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Sturm–Liouville and Riccati Conformable Dynamic Equations

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Abstract

We define Sturm–Liouville and Riccati equations for the general conformable calculus on time scales. We show existence of solutions, establish the relationship between the Riccati and Sturm–Liouville form, provide some examples, and prove a conformable analogue of Wintner's theorem.

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1 Introduction

We define what we call Sturm–Liouville conformable dynamic equations on time scales, which take the nonhomogeneous form

$$\left(px^{\Delta_\alpha}\right)^{\Delta_\alpha} + qx^\sigma = f, \tag{1.1}$$

or the homogeneous form

$$\left(px^{\Delta_\alpha}\right)^{\Delta_\alpha} + qx^\sigma = 0, \tag{1.2}$$

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where p, q and f are rd-continuous for all $t \in \mathbb{T}$. We will demonstrate that this definition aligns well with the traditional notion Sturm–Liouville dynamic equations.

The continuous conformable calculus extends the usual calculus and has yielded several articles such as [8, 11–13]. In the literature, there have been few discussions on Sturm–Liouville equations for the conformable calculus. In [3], a conformable Sturm–Liouville differential equation was explored, but the form of the equation is somewhat atypical because the term with the second order derivative of the unknown function contains the κ_1 function. More conventional conformable Sturm–Liouville differential equations were defined in [1] which considered eigenvalue problems and further studied in [2] which developed Green's functions and eigenfunction expansions.

There are numerous results in the literature relating to conformable calculus on time scales (see, e.g., [4–6, 9, 10, 14]). The paper [10] was the first to study what they called a Sturm–Liouville equation for conformable calculus on time scales. They used the original definition of a conformable derivative on time scales from [5], their operator is equipped with boundary conditions, and the second order term has $p \equiv 1$.

A conformable derivative on time scales was first introduced in [5] by the formula

$$f^{\Delta_\alpha}(t) = \begin{cases} \dfrac{f(\sigma(t)) - f(t)}{\mu(t)} t^{1-\alpha}, & \sigma(t) > t, \\ \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s} t^{1-\alpha}, & \sigma(t) = t. \end{cases} \tag{1.3}$$

In [15], an alternative definition of a conformable derivative on time scales was given by the formula

$$f^{\Delta_\alpha}(t) = \begin{cases} \dfrac{f(\sigma(t)) - f(t)}{\sigma(t)^\alpha - t^\alpha}, & \sigma(t) > t, \\ \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t^\alpha - s^\alpha}, & \sigma(t) = t. \end{cases} \tag{1.4}$$

Both of these definitions have shortcomings that are explored in detail in [4]. In [3, Remark 1.2], it was pointed out that a proportional-derivative controller inspired a generalization of the continuous conformable derivative, which was extended in [4] to fix the shortcomings of (1.3) and (1.4) as follows.

Definition 1.1. Let $\mathbb{T}$ be a time scale, $\alpha \in [0, 1]$, and let $\kappa_0, \kappa_1 : \mathbb{T} \times [0, 1] \to [0, \infty)$ be rd-continuous functions such that for all $t \in \mathbb{T}$,

$$\begin{cases} \lim\limits_{\alpha \to 0^+} \kappa_1(t; \alpha) = 1, & \lim\limits_{\alpha \to 0^+} \kappa_0(t; \alpha) = 0, \\ \lim\limits_{\alpha \to 1^-} \kappa_1(t; \alpha) = 0, & \lim\limits_{\alpha \to 1^-} \kappa_0(t; \alpha) = 1, \\ \kappa_1(t; \alpha) \neq 0, \quad \kappa_0(t; \alpha) \neq 0, \quad \alpha \in (0, 1]. \end{cases} \tag{1.5}$$

The conformable derivative of a function $f \colon \mathbb{T} \to \mathbb{R}$ (with respect to the functions κ_0 and κ_1) is defined by

$$f^{\Delta_\alpha}(t) = \kappa_1(t; \alpha) f(t) + \kappa_0(t; \alpha) f^\Delta(t). \tag{1.6}$$

After the definition (1.6), many properties of this "general" conformable derivative on time scales were explored. In particular, solutions to first order conformable dynamic equations

$$y^{\Delta_\alpha} = py$$

and second order dynamic equations

$$y^{\Delta_\alpha \Delta_\alpha} + ay^{\Delta_\alpha} + by = 0$$

were found and investigated in great detail. The theory that arose greatly resembles that of the traditional calculus on time scales.

2 Preliminary Definitions

See [7] for the basic definitions of time scale calculus, which we now summarize. A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$. The forward jump $\sigma\colon \mathbb{T} \to \mathbb{R}$ is defined by the formula $\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\}$ and the graininess ("stepsize") $\mu\colon \mathbb{T} \to [0, \infty)$ obeys $\mu(t) = \sigma(t) - t$. A backwards jump $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T}\colon s < t\}$. If $\mathbb{T}$ has a maximum M such that $\rho(M) < M$, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. For a function $f\colon \mathbb{T} \to \mathbb{R}$, the Δ-derivative is defined for $t \in \mathbb{T}^\kappa$ that obeys

$$f^\Delta(t) = \begin{cases} \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, & \sigma(t) > t, \\ \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s}, & \sigma(t) = t. \end{cases}$$

The Δ-integral operation is defined so that the fundamental theorem of calculus holds, i.e.,

$$\int_s^t f^\Delta(t)\Delta t = f(s) - f(t),$$

The Δ-integral obeys

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t) f(t). \tag{2.1}$$

A function $p\colon \mathbb{T} \to \mathbb{C}$ is called a regressive function provided that for all $t \in \mathbb{T}$, $1 + \mu(t)p(t) \neq 0$. The cylinder transformation ξ_h given by the formula

$$\xi_h(z) = \begin{cases} \dfrac{1}{h}\mathrm{Log}(1 + hz), & h > 0 \\ z, & h = 0. \end{cases} \tag{2.2}$$

If p is regressive and rd-continuous, then the exponential function $e_p\colon \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ is defined by

$$e_p(t, t_0) = \exp\left(\int_{t_0}^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right). \tag{2.3}$$

It is known that

$$e_p(\sigma(t), t_0) = (1 + \mu(t)p(t))e_p(t, t_0). \tag{2.4}$$

The following "simple useful formula" [7, Theorem 1.16, iv] will be needed in the sequel:

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t). \tag{2.5}$$

A $n \times n$-valued function is said to be in $\mathcal{R}$ provided that for all $t \in \mathbb{T}^{\kappa}$, $I + \mu(t)A(t)$ is invertible and it is componentwise rd-continuous. The following existence and uniqueness theorem is well-known.

Theorem 2.1 (See [7, Theorem 5.24])**.** *Let $A \in \mathcal{R}$ be an $n \times n$-matrix valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^{\Delta} = A(t)y + f(t),\ y(t_0) = y_0$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$.

The following information for the general conformable calculus on time scales may be found in [4]. The conformable derivative on time scales obeys a product rule of the form

$$(fg)^{\Delta_\alpha}(t) = f(t)g^{\Delta_\alpha}(t) + g(\sigma(t))f^{\Delta_\alpha}(t) - f(t)g(\sigma(t))\kappa_1(t;\alpha), \tag{2.6}$$

and a quotient rule of the form

$$\left(\frac{f}{g}\right)^{\Delta_\alpha}(t) = \frac{g(t)f^{\Delta_\alpha}(t) - f(t)g^{\Delta_\alpha}(t)}{g(t)g(\sigma(t))} + \frac{f(t)\kappa_1(t;\alpha)}{g(t)}. \tag{2.7}$$

The following formula is immediate from algebraic rearrangement of (1.6):

$$f^{\Delta}(t) = \frac{1}{\kappa_0(t;\alpha)}f^{\Delta_\alpha}(t) - \frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}f(t). \tag{2.8}$$

Let $\alpha \in (0,1]$, $s, t \in \mathbb{T}$ with $s \leq t$ and let function $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous. Let κ_0, κ_1 be as in (1.5) with the function $t \mapsto \dfrac{p(t) - \kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}$ a regressive rd-continuous function. Then, the conformable exponential function $E_p \colon \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ is defined by

$$E_p(t, t_0;\alpha) = e_{\frac{p(\cdot)-\kappa_1(\cdot;\alpha)}{\kappa_0(\cdot;\alpha)}}(t, t_0). \tag{2.9}$$

The conformable exponential function obeys the formula

$$E_p^{\Delta_\alpha}(t, s;\alpha) = p(t)E_p(t, s;\alpha), \tag{2.10}$$

showing that it is an eigenfunction of the conformable derivative operator. The conformable Δ-integral of f is defined by its relation to the standard Δ-integral on time scales:

$$\int_a^t f(s)E_0(\sigma(t), s;\alpha)\Delta_\alpha s := \int_a^t \frac{f(s)E_0(t, s;\alpha)}{\kappa_0(s;\alpha)}\Delta s \tag{2.11}$$

The fundamental theorem of calculus for the conformable Δ-integral does not take the conventional form. It requires the presence of the exponential function E_0:

$$\int_a^t f^{\Delta_\alpha}(s)E_0(t,\sigma(s);\alpha)\Delta_\alpha s$$
$$= f(t) - f(a)E_0(t,a;\alpha) + \int_a^t f(s)E_0(t,\sigma(s);\alpha)\kappa_1(s;\alpha)\Delta_\alpha s. \quad (2.12)$$

3 Some New Elementary Results

In this section, we prove some natural formulas for the conformable dynamic calculus that will be useful. The following proposition is the conformable analogue of (2.1).

Proposition 3.1. *The following formula holds:*

$$\int_t^{\sigma(t)} f(\tau)\Delta_\alpha\tau = \frac{f(t)\mu(t)}{\kappa_0(t;\alpha) - \mu(t)\kappa_1(t;\alpha)}.$$

Proof. Using (2.1), (2.4), (2.9), and (2.11), compute

$$\begin{aligned}
\int_t^{\sigma(t)} f(s)\Delta_\alpha s &= \int_t^{\sigma(t)} \frac{f(s)E_0(\sigma(t),s)}{E_0(\sigma(t),s)}\Delta_\alpha s \\
&= \int_t^{\sigma(t)} \frac{f(s)E_0(t,s)}{\kappa_0(s;\alpha)E_0(\sigma(t),s)}\Delta s \\
&= \frac{\mu(t)f(t)E_0(t,t)}{\kappa_0(t;\alpha)E_0(\sigma(t),t)} \\
&= \frac{\mu(t)f(t)}{\kappa_0(t;\alpha)\left(1 - \frac{\mu(t)\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}\right)} \\
&= \frac{f(t)\mu(t)}{\kappa_0(t;\alpha) - \mu(t)\kappa_1(t;\alpha)},
\end{aligned}$$

completing the proof. □

Now we investigate how the conformable exponential behaves when taking the limit as $\alpha \to 1^-$.

Lemma 3.2. *If $0 < p(t) \leq M$, then for all but at most finitely many $t \in \mathbb{T}$,*

$$\lim_{\alpha\to 1^-} \xi_{\mu(t)}\left(\frac{p(t) - \kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}\right) \leq M.$$

Proof. Fix a $t \in \mathbb{T}$. By (1.5), for every $\epsilon > 0$ there is a constant $\delta_t > 0$ such that if $0 < |\alpha - 1| < \delta_t$, then $|\kappa_0(t;\alpha) - 1| < \epsilon$ and $|\kappa_1(t;\alpha)| < \epsilon$. From (2.2), the observation that there are at most finitely many $\tau \in [t, t_0] \cap \mathbb{T}$ so that $1 + \mu(\tau)\dfrac{p(\tau) - \kappa_1(\tau;\alpha)}{\kappa_0(\tau;\alpha)} < 0$, and the elementary estimate $\operatorname{Log}(1+x) \leq x$ whenever $x > -1$, we conclude that for all but finitely many $\tau \in [t, t_0] \cap \mathbb{T}$,

$$\left| \xi_{\mu(t)} \left(\frac{p(t) - \kappa_1(t;\alpha)}{\kappa_0(t;\alpha)} \right) \right| \leq \left| \frac{p(t) - \kappa_1(t;\alpha)}{\kappa_0(t;\alpha)} \right| \leq \frac{M}{|\kappa_0(t;\alpha)|} + \left| \frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)} \right|,$$

hence taking the limit as $\alpha \to 1^-$ on each side completes the proof. □

We may now use the lemma to establish a limiting property of the conformable exponential as $\alpha \to 1^-$.

Proposition 3.3. *The following formula holds:*

$$\lim_{\alpha \to 1^-} E_0(t, t_0; \alpha) = 1.$$

Proof. By Proposition 3.2, we know that the cylinder transformation given by $t \mapsto \xi_{\mu(t)} \left(\dfrac{-\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)} \right)$ is bounded above by a constant, and using continuity, (2.2) shows

$$\lim_{\alpha \to 1^-} \xi_{\mu(t)} \left(\frac{-\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)} \right) = 0.$$

Therefore by (2.3) and the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{\alpha \to 1^-} E_0(t, t_0) = \lim_{\alpha \to 1^-} \exp \left(\int_{t_0}^{t} \xi_{\mu(\tau)} \left(\frac{-\kappa_1(\tau;\alpha)}{\kappa_0(\tau;\alpha)} \right) \Delta\tau \right) = \exp(0) = 1,$$

completing the proof. □

4 Sturm–Liouville and Riccati Equations

We now study the conformable Sturm–Liouville dynamic equation (1.1). First we establish that the nonhomogeneous conformable Sturm–Liouville equation (1.1) has a unique solution using a proof resembling that of [3, Theorem 4.1].

Theorem 4.1. *Equation* (1.1) *equipped with the initial conditions* $x(t_0) = x_0$ *and* $x^{\Delta_\alpha}(t_0) = x_1$ *has a unique solution.*

Proof. Define $y(t) = p(t)x^{\Delta_\alpha}(t)$. Immediately,

$$x^{\Delta_\alpha}(t) = \frac{y(t)}{p(t)}. \tag{4.1}$$

Using (1.1), (2.5), (2.8), and (4.1), we observe

$$\begin{aligned}
y^{\Delta_\alpha}(t) &= \left(px^{\Delta_\alpha}\right)^{\Delta_\alpha}(t) \\
&= f(t) - q(t)x^\sigma(t) \\
&= f(t) - q(t)\left(x(t) + \mu(t)x^\Delta(t)\right) \\
&= f(t) - q(t)x(t) - q(t)\mu(t)\left(\frac{x^{\Delta_\alpha}(t)}{\kappa_0(t;\alpha)} - \frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}x(t)\right) \\
&= f(t) - q(t)x(t) - \frac{q(t)\mu(t)}{\kappa_0(t;\alpha)} \cdot \frac{y(t)}{p(t)} + q(t)\mu(t)\frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}x(t) \\
&= \left(-q(t) + q(t)\mu(t)\frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}\right)x(t) + \left(-\frac{q(t)\mu(t)}{\kappa_0(t;\alpha)p(t)}\right)y(t) + f(t).
\end{aligned}$$

Let $z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Using (2.8), we compute

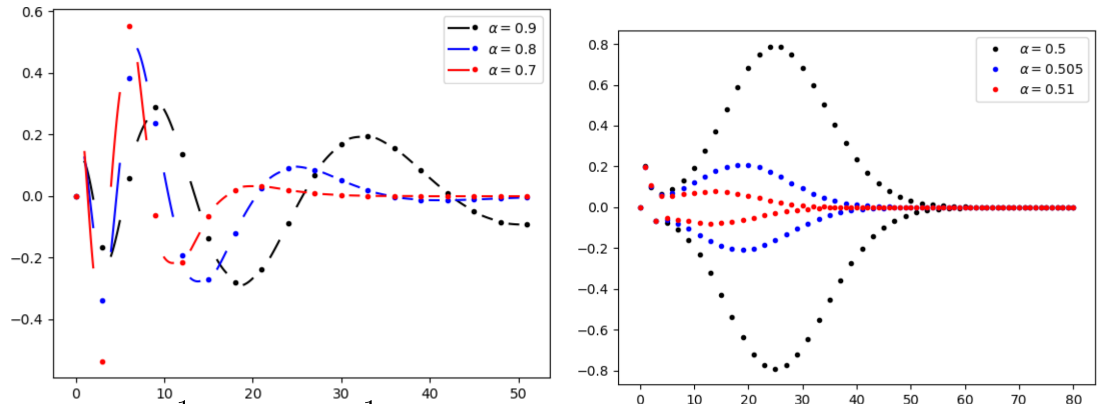
$$\begin{aligned}
z^\Delta(t) &= \begin{bmatrix} \frac{1}{\kappa_0(t;\alpha)}x^{\Delta_\alpha}(t) - \frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}x(t) \\ \frac{1}{\kappa_0(t;\alpha)}y^{\Delta_\alpha}(t) - \frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}y(t) \end{bmatrix} \\
&= \frac{1}{\kappa_0(t;\alpha)}\begin{bmatrix} \frac{y(t)}{p(t)} - \kappa_1(t;\alpha)x(t) \\ q(t)\left(\mu(t)\frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)} - 1\right)x(t) - \frac{q(t)\mu(t)}{\kappa_0(t;\alpha)p(t)}y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f(t)}{\kappa_0(t;\alpha)} \end{bmatrix} \\
&= \frac{1}{\kappa_0(t;\alpha)}\begin{bmatrix} -\kappa_1(t;\alpha) & \frac{1}{p(t)} \\ q(t)\left(\mu(t)\frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)} - 1\right) & -\frac{q(t)\mu(t)}{\kappa_0(t;\alpha)p(t)} \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f(t)}{\kappa_0(t;\alpha)} \end{bmatrix}.
\end{aligned}$$

This is a system of dynamic equations whose components are rd-continuous. Therefore by Theorem 2.1, a unique solution exists. □

The following theorem expresses the second order conformable dynamic equations in Sturm–Liouville form, similar to [7, Theorem 4.12 and Theorem 4.17].

Theorem 4.2. *The second order conformable dynamic equation*

$$x^{\Delta_\alpha\Delta_\alpha} + a(t)x^{\Delta_\alpha} + bx^\sigma(t) = 0$$



(a) $p(t) = \dfrac{1}{1+t}$, $q(t) = \dfrac{1}{1+t^2}$ (b) $p(t) = t + 1$, $q(t) = t^{0.8}$

Figure 4.1: Solutions of (1.1) on the time scales $\mathbb{T} = \displaystyle\bigcup_{k=0}^{\infty}\{3k\} \cup \bigcup_{k=0}^{\infty}\Big[3k+1, 3k+2\Big]$ and $\mathbb{T} = \mathbb{N}_0$ respectively, with initial conditions $x(0) = 0$ and $x^{\Delta}(0) = 0.1$ along with $\kappa_0 \equiv \alpha$, $\kappa_1 \equiv 1 - \alpha$, and $f(t) = 0$.

can be written in self-adjoint form (1.2) *where*

$$p(t) = E_{\beta}(t, t_0; \alpha), \tag{4.2}$$

$$\beta(t) = \frac{(a(t) + \kappa_1(\alpha, t))(\kappa_0(t;\alpha) - \mu(t)\kappa_1(t;\alpha))}{\kappa_0(t;\alpha) - (a(t) + \kappa_1(t;\alpha))\mu(t)},$$

and

$$q(t) = b(t)E_{\beta}^{\sigma}(t, t_0; \alpha).$$

Proof. Using (2.6), we rewrite (1.1) as

$$x^{\Delta_\alpha \Delta_\alpha} + \left(\frac{p^{\Delta_\alpha} - p^{\sigma}\kappa_1}{p^{\sigma}}\right)x^{\Delta_\alpha} + \frac{q}{p^{\sigma}}x^{\sigma} = 0.$$

If we denote $\dfrac{p^{\Delta_\alpha}}{p^{\sigma}} - \kappa_1 = a$ and $\dfrac{q}{p^{\sigma}} = b$, then we use (2.5) and (2.8) to compute

$$\begin{aligned} p^{\Delta_\alpha}(t) &= (a(t) + \kappa_1(t;\alpha))\, p(\sigma(t)) \\ &= (a(t) + \kappa_1(t;\alpha))\, \big(p(t) + \mu(t)p^{\Delta}(t)\big) \\ &= \Big(a(t) + \kappa_1(t;\alpha)\Big)\left(p(t) + \mu(t)\left(\frac{p^{\Delta_\alpha}(t)}{\kappa_0(t;\alpha)} - \frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}p(t)\right)\right) \\ &= (a + \kappa_1)p + \mu(a + \kappa_1)\frac{p^{\Delta_\alpha}}{\kappa_0} - \mu(a + \kappa_1)\frac{\kappa_1}{\kappa_0}p. \end{aligned}$$

From here, we get

$$p^{\Delta_\alpha}\left(1 - \frac{(a + \kappa_1)\mu}{\kappa_0}\right) = \left(a + \kappa_1 - (a + \kappa_1)\mu\frac{\kappa_1}{\kappa_0}\right)p,$$

and this yields

$$p^{\Delta_\alpha} = \left(\frac{(a+\kappa_1)\Big(1-\mu\dfrac{\kappa_1}{\kappa_0}\Big)}{1-\dfrac{a+\kappa_1}{\kappa_0}\mu} \right) p = \left(\frac{(a+\kappa_1)(\kappa_0-\mu\kappa_1)}{\kappa_0-(a+\kappa_1)\mu} \right) p.$$

So we have arrived at $p^{\Delta_\alpha} = \beta p$, which establishes (4.2) via (2.10), completing the proof. □

We say that a solution x of (1.2) has a generalized zero at $t \in \mathbb{T}$ if $x(t) = 0$ or if t is left-scattered and $p(\rho(t))x(\rho(t))x(t) < 0$. We now show that Sturm–Liouville equations whose solution contains no generalized zeros may be converted into a dynamic conformable Riccati equation.

Theorem 4.3. *Let x be a solution of* (1.2)*. If x has no generalized zeros, then z defined by*

$$z = \frac{px^{\Delta_\alpha}}{x} \tag{4.3}$$

is a solution of the Riccati equation

$$z^{\Delta_\alpha} + q(t) + \frac{z^2\kappa_0(t;\alpha)}{p(t)\kappa_0(t;\alpha)+\mu(t)\Big(z-\kappa_1(t;\alpha)p(t)\Big)} - z\kappa_1(t;\alpha) = 0, \tag{4.4}$$

satisfying

$$p(t)+\mu(t)\left(\frac{z(t)}{\kappa_0(t;\alpha)} - \frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}p(t)\right) > 0. \tag{4.5}$$

Proof. Assume that x is a solution of (1.2) with no generalized zeros. If we define z by

$$z = \frac{px^{\Delta_\alpha}}{x},$$

then

$$p+\mu\left(\frac{z}{\kappa_0} - \frac{\kappa_1}{\kappa_0}p\right) = \frac{p}{x}\left(x+\mu x^{\Delta}\right) = \frac{p}{x}x^{\sigma} > 0 \tag{4.6}$$

holds. Using (2.7), we have

$$\begin{aligned} z^{\Delta_\alpha} &= \frac{x\left(px^{\Delta_\alpha}\right)^{\Delta_\alpha} - \left(px^{\Delta_\alpha}\right)x^{\Delta_\alpha}}{xx^{\sigma}} + \frac{px^{\Delta_\alpha}}{x}\kappa_1 \\ &= \frac{x\left(-qx^{\sigma}\right)}{xx^{\sigma}} - \frac{p\left(x^{\Delta_\alpha}\right)^2}{xx^{\sigma}} + \frac{px^{\Delta_\alpha}}{x}\kappa_1 \\ &= -q - \frac{x}{px^{\sigma}}\left(\frac{px^{\Delta_\alpha}}{x}\right)^2 + \frac{px^{\Delta_\alpha}}{x}\kappa_1 \\ &= -q - \frac{x}{px^{\sigma}}z^2 + z\kappa_1 \end{aligned}$$

where we apply the negative reciprocal of (4.6) to the coefficient of z^2 in the last equation. Hence x is a solution of the Riccati equation (4.4) satisfying (4.5). □

The following is a conformable analogue of Wintner's theorem on time scales [7, Theorem 4.45].

Theorem 4.4. *If* $\sup \mathbb{T} = \infty, a \in \mathbb{T}, \mu(t) \geq K > 0, 0 < p(t) \leq M$ *for all* $t \in [a,\infty)$*, and* $\displaystyle\int_a^\infty q(s)E_0(t,\sigma(s);\alpha)\Delta_\alpha s = \infty$*, then there exists* $\epsilon > 0$ *so that for all* $\alpha \in (1-\epsilon, 1]$*, (1.2) has oscillatory solution on* $[a,\infty)$*.*

Proof. Without loss of generality, assume that x is an always positive solution. Consider z from (4.3), which we write as $z(\cdot;\alpha)$ to emphasize the dependence on the parameter α. Multiply (4.4) on both sides by $E_0(t,\sigma(s);\alpha)$ and Δ_α-integrate from t_0 to t with respect to s to arrive at

$$\begin{aligned}\int_{t_0}^t z^{\Delta_\alpha}(s;\alpha)E_0(t,\sigma(s);\alpha)\Delta_\alpha s = &-\int_{t_0}^t q(s)E_0(t,\sigma(s);\alpha)\Delta_\alpha s \\ &-\int_{t_0}^t \frac{z(s;\alpha)^2E_0(t,\sigma(s);\alpha)}{p(s)\kappa_0(t;\alpha)+\mu(s)\left(z(s;\alpha)-\kappa_1(s;\alpha)p(s)\right)}\Delta_\alpha s \\ &+\int_{t_0}^t z(s;\alpha)\kappa_1(s;\alpha)E_0(t,\sigma(s);\alpha)\Delta_\alpha s \end{aligned} \tag{4.7}$$

Using (2.12), the left-hand side of (4.7) equals

$$z(t;\alpha) - z(t_0;\alpha)E_0(t,t_0;\alpha) + \int_{t_0}^t z(s;\alpha)E_0(t,\sigma(s);\alpha)\kappa_1(s;\alpha)\Delta_\alpha s,$$

hence (4.7) becomes

$$\begin{aligned} z(t;\alpha) = z(t_0;\alpha)E_0(t,t_0) &- \int_{t_0}^t q(s)E_0(t,\sigma(s);\alpha)\Delta_\alpha s \\ &- \int_{t_0}^t \frac{z(s;\alpha)^2E_0(t,\sigma(s);\alpha)}{\kappa_0(s;\alpha)p(s)+\mu(s)\Big(z(s;\alpha)-\kappa_1(s;\alpha)p(s)\Big)}\Delta_\alpha s. \end{aligned} \tag{4.8}$$

By Proposition 3.3, there is an $\alpha_0 < 1$ so close to 1 that $E_0(t,\sigma(s);\alpha)$ is positive for all $\alpha_0 < \alpha < 1$. By (1.5), there is an α_1 so to close to 1 that $|\kappa_1(s;\alpha)p(s)| < |z(s;\alpha)|$ for all $\alpha_1 < \alpha < 1$. For any $\max\{\alpha_0,\alpha_1\} < \alpha < 1$, we bound (4.8) as

$$z(t;\alpha) \leq z(t_0;\alpha)E_0(\sigma(s),s) - \int_{t_0}^t q(s)E_0(t,\sigma(s);\alpha)\Delta s.$$

Taking the limit of this expression as $t \to \infty$ suggests that

$$\lim_{t\to\infty} z(t;\alpha) = -\infty. \tag{4.9}$$

By equation (4.5),

$$p(t) + \mu(t)\left[\frac{z(t;\alpha)}{\kappa_0(t;\alpha)} - \frac{\kappa_1(t;\alpha)}{\kappa_0(t;\alpha)}p(t)\right] > 0,$$

and routine algebra, we see this is equivalent to

$$z(t;\alpha) > -\frac{p(t)}{\mu(t)}\kappa_0(t;\alpha) \geq \frac{-M\kappa_0(t;\alpha)}{K},$$

which contradicts (4.9), completing the proof. □

Example 4.5. Consider the time scale $\mathbb{T} = \mathbb{N}_0$. It is well-known that on this time scale, the integration reduces to summation and if $\beta \in \mathbb{R}$ such that $\beta \neq -1$, then $e_\beta(t,s) = (1+\beta)^{t-s}$. Let $a = 0$, $\kappa_0(t;\alpha) = \alpha$, $\kappa_1(t;\alpha) = 1-\alpha$, $q(t) = 0.1$, and $p(t) = 0.5$. Then by (2.9) and (2.11),

$$\begin{aligned}
\int_0^\infty q(s)E_0(t,\sigma(s);\alpha)\Delta_\alpha s &= \int_0^\infty \frac{q(s)E_0(t,s;\alpha)}{\kappa_0(s;\alpha)}\Delta s \\
&= \frac{0.1}{\alpha}\sum_{k=0}^{\infty} e_{-\frac{1-\alpha}{\alpha}}(t,k) \\
&= \frac{0.1}{\alpha}\sum_{k=0}^{\infty}\left(1 - \frac{1-\alpha}{\alpha}\right)^{t-k} \\
&= \frac{0.1}{\alpha}\left(\frac{2\alpha-1}{\alpha}\right)^t \sum_{k=0}^{\infty}\left(\frac{\alpha}{2\alpha-1}\right)^k.
\end{aligned}$$

Since the map $\alpha \mapsto \dfrac{\alpha}{2\alpha-1} \geq 1$ for any $0.5 < \alpha \leq 1$, we observe that the integral diverges for such α, implying that the solution of (1.2) will be oscillatory.

5 Conclusion

We have introduced and investigated Sturm–Liouville conformable dynamic equations. We have shown that they are equivalent to certain systems of dynamic equations, we found its Riccati equation, and we proved an oscillation theorem for time scales whose graininess are bounded below by a positive number. We have plotted solutions of these equations using the `timescalecalculus`[1] Python package. There are many directions research can develop into, including better (non)oscillation theorems, stability theorems, treatment of boundary value problems, and investigation of eigenvalue problems.

[1] the Python package used to plot systems of dynamic equations in this paper may be found on Github at `https://github.com/tomcuchta/timescalecalculus`; commit `2ffb9ed` was used

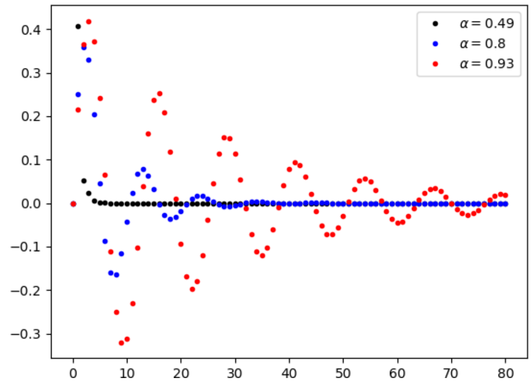


Figure 4.2: Solutions to (1.2) on the time scale $\mathbb{N}_0$ and initial conditions $x(0) = 0$ and $x^{\Delta}(0) = 0.1$ with parameter functions $\kappa_0 \equiv \alpha$, $\kappa_1 \equiv 1 - \alpha$, $q(t) = 0.1$, and $p(t) = 0.5$.

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