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Hyers–Ulam Stability of Volterra Type Integral Equations on Time Scales

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Abstract

In this paper, we study Hyers–Ulam stability of general Volterra type integral equations on unbounded and bounded time scales. We give an existence and uniqueness conditions of the solutions of Volterra type integral equations on time scales using Banach contraction principle, Bielecki type norm and Lipschitz type functions. Furthermore it allows to get sufficient conditions for Hyers–Ulam stability.

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1 Introduction

In 1940, S.M. Ulam [25] at the Mathematics Club of the University of Wisconsin raised the question when a solution of an equation, differing slightly from a given one, must be somehow near to the exact solution of the given equation. In the following year, D.H. Hyers [12] gave an affirmative answer to the question of S.M. Ulam for Cauchy additive functional equation in a Banach space. So the stability concept proposed by S.M. Ulam and D.H. Hyers was named as *Hyers–Ulam stability*. Afterwards Th.M. Rassias [18] introduced new ideas of Hyers–Ulam stability using unbounded right-hand

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side in the involved inequalities, depending on certain functions, introducing therefore the so-called *Hyers–Ulam–Rassias stability*. However, we will use only the term Hyers–Ulam stability in this article.

In 2007, S.M. Jung [15] proved, using a fixed point approach, that the Volterra nonlinear integral equation is Hyers–Ulam–Rassias stable, on a compact interval under certain conditions. Then several authors [6, 13, 14] generalized the previous result on the Volterra integral equations to infinite interval in the case when the integrand is Lipschitz with a fixed Lipschitz constant. In the near past many research papers have been published about Ulam-Hyers stability of Volterra integral equations of different type including nonlinear Volterra integro-differential equations, mixed integral dynamic system with impulses etc. [7, 8, 21–23, 26].

The theory of time scales analysis has been rising fast and has acknowledged a lot of interest. The pioneer of this theory was S. Hilger [10]. He introduced this theory in 1988 with the inspiration to unify continuous and discrete calculus. For the introduction to the calculus on time scales and to the theory of dynamic equations on time scales, we recommend the books [4] and [5] by M. Bohner and A. Peterson.

T. Kulik and C.C. Tisdell [16, 24] gave the basic qualitative and quantitative results to Volterra integral equations on time scales in the case when the integrand is Lipschitz with a fixed Lipschitz constant. A. Reinfelds and S. Christian [19, 20] generalized previous results using Lipschitz functions, whose Lipschitz coefficients can be unbounded.

To the best of our knowledge, the first ones who pay attention to Hyers–Ulam stability for Volterra integral equations on time scales are S. Andras, A.R. Meszaros [1] and L. Hua, Y. Li, J. Feng [11]. However they restricted their research to the case when integrand satisfies Lipschitz conditions with some Lipschitz constant. We generalize the results of [1, 11] using Lipschitz functions, whose Lipschitz coefficients can be unbounded, and the Banach fixed point theorem for an appropriate functional space with Bielecki type norm.

D.B. Pachpatte [17] studied qualitative properties of solutions of general nonlinear Volterra integral equation

$$x(t) = f\left(t, x(t), \int_a^t K(t, s, x(s))\right) \Delta s$$

on time scales. In the present paper, using the methods developed at [19, 20], we give new existence and uniqueness conditions of solutions and analyze Hyers–Ulam stability for the following class of Volterra type integral equation on an arbitrary time scales $\mathbb{T}$

$$x(t) = f\left(t, x(t), x(\sigma(t)), \int_a^t K(t, s, x(s), x(\sigma(s)))\Delta s\right), \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}. \quad (1.1)$$

This type of integral equations could be very useful for modelling economic process, for example, a Keynesian–Cross model with "lagged" income [9, 24].

2 Preliminaries on Time Scales

In this section, we present some basic notation, definitions and properties concerning the calculus on time scales, for more details the reader is referred to [4,5]. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. Since a time scales may or may not be connected, we need the concept of jump operators. For $t \in \mathbb{T}$, the *forward jump operator* $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\}$, while the *backward jump operator* $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T}\colon s < t\}$. In this definitions we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. Using these operators we can classify the points of time scale $\mathbb{T}$ as *left dense*, *left scattered*, *right dense* and *right scattered* according to whether $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$ and $\sigma(t) > t$ respectively. If $\mathbb{T}$ has a left scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$, otherwise set $\mathbb{T}^\kappa = \mathbb{T}$. The function $g\colon \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at every right dense points in $\mathbb{T}$ and its left sided limits exist (finite) at every left dense points in $\mathbb{T}$. The *graininess function* $\mu\colon \mathbb{T} \to [0, +\infty)$ is defined by $\mu(t) = \sigma(t) - t$. The function $g\colon \mathbb{T} \to \mathbb{R}$ is *regressive* if

$$1 + \mu(t)g(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa.$$

Assume $g\colon \mathbb{T} \to \mathbb{R}$ is a function and fix $t \in \mathbb{T}^\kappa$. The *delta derivative* (also Hilger derivative) $g^\Delta(t)$ exists if for every $\varepsilon > 0$ there exists a neighbourhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$ such that

$$\left|(g(\sigma(t)) - g(s)) - g^\Delta(t)(\sigma(t) - s)\right| \leq \varepsilon \left|\sigma(t) - s\right| \text{ for all } s \in U.$$

If g is rd-continuous, then there is function F [4,5] such that $F^\Delta(t) = g(t)$. In this case, we define the (Cauchy) delta integral by

$$\int_r^s g(t)\, \Delta t = F(s) - F(r), \text{ for all } r, s \in \mathbb{T}.$$

Let $\beta\colon \mathbb{T} \to \mathbb{R}$ be a nonnegative (and therefore regressive) and rd-continuous scalar function. The Cauchy initial value problem for scalar linear equation

$$x^\Delta = \beta(t)x, \quad x(a) = 1, \quad a \in \mathbb{T}$$

has the unique solution $e_\beta(\cdot, a)\colon \mathbb{T} \to \mathbb{R}$ [4, 5]. More explicitly, using the cylinder transformation the *exponential function* $e_\beta(\cdot, a)$ is given by

$$e_\beta(t, a) = \exp\left(\int_a^t \xi_{\mu(s)}(\beta(s))\, \Delta s\right),$$

where

$$\xi_h(z) = \begin{cases} z, & h = 0; \\ \dfrac{1}{h}\log(1 + hz), & h > 0. \end{cases}$$

We will use the following property of exponential functions [4, 5]

$$e_\beta(\sigma(t), a) = (1 + \mu(t)\beta(t))e_\beta(t, a).$$

Observe that we also have the Bernoulli type estimate

$$1 + \int_a^t \beta(s)\,\Delta s \le e_\beta(t, a) \le \exp\left(\int_a^t \beta(s)\,\Delta s\right)$$

for all $t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}$.

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^n$. We will consider the linear space of continuous functions $C(I_{\mathbb{T}}; \mathbb{R}^n)$ such that

$$\sup_{t \in I_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t, a)} < \infty$$

and denote this special space by $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$. The space $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$ endowed with the Bielecki type norm (see [2, 24])

$$\|x\|_\beta = \sup_{t \in I_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t, a)}$$

is a Banach space.

3 Volterra Type Integral Equations

Let us introduce new existence and uniqueness conditions of solutions and analyze Hyers–Ulam stability for the class of Volterra type integral equations (1.1). We assume that the Lipschitz coefficients L_1 and L_2 can be unbounded rd-continuous functions. The use of Bielecki type norms related to Lipschitz coefficients allows to choose a suitable functional space to prove the following theorem.

Theorem 3.1. *Consider the integral equation* (1.1). *Let* $K\colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous in its first, third and fourth variables and rd-continuous in its second variable,* $f\colon I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous,* $L_1, L_2\colon I_{\mathbb{T}} \to \mathbb{R}$ *be rd-continuous,* $\sup_{s \in I_{\mathbb{T}}} |L_1(s)\mu(s)| = q < \infty$, $\sup_{s \in I_{\mathbb{T}}} |L_2(s)\mu(s)| = r < 1$, $1 < \gamma < r^{-1}$ *and*

$$\beta(s) = \frac{[L_1(s) + L_2(s)]\gamma}{1 - r\gamma}.$$

If

$$|f(t, x, x', x'') - f(t, \bar{x}, \bar{x}', \bar{x}'')| \le M(|x - \bar{x}| + |x' - \bar{x}'| + |x'' - \bar{x}''|),$$

where

$$M\left(1 + \frac{1 + p\gamma}{1 - r\gamma} + \frac{1}{\gamma}\right) < 1,$$

$$|K(t,s,x,\bar{x}) - K(t,s,x',\bar{x}')| \le L_1(s)|x-x'| + L_2(s)|\bar{x}-\bar{x}'|,\ s<t,$$

and

$$m = \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| f\left(t,0,0,\int_a^t K(t,s,0,0)\,\Delta s\right)\right| < \infty,$$

then the integral equation (1.1) *has a unique solution* $x \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$.

Proof. Let $L_1, L_2 \colon I_{\mathbb{T}} \to \mathbb{R}$ be the Lipschitz coefficients and let

$$\beta(s) = \frac{[L_1(s)+L_2(s)]\gamma}{1-\gamma r}, \quad \text{where} \quad 1<\gamma<r^{-1}.$$

It follows that

$$1+\mu(s)\beta(s) = 1 + \frac{\mu(s)(L_1(s)+L_2(s))\gamma}{1-\gamma r} = 1 + \frac{(q+r)\gamma}{1-\gamma r} = \frac{1+q\gamma}{1-r\gamma}$$

and

$$\begin{aligned} L_1(s)+L_2(s)(1+\mu(s)\beta(s)) &= L_1(s)+L_2(s)+L_2(s)\mu(s)\beta(s) \\ &\le L_1(s)+L_2(s)+\frac{r\,[L_1(s)+L_2(s)]\,\gamma}{1-r\gamma} \\ &= \frac{L_1(s)+L_2(s)}{1-r\gamma} = \frac{\beta(s)}{\gamma}. \end{aligned}$$

Consider the Banach space $C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. To prove the result, we define an operator $F\colon C_\beta(I_{\mathbb{T}};\mathbb{R}^n) \to C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ by the expression

$$[Fx](t) = f\left(t, x(t), x(\sigma(t)), \int_a^t K(t,s,x(s),x(\sigma(s)))\,\Delta s\right).$$

We show that for any $u,v \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$

$$\begin{aligned} \|Fu-Fv\|_\beta &= \sup_{t\in I_{\mathbb{T}}} \frac{|[Fu](t)-[Fv](t)|}{e_\beta(t,a)} \\ &= \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| f\left(t,u(t),u(\sigma(t)),\int_a^t K(t,s,u(s),u(\sigma(s)))\,\Delta s\right)\right. \\ &\quad \left. - f\left(t,v(t),v(\sigma(t)),\int_a^t K(t,s,v(s),v(\sigma(s)))\,\Delta s\right)\right| \\ &\le M \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left(|u(t)-v(t)| + |u(\sigma(t))-v(\sigma(t))|\right. \\ &\quad \left. + \left|\int_a^t K(t,s,u(s),u(\sigma(s)))\,\Delta s - \int_a^t K(t,s,v(s),v(\sigma(s)))\,\Delta s\right|\right) \\ &= I_1+I_2+I_3. \end{aligned}$$

We get

$$I_1 = M \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} |u(t) - v(t)| = M\|u - v\|_\beta,$$

$$I_2 = M \sup_{t\in I_{\mathbb{T}}} \frac{1+\mu(t)\beta(t)}{e_\beta(\sigma(t),a)} |u(\sigma(t)) - v(\sigma(t))| = M\frac{1+q\gamma}{1-r\gamma}\|u - v\|_\beta,$$

$$\begin{aligned}
I_3 &= M \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| \int_a^t K(t,s,u(s),u(\sigma(s)))\,\Delta s - \int_a^t K(t,s,v(s),v(\sigma(s)))\,\Delta s \right| \\
&\le M \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| \int_a^t [L_1(s)|u(s)-v(s)| + L_2(s)|u(\beta(s)) - v(\beta(s))|]\,\Delta s \right| \\
&\le M\|u-v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t [L_1(s) + L_2(s)(1+\mu(s)\beta(s))]\, e_\beta(s,a)\,\Delta s] \\
&\le \frac{M}{\gamma}\|u-v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \beta(s) e_\beta(s,a)\,\Delta s \\
&= \frac{M}{\gamma}\|u-v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta^\Delta(s,a)\,\Delta s \\
&= \frac{M}{\gamma}\|u-v\|_\beta \sup_{x\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} (e_\beta(t,a) - 1) \\
&= \frac{M}{\gamma}\|u-v\|_\beta \sup_{t\in I_{\mathbb{T}}} \left(1 - \frac{1}{e_\beta(t,a)}\right) = \frac{M}{\gamma}\|u-v\|_\beta.
\end{aligned}$$

It follows that

$$\|Fu - Fv\|_\beta \le M\|u-v\|_\beta \left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right).$$

We show that $F\colon C_\beta(I_{\mathbb{T}};\mathbb{R}^n) \to C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. Let $x \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. Taking norms, we obtain

$$\begin{aligned}
\|Fx\|_\beta &= \|Fx - F0 + F0\|_\beta \le \|Fx - F0\|_\beta + \|F0\|_\beta \\
&\le M\|x\|_\beta \left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right) + m.
\end{aligned}$$

As $M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right) < 1$, we see that F is a contraction map and so Banach's fixed point theorem applies, yielding the existence of a unique fixed point x of the map F. □

Definition 3.2. We say that integral equation (1.1) is Hyers–Ulam stable if there exists a constant $C > 0$ such that for each real number $\varepsilon > 0$ and for each solution $x \in C_\beta(I_\mathbb{T}; \mathbb{R}^n)$ of the inequality

$$\sup_{t\in I_\mathbb{T}} \frac{\left|x(t) - f\left(t, x(t), x(\sigma(t)), \int_a^t K(t,s,x(s),x(\sigma(s)))\,\Delta s\right)\right|}{e_\beta(t,a)} = \|x - Fx\|_\beta \le \varepsilon,$$

there exists a solution $x_0 \in C_\beta(I_\mathbb{T}; \mathbb{R}^n)$ of the integral equation (1.1) with the property

$$\|x - x_0\|_\beta \le C\varepsilon.$$

Let us find a sufficient condition for the Volterra type integral equation (1.1) to be Hyers–Ulam stable.

Theorem 3.3. *If* $x_0 \in C_\beta(I_\mathbb{T}; \mathbb{R}^n)$ *is a solution of the Volterra type integral equation* (1.1) *and*

$$M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right) < 1,$$

then the Volterra type integral equation (1.1) *is Hyers–Ulam stable.*

Proof. According to Theorem 3.1, there is a unique solution $x_0 \in C_\beta(I_\mathbb{T}; \mathbb{R}^n)$ to the Volterra type integral equation (1.1) in Banach space. Therefore we get the estimate

$$\begin{aligned}\|x - x_0\|_\beta &\le \|x - Fx\|_\beta + \|Fx - Fx_0\|_\beta \\ &\le \varepsilon + M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right)\|x - x_0\|_\beta.\end{aligned}$$

Hence,

$$\|x - x_0\|_\beta \le C\varepsilon,$$

where $C = \left(1 - M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right)\right)^{-1}$. □

4 The Case of Bounded Time Scales

In the case of a bounded (compact) time scale $a, b \in I_\mathbb{T} = [a,b] \cap \mathbb{T}$, we have

$$1 \le \sup_{t\in I_\mathbb{T}} e_\beta(t,a) \le \sup_{t\in I_\mathbb{T}} \exp \int_a^t \beta(s)\,\Delta s = N < \infty.$$

Let us note that every rd-continuous function on a compact interval is bounded. Therefore, the supremum norm and the Bielecki type norm in the Banach space $C_\beta(I_\mathbb{T}; \mathbb{R}^n)$ are equivalent, i.e.,

$$\sup_{t\in I_\mathbb{T}} |x(t)| \le N\|x\|_\beta \le N \sup_{t\in I_\mathbb{T}} |x(t)|.$$

We can take also $\gamma = 1$. Then $\beta(s) = \dfrac{L_1(t) + L_2(t)}{1 - r}$ and we get estimates

$$\|Fx - Fx_0\|_\beta \le M\left(1 + \frac{1+q}{1-r} + (1 - N^{-1})\right)\|x - x_0\|_\beta$$

and

$$\|x - Fx\|_\beta \le \sup_{t\in I_{\mathbb{T}}}\left|x(t) - f\left(t, x(t), x(\sigma(t)), \int_a^t K(t,s,x(s),x(\sigma(s)))\,\Delta s\right)\right| \le \varepsilon.$$

From Theorem 3.3, we get

$$\begin{aligned}\|x - x_0\|_\beta &\le \|x - Fx\|_\beta + \|Fx - Fx_0\|_\beta \\ &\le \varepsilon + M\left(1 + \frac{1+q\gamma}{1-r\gamma} + 1 - N^{-1}\right)\|x - x_0\|_\beta.\end{aligned}$$

It follows that

$$\sup_{t\in I_{\mathbb{T}}} |x(t) - x_0(t)| \le N\|x - x_0\|_\beta \le C\varepsilon.$$

Here,

$$C = N\left(1 - M\left(2 + \frac{1+q\gamma}{1-r\gamma} - N^{-1}\right)\right)^{-1}.$$

It follows that integral equation (1.1) on bounded time scales is also Hyers–Ulam stable in Banach space with supremum norm.

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