$$h(t) = \int_a^T G(t,s)v(s)ds;\ i = 1,2.$$

If $t \in [a-r, a]$, then

$$|h(t)| \le \|\phi\|_{[a-r,a]} \le R,$$

and if $t \in [T, T+\beta]$, then

$$|h(t)| \le \|\psi\|_{[T,T+\beta]} \le R.$$

For each $t \in I$, we have

$$|h(t)| \ \le \ \int_a^T |G(t,s)||v(s)|ds.$$

By (H_2), we have

$$\begin{aligned} |h(t)| &\le \int_a^T |G(t,s)|p(t)\Omega(\|u\|_{[-r,\beta]})ds \\ &\le p^*\Omega(R)\int_a^T |G(t,s)|ds \\ &\le p^*\Omega(R)\widetilde{G} \\ &\le R, \end{aligned}$$

from which it follows that for each $t \in [a-r, T+\beta]$, we have $|h(t)| \le R$, which implies that $\|h\|_{\mathcal{C}} \le R$, and so $N(D_R) \subset D_R$.

Step 3. *N maps bounded sets in $\mathcal{C}$ into equicontinuous sets.* By Step 2 we have $N(D_R) \subset D_R$. Now let $t_1, t_2 \in I = [a, T], t_1 < t_2,$ and let $u \in D_R$, $h \in N(u)$. Then, there exists $v \in S_{F\circ u}$ such that

$$\begin{aligned} |h(t_2) - h(t_1)| &\le \int_a^T |G(t_2,s) - G(t_1,s)||v(s)|ds \\ &\le p^*\Omega(R)\int_a^T |G(t_2,s) - G(t_1,s)|ds. \end{aligned}$$

As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero. As consequence of Step 1 to Step 3, together withe the Arzela–Ascoli theorem, we can conclude that N is completely continuous multi-valued operator.

Step 4. *The graph of N is closed.* Let $(u_n, h_n) \in \operatorname{graph}(N)$, $n \geq 1$, with $(\|u_n - u\|, \|h_n - h\|) \to (0,0)$, as $n \to \infty$. We have to show that $(u, h) \in \operatorname{graph}(N)$. $(u_n, h_n) \in \operatorname{graph}(N)$ means that $h_n \in N(u_n)$, which implies that there exists $v_n \in S_{F \circ u_n}$, such that for each $t \in I$,

$$h_n(t) = \int_a^T G(t,s) v_n(s) ds.$$

Consider the continuous linear operator $\Theta : L^1(I) \to \mathcal{C}$,

$$\Theta(v)(t) \mapsto h_n(t) = \int_a^T G(t,s) v_n(s) ds.$$

Clearly, $\|h_n(t) - h(t)\| \to 0$ as $n \to \infty$. From Lemma 2.11 it follows that $\Theta \circ S_F$ is a closed graph operator. Moreover, $h_n(t) \in \Theta(S_{F \circ u_n})$. Since $u_n \to u$, Lemma 2.11 implies

$$h(t) = \int_a^T G(t,s) v(s) ds,$$

for some $v \in S_{F \circ u}$.

Step 5. A priori bounds on solutions. Let $u \in \mathcal{C}$ be such that $u \in \lambda N(u)$ for all $\lambda \in (0,1)$. Then, there exists $v \in S_{F \circ u}$ such that for each $t \in I$, we have

$$u(t) = -\lambda \int_a^T G(t,s) v(s) ds, \tag{3.10}$$

This implies, by (3.10) that for each $t \in I$ we have

$$\begin{aligned} |u(t)| &\leq \int_a^T |G(t,s)| p(t) \Omega(\|u\|_{[-\alpha,\beta]}) ds \\ &\leq \widetilde{G} p^* \Omega(\|u\|_{[-\alpha,\beta]}). \end{aligned}$$

Thus

$$\frac{\|u\|_{\mathcal{C}}}{\widetilde{G} p^* \Omega(\|u\|_{\mathcal{C}})} \leq 1.$$

By (H_4), we have $\|u\|_{\mathcal{C}} \neq K_1$. Set

$$U = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} < K_1 + 1\}.$$

From the choice of U there is no $u \in \partial U$ such that $u \in \lambda N(u)$ for some $\lambda \in (0,1)$. As a consequence of Lemma 2.12, we deduce that N has a fixed point u in U which is a solution of (1.1)–(1.3). $\square$

We now prove an existence result for (1.1)–(1.3) with non-convex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler.

Theorem 3.4. *Assume* (H_3) *and*

(H_5) $F : [a,T] \times C([-r,\beta],\mathbb{R}) \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot,u) : [a,T] \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $u \in C([-r,\beta],\mathbb{R})$.

If

$$\widetilde{G}\|l\|_{[a,T]} < 1, \tag{3.11}$$

then problem (1.1)–(1.3) *has at least one solution.*

Proof. We shall show that N, as defined in the proof of Theorem 3.3, satisfies the assumptions of Lemma 2.13. The proof will be given in two steps.

Step 1. $N(\cdot)$ is closed valued.

$N(u) \in \mathcal{P}_{cl}(D_R)$ *for each* $u \in D_R$. Let $\{u_n\}_{n\geq 0} \in N(u)$ such that $u_n \longrightarrow \tilde{u}$ in $\mathcal{C}$. Then, $\tilde{u} \in D_R$ and there exists $g_n \in S_{F\circ u}$ be such that, for each $t \in I$, we have

$$u_n(t) = \int_a^T G(t,s)g_n(s)ds.$$

Using (H_5) together with the fact that F has compact values, we may pass to a subsequence to see that g_n converges to g in $L^1(I)$, and hence $g \in S_{F\circ u}$. Then, for each $t \in I$, we get

$$u_n(t) \longrightarrow \tilde{u}(t) = \int_a^T G(t,s)g(s)ds.$$

So, $\tilde{u} \in N(u)$.

Step 2. There exist $\gamma < 1$ such that $H_d(N(u),N(\bar{u})) \leq \gamma\|u-\bar{u}\|_{\mathcal{C}}$ for each $u,\bar{u} \in \mathcal{C}$.

Let $u,\bar{u} \in \mathcal{C}$ and $h_1 \in N(u)$. Then, there exists $v_1 \in F(t,u^t)$ such that for each $t \in I$

$$u_1 = \int_a^T G(t,s)v_1(s)ds.$$

From (H_3) it follows that

$$H_d(F(t,u^t),F(t,\bar{u^t})) \leq l(t)\|u^t - \bar{u^t}\|.$$

Hence, there exists $w \in F(t,\bar{u^t})$ such that

$$|v_1 - w| \leq l(t)\|u^t - \bar{u^t}\|, \ \ t \in I.$$

Consider $U : I \longrightarrow \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1 - w| \leq l(t)\|u^t - \bar{u^t}\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{u}^t)$ is measurable, there exists a function $v_2(t)$ which is measurable selection for V. So, $v_2 \in F(t, \bar{u}^t)$, and for each $t \in I$

$$|v_1 - v_2| \leq l(t)\|u^t - \bar{u^t}\|.$$

Let us define for each $t \in I$

$$u_2 = \int_a^T G(t,s)v_2(s)ds.$$

For $t \in I$, we have

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \int_a^T |G(t,s)||v_1(s) - v_2(s)|ds \\ &\leq \int_a^T |G(t,s)|l(t)\|u^s - \bar{u^s}\|ds \\ &\leq \int_a^T |G(t,s)|l(t)\|u - \bar{u}\|_{[-r,\beta]}ds \\ &\leq \widetilde{G}\|l\|_{[a,T]}\|u - \bar{u}\|_{\mathcal{C}}. \end{aligned}$$

Thus

$$\|h_1 - h_2\|_{\mathcal{C}} \leq \widetilde{G}\|l\|_{[a,T]}\|u - \bar{u}\|_{\mathcal{C}}.$$

Analogously, interchanging the roles of u and $\bar{u}$, we obtain

$$H_d(N(u), N(\bar{u})) \leq \widetilde{G}\|l\|_{[a,T]}\|u - \bar{u}\|_{\mathcal{C}}.$$

Since N is a contraction, it follows by Lemma 2.13 that N has a fixed point u which is a solution of (1.1)–(1.3). This completes the proof. $\square$

4 An Example

Consider the boundary value problem of Caputo type modification of the Erdélyi–Kober fractional differential inclusion:

$$\begin{cases} y(t) = e^{t-2} - 1, \quad t \in [1,2], \\ \\ {}_c^{\frac{1}{2}}D_{2+}^{\frac{3}{2}} y(t) \in F(t, u^t), \quad t \in I = [2,4] \\ \\ y(t) = t - 4, \quad t \in [4,6]. \end{cases} \tag{4.1}$$

Set

$$F(t, u^t) = \{v \in \mathbb{R} : 0 \leq v \leq \frac{1}{t+1}(\|u\|_{[-r,\beta]} + 1)\}, \quad t \in [2,4], \; u \in C([-r,\beta]),$$

and

$$\alpha = \frac{3}{2},\ \rho = \frac{1}{2},\ r = 1,\ \beta = 2.$$

For each $u \in C([-r,\beta])$, $t \in [2,4]$, we have

$$\|F(t,u^t)\| \ \leq \ \frac{1}{t+1}(\|u\|_{[-r,\beta]} + 1).$$

Therefore, (H_2) is verified with $p^* = \dfrac{1}{3}$. For each $t \in I$ we have

$$\begin{aligned}\int_a^T |G(t,s)|ds \ &\leq \ \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{T^\rho - a^\rho}\right)\int_a^T \left|\left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds \\ &+ \ \frac{1}{\Gamma(\alpha)}\int_a^t \left|\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\right| ds.\end{aligned}$$

Then

$$\int_a^T |G(t,s)|ds \leq \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^\alpha.$$

Therefore

$$\widetilde{G} \leq \frac{2}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^\alpha.$$

The condition (3.7) is satisfied. Indeed, we have

$$\begin{aligned}\widetilde{G}p^* \ &\leq \ \frac{2}{3\Gamma(\frac{3}{2}+1)}\left(\frac{4^{\frac{1}{2}} - 2^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{3}{2}} \\ &\approx \ 0.6359551731 \\ &< \ 1,\end{aligned}$$

with $T = 4, a = 2$ and $\alpha = \dfrac{3}{2}$. Hence all conditions of Theorem 3.3 are satisfied, and F is compact, convex valued, and upper semi-continuous. It follows that the problem (4.1) admit a least one solution.

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