Hyers–Ulam Stability of Volterra Type Integral Equations on Time Scales

Andrejs Reinfelds
Institute of Mathematics and Computer Science
University of Latvia, Rīga, Latvia
reinf@latnet.lv

Shraddha Christian
Faculty of Physics, Mathematics and Optometry
University of Latvia, Rīga, Latvia
sc16024@students.lu.lv

Abstract

In this paper, we study Hyers–Ulam stability of general Volterra type integral equations on unbounded and bounded time scales. We give an existence and uniqueness conditions of the solutions of Volterra type integral equations on time scales using Banach contraction principle, Bielecki type norm and Lipschitz type functions. Furthermore it allows to get sufficient conditions for Hyers–Ulam stability.

AMS Subject Classifications: 45D05, 45G10, 34N05.
Keywords: Volterra integral equation, Hyers–Ulam stability, time scales.

1 Introduction

In 1940, S.M. Ulam [25] at the Mathematics Club of the University of Wisconsin raised the question when a solution of an equation, differing slightly from a given one, must be somehow near to the exact solution of the given equation. In the following year, D.H. Hyers [12] gave an affirmative answer to the question of S.M. Ulam for Cauchy additive functional equation in a Banach space. So the stability concept proposed by S.M. Ulam and D.H. Hyers was named as Hyers–Ulam stability. Afterwards Th.M. Rassias [18] introduced new ideas of Hyers–Ulam stability using unbounded right-hand
side in the involved inequalities, depending on certain functions, introducing therefore
the so-called Hyers–Ulam–Rassias stability. However, we will use only the term Hyers–
Ulam stability in this article.

In 2007, S.M. Jung [15] proved, using a fixed point approach, that the Volterra non-
linear integral equation is Hyers–Ulam–Rassias stable, on a compact interval under cer-
tain conditions. Then several authors [6, 13, 14] generalized the previous result on the
Volterra integral equations to infinite interval in the case when the integrand is Lipschitz
with a fixed Lipschitz constant. In the near past many research papers have been
published about Ulam-Hyers stability of Volterra integral equations of different type
including nonlinear Volterra integro-differential equations, mixed integral dynamic sys-
tem with impulses etc. [7, 8, 21–23, 26].

The theory of time scales analysis has been rising fast and has acknowledged a lot
of interest. The pioneer of this theory was S. Hilger [10]. He introduced this theory in
1988 with the inspiration to unify continuous and discrete calculus. For the introduction
to the calculus on time scales and to the theory of dynamic equations on time scales, we

T. Kulik and C.C. Tisdell [16, 24] gave the basic qualitative and quantitative results
to Volterra integral equations on time scales in the case when the integrand is Lipschitz
with a fixed Lipschitz constant. A. Reinfelds and S. Christian [19, 20] generalized pre-
vious results using Lipschitz functions, whose Lipschitz coefficients can be unbounded.

To the best of our knowledge, the first ones who pay attention to Hyers–Ulam sta-
bility for Volterra integral equations on time scales are S. Andras, A.R. Meszaros [1]
and L. Hua, Y. Li, J. Feng [11]. However they restricted their research to the case when
integrand satisfies Lipschitz conditions with some Lipschitz constant. We generalize
the results of [1, 11] using Lipschitz functions, whose Lipschitz coefficients can be un-
bounded, and the Banach fixed point theorem for an appropriate functional space with
Bielecki type norm.

D.B. Pachpatte [17] studied qualitative properties of solutions of general nonlinear
Volterra integral equation

$$x(t) = f \left( t, x(t), \int_a^t K(t, s, x(s)) \Delta s \right)$$

on time scales. In the present paper, using the methods developed at [19, 20], we give
new existence and uniqueness conditions of solutions and analyze Hyers–Ulam stability
for the following class of Volterra type integral equation on an arbitrary time scales $\mathbb{T}$

$$x(t) = f \left( t, x(t), x(\sigma(t)), \int_a^t K(t, s, x(s), x(\sigma(s))) \Delta s \right),$$

$$a, t \in I_a = [a, +\infty) \cap \mathbb{T}. \quad (1.1)$$

This type of integral equations could be very useful for modelling economic process,
for example, a Keynesian–Cross model with “lagged” income [9, 24].
2 Preliminaries on Time Scales

In this section, we present some basic notation, definitions and properties concerning
the calculus on time scales, for more details the reader is referred to [4, 5]. A time scale
\( \mathbb{T} \) is an arbitrary nonempty closed subset of real numbers \( \mathbb{R} \). Since a time scales may or
may not be connected, we need the concept of jump operators. For \( t \in \mathbb{T} \), the forward
jump operator \( \sigma: \mathbb{T} \to \mathbb{T} \) is defined by \( \sigma(t) = \inf\{s \in \mathbb{T}: s > t\} \), while the backward
jump operator \( \rho: \mathbb{T} \to \mathbb{T} \) is defined by \( \rho(t) = \sup\{s \in \mathbb{T}: s < t\} \). In this definitions
we put \( \inf \emptyset = \sup \mathbb{T} \) and \( \sup \emptyset = \inf \mathbb{T} \), where \( \emptyset \) denotes the empty set. Using these
operators we can classify the points of time scale \( \mathbb{T} \) as left dense, left scattered, right
dense and right scattered according to whether \( \rho(t) = t \), \( \rho(t) < t \), \( \sigma(t) = t \) and \( \sigma(t) > t \)
respectively. If \( \mathbb{T} \) has a left scattered maximum \( m \), then \( \mathbb{T}^\kappa = \mathbb{T} \setminus \{m\} \), otherwise set
\( \mathbb{T}^\kappa = \mathbb{T} \). The function \( g: \mathbb{T} \to \mathbb{R} \) is called rd-continuous
provided it is continuous at every right dense points in \( \mathbb{T} \) and its left sided limits exist (finite) at every left dense
points in \( \mathbb{T} \). The graininess function \( \mu: \mathbb{T} \to [0, +\infty) \) is defined by
\( \mu(t) = \sigma(t) - t \).

Assume \( g: \mathbb{T} \to \mathbb{R} \) is a function and fix \( t \in \mathbb{T}^\kappa \). The delta derivative
(also Hilger derivative) \( g(\Delta)(t) \) exists if for every \( \varepsilon > 0 \) there exists a neighbourhood
\( U = (t - \delta, t + \delta) \cap \mathbb{T} \) for some \( \delta > 0 \) such that
\[ |(g(\sigma(t)) - g(s)) - g(\Delta)(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U. \]

If \( g \) is rd-continuous, then there is function \( F [4, 5] \) such that \( F(\Delta)(t) = g(t) \). In this case,
we define the (Cauchy) delta integral by
\[ \int_r^s g(t) \Delta t = F(s) - F(r), \text{ for all } r, s \in \mathbb{T}. \]

Let \( \beta: \mathbb{T} \to \mathbb{R} \) be a nonnegative (and therefore regressive) and rd-continuous scalar
function. The Cauchy initial value problem for scalar linear equation
\[ x(\Delta) = \beta(t)x, \quad x(a) = 1, \quad a \in \mathbb{T} \]
has the unique solution \( e_\beta(\cdot, a): \mathbb{T} \to \mathbb{R} [4, 5] \). More explicitly, using the cylinder
transformation the exponential function \( e_\beta(\cdot, a) \) is given by
\[ e_\beta(t, a) = \exp \left( \int_a^t \xi_{\beta(s)}(\beta(s)) \Delta s \right), \]
where
\[ \xi_h(z) = \begin{cases} z, & h = 0; \\ \frac{1}{h} \log(1 + hz), & h > 0. \end{cases} \]
We will use the following property of exponential functions \([4, 5]\)

\[ e^{\beta(\sigma(t), a)} = (1 + \mu(t)\beta(t))e^{\beta(t, a)}. \]

Observe that we also have the Bernoulli type estimate

\[
1 + \int_a^t \beta(s) \Delta s \leq e^{\beta(t, a)} \leq \exp\left(\int_a^t \beta(s) \Delta s\right)
\]

for all \(t \in I_T = [a, +\infty) \cap \mathbb{T}.\)

Let \(|·|\) denote the Euclidean norm on \(\mathbb{R}^n\). We will consider the linear space of continuous functions \(C(I_T; \mathbb{R}^n)\) such that

\[
\sup_{t \in I_T} \left| x(t) \right| e^{\beta(t, a)} < \infty
\]

and denote this special space by \(C_\beta(I_T; \mathbb{R}^n)\). The space \(C_\beta(I_T; \mathbb{R}^n)\) endowed with the Bielecki type norm (see \([2, 24]\))

\[
\|x\|_\beta = \sup_{t \in I_T} \frac{|x(t)|}{e^{\beta(t, a)}
\]

is a Banach space.

### 3 Volterra Type Integral Equations

Let us introduce new existence and uniqueness conditions of solutions and analyze Hyers–Ulam stability for the class of Volterra type integral equations (1.1). We assume that the Lipschitz coefficients \(L_1\) and \(L_2\) can be unbounded rd-continuous functions. The use of Bielecki type norms related to Lipschitz coefficients allows to choose a suitable functional space to prove the following theorem.

**Theorem 3.1.** Consider the integral equation (1.1). Let \(K : I_T \times I_T \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) be continuous in its first, third and fourth variables and rd-continuous in its second variable, \(f : I_T \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) be continuous, \(L_1, L_2 : I_T \to \mathbb{R}\) be rd-continuous, \(\sup_{s \in I_T} |L_1(s)\mu(s)| = q < \infty, \sup_{s \in I_T} |L_2(s)\mu(s)| = r < 1, 1 < \gamma < r^{-1}\) and

\[
\beta(s) = \frac{[L_1(s) + L_2(s)]\gamma}{1 - r\gamma}.
\]

If

\[
|f(t, x, x', x'') - f(t, \bar{x}, \bar{x}', \bar{x}'')| \leq M(|x - \bar{x}| + |x' - \bar{x}'| + |x'' - \bar{x}'''|),
\]

where

\[
M \left( 1 + \frac{1 + p\gamma}{1 - r\gamma} + \frac{1}{\gamma} \right) < 1,
\]
\[ |K(t, s, x, \bar{x}) - K(t, s, x', \bar{x}')| \leq L_1(s)|x - x'| + L_2(s)|\bar{x} - \bar{x}'|, \ s < t, \]
and
\[ m = \sup_{t \in I_T} \frac{1}{e_{\beta}(t, a)} \left| f \left( t, 0, 0, \int_a^t K(t, s, 0, 0) \Delta s \right) \right| < \infty, \]
then the integral equation (1.1) has a unique solution \( x \in C_{\beta}(I_T; \mathbb{R}^n) \).

**Proof.** Let \( L_1, L_2 : I_T \to \mathbb{R} \) be the Lipschitz coefficients and let
\[ \beta(s) = \frac{[L_1(s) + L_2(s)]\gamma}{1 - \gamma r}, \quad \text{where} \quad 1 < \gamma < r^{-1}. \]
It follows that
\[ 1 + \mu(s)\beta(s) = 1 + \frac{\mu(s)(L_1(s) + L_2(s))\gamma}{1 - \gamma r} = 1 + \frac{(q + r)\gamma}{1 - \gamma r} = \frac{1 + q\gamma}{1 - r\gamma} \]
and
\[ L_1(s) + L_2(s)(1 + \mu(s)\beta(s)) = L_1(s) + L_2(s) + L_3(s)\mu(s)\beta(s) \leq L_1(s) + L_2(s) + \frac{r[L_1(s) + L_2(s)]\gamma}{1 - \gamma r} = \frac{L_1(s) + L_2(s)}{1 - r\gamma} \frac{\beta(s)}{\gamma}. \]
Consider the Banach space \( C_{\beta}(I_T; \mathbb{R}^n) \). To prove the result, we define an operator \( F : C_{\beta}(I_T; \mathbb{R}^n) \to C_{\beta}(I_T; \mathbb{R}^n) \) by the expression
\[ [Fx](t) = f \left( t, x(t), x(\sigma(t)), \int_a^t K(t, s, x(s), x(\sigma(s))) \Delta s \right). \]
We show that for any \( u, v \in C_{\beta}(I_T; \mathbb{R}^n) \)
\[ \|Fu - Fv\|_{\beta} = \sup_{t \in I_T} \frac{\|F[u](t) - [Fv](t)\|}{e_{\beta}(t, a)} \]
\[ = \sup_{t \in I_T} \frac{1}{e_{\beta}(t, a)} \left| f \left( t, u(t), u(\sigma(t)), \int_a^t K(t, s, u(s), u(\sigma(s))) \Delta s \right) \right| \]
\[ - f \left( t, v(t), v(\sigma(t)), \int_a^t K(t, s, v(s), v(\sigma(s))) \Delta s \right) \left| \right| \]
\[ \leq M \sup_{t \in I_T} \frac{1}{e_{\beta}(t, a)} \left( |u(t) - v(t)| + |u(\sigma(t)) - v(\sigma(t))| \right) \]
\[ + \left| \int_a^t K(t, s, u(s), u(\sigma(s))) \Delta s - \int_a^t K(t, s, v(s), v(\sigma(s))) \Delta s \right| \]
\[ = I_1 + I_2 + I_3. \]
We get
\[ I_1 = M \sup_{t \in I_T} \frac{1}{e^\beta(t, a)} |u(t) - v(t)| = M \| u - v \|_\beta, \]
\[ I_2 = M \sup_{t \in I_T} \frac{1 + \mu(t) \beta(t)}{e^\beta(s, a)} |u(s(t)) - v(s(t))| = M \frac{1 + q \gamma}{1 - r \gamma} \| u - v \|_\beta, \]
\[ I_3 = M \sup_{t \in I_T} \frac{1}{e^\beta(t, a)} \left| \int_a^t K(t, s, u(s), u(s(s))) \Delta s - \int_a^t K(t, s, v(s), v(s(s))) \Delta s \right| \]
\[ \leq M \sup_{t \in I_T} \frac{1}{e^\beta(t, a)} \left| \int_a^t [L_1(s)|u(s) - v(s)| + L_2(s)|u(s(s)) - v(s(s))|] \Delta s \right| \]
\[ \leq M \| u - v \|_\beta \sup_{t \in I_T} \frac{1}{e^\beta(t, a)} \int_a^t [L_1(s) + L_2(s)(1 + \mu(s) \beta(s))] e^\beta(s, a) \Delta s \]
\[ \leq M \gamma \| u - v \|_\beta \sup_{t \in I_T} \frac{1}{e^\beta(t, a)} \int_a^t \beta(s) e^\beta(s, a) \Delta s \]
\[ = M \gamma \| u - v \|_\beta \sup_{t \in I_T} \frac{1}{e^\beta(t, a)} \int_a^t e^\Delta(s, a) \Delta s \]
\[ = M \gamma \| u - v \|_\beta \sup_{x \in I_T} \frac{1}{e^\beta(t, a)} (e^\beta(t, a) - 1) \]
\[ = M \gamma \| u - v \|_\beta \sup_{t \in I_T} \left( 1 - \frac{1}{e^\beta(t, a)} \right) = M \gamma \| u - v \|_\beta. \]

It follows that
\[ \| Fu - Fv \|_\beta \leq M \| u - v \|_\beta \left( 1 + \frac{1 + q \gamma}{1 - r \gamma} + \frac{1}{\gamma} \right). \]

We show that \( F : C_\beta(I_T; \mathbb{R}^n) \to C_\beta(I_T; \mathbb{R}^n) \). Let \( x \in C_\beta(I_T; \mathbb{R}^n) \). Taking norms, we obtain
\[ \|Fx\|_\beta = \|Fx - F0 + F0\|_\beta \leq \|Fx - F0\|_\beta + \|F0\|_\beta \]
\[ \leq M \|x\|_\beta \left( 1 + \frac{1 + q \gamma}{1 - r \gamma} + \frac{1}{\gamma} \right) + m. \]

As \( M \left( 1 + \frac{1 + q \gamma}{1 - r \gamma} + \frac{1}{\gamma} \right) < 1 \), we see that \( F \) is a contraction map and so Banach’s fixed point theorem applies, yielding the existence of a unique fixed point \( x \) of the map \( F \). \( \square \)
Definition 3.2. We say that integral equation (1.1) is Hyers–Ulam stable if there exists a constant \( C > 0 \) such that for each real number \( \varepsilon > 0 \) and for each solution \( x \in C_\beta(I_T; \mathbb{R}^n) \) of the inequality
\[
\sup_{t \in I_T} \left| \frac{x(t) - f\left(t, x(t), x(\sigma(t)), \int_a^t K(t, s, x(s), x(\sigma(s))) \Delta s\right)}{e_\beta(t, a)} \right| = \|x - Fx\|_\beta \leq \varepsilon,
\]
there exists a solution \( x_0 \in C_\beta(I_T; \mathbb{R}^n) \) of the integral equation (1.1) with the property
\[
\|x - x_0\|_\beta \leq C\varepsilon.
\]

Let us find a sufficient condition for the Volterra type integral equation (1.1) to be Hyers–Ulam stable.

Theorem 3.3. If \( x_0 \in C_\beta(I_T; \mathbb{R}^n) \) is a solution of the Volterra type integral equation (1.1) and
\[
M \left( 1 + \frac{1 + q\gamma}{1 - r\gamma} + \frac{1}{\gamma} \right) < 1,
\]
then the Volterra type integral equation (1.1) is Hyers–Ulam stable.

Proof. According to Theorem 3.1, there is a unique solution \( x_0 \in C_\beta(I_T; \mathbb{R}^n) \) to the Volterra type integral equation (1.1) in Banach space. Therefore we get the estimate
\[
\|x - x_0\|_\beta \leq \|x - Fx\|_\beta + \|Fx - Fx_0\|_\beta \\
\leq \varepsilon + M \left( 1 + \frac{1 + q\gamma}{1 - r\gamma} + \frac{1}{\gamma} \right) \|x - x_0\|_\beta.
\]
Hence,
\[
\|x - x_0\|_\beta \leq C\varepsilon,
\]
where \( C = \left( 1 - M \left( 1 + \frac{1 + q\gamma}{1 - r\gamma} + \frac{1}{\gamma} \right) \right)^{-1}. \]

4 The Case of Bounded Time Scales

In the case of a bounded (compact) time scale \( a, b \in I_T = [a, b] \cap \mathbb{T} \), we have
\[
1 \leq \sup_{t \in I_T} e_\beta(t, a) \leq \sup_{t \in I_T} \exp \int_a^t \beta(s) \Delta s = N < \infty.
\]

Let us note that every rd-continuous function on a compact interval is bounded. Therefore, the supremum norm and the Bielecki type norm in the Banach space \( C_\beta(I_T; \mathbb{R}^n) \) are equivalent, i.e.,
\[
\sup_{t \in I_T} |x(t)| \leq N\|x\|_\beta \leq N \sup_{t \in I_T} |x(t)|.
\]
We can take also $\gamma = 1$. Then $\beta(s) = \frac{L_1(t) + L_2(t)}{1 - r}$ and we get estimates
\[
\|Fx - Fx_0\|_\beta \leq M \left( 1 + \frac{1 + q}{1 - r} + (1 - N^{-1}) \right) \|x - x_0\|_\beta
\]
and
\[
\|x - Fx\|_\beta \leq \sup_{t \in I} \left| x(t) - f \left( t, x(t), x(\sigma(t)), \int_a^t K(t, s, x(s), x(\sigma(s))) \Delta s \right) \right| \leq \varepsilon.
\]
From Theorem 3.3, we get
\[
\|x - x_0\|_\beta \leq \|x - Fx\|_\beta + \|Fx - Fx_0\|_\beta \\
\leq \varepsilon + M \left( 1 + \frac{1 + q\gamma}{1 - r\gamma} + 1 - N^{-1} \right) \|x - x_0\|_\beta.
\]
It follows that
\[
\sup_{t \in I} |x(t) - x_0(t)| \leq N \|x - x_0\|_\beta \leq C\varepsilon.
\]
Here,
\[
C = N \left( 1 - M \left( 2 + \frac{1 + q\gamma}{1 - r\gamma} - N^{-1} \right) \right)^{-1}.
\]
It follows that integral equation (1.1) on bounded time scales is also Hyers–Ulam stable in Banach space with supremum norm.

**References**


