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Existence and Uniqueness of Mild Solutions of Stochastic Partial Integro-Differential Impulsive Equations with Infinite Delay via Resolvent Operator

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Abstract

In this paper, we investigate the existence of mild solutions for a class of stochastic functional differential impulsive equations with infinite delay on Hilbert space. The results are obtained by using the Banach fixed point theorem and Krasnoselskii–Schaefer type fixed point theorem combined with theories of resolvent operators. In the end as an application, an example has been presented to illustrate the results obtained.

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1 Introduction

In this paper, we consider the stochastic integro-differential equation with impulses and infinite delay in a separable Hilbert space X with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$

$$\begin{cases} d\left[u(t)-G\left(t,u_t,\int_0^t g(t,s,u_s)ds\right)\right]=A\left[u(t)-G\left(t,u_t,\int_0^t g(t,s,u_s)ds\right)\right]dt \\ \quad+\int_0^t \Gamma(t-s)\left[u(s)-G\left(s,u_s,\int_0^s g(s,r,u_r)dr\right)\right]dsdt \\ \quad+F\left(t,u_t,\int_0^t f(t,s,u_s)ds\right)dt+H\left(t,u_t,\int_0^t \varrho(t,s,u_s)ds\right)dw(t), \\ \quad t\in[0,T]; t\neq t_i; i=1,2,\cdots,m, m\in\mathbb{N}, \\ \Delta u(t_i)=I_k(u(t_i^-)), i=1,2,\cdots,m \\ u_0(\cdot)=\varphi(\cdot)\in\mathfrak{B}_h, \end{cases} \tag{1.1}$$

where $0<T<\infty$, the state $u(\cdot)$ takes values in a separable real Hilbert space X and $A:D(A)\subset X\to X$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X, for $t\geq 0$, $\Gamma(t)$ is a closed linear operator with domain $D(A)\subset D(\Gamma(t))$; $0=t_0\leq t_1\leq ...\leq t_m\leq t_m+1=T$, are pre-fixed points and the symbol $\Delta u(t_i)=u(t_i^+)-u(t_i^-)$, where $u(t_i^+)$ and $u(t_i^-)$ represent the right and left limits of $u(t)$ at $t=t_i$, respectively. The functions $g:D_1\times\mathfrak{B}_h\to X$, $f:D_1\times\mathfrak{B}_h\to X$, $G:[0,T]\times\mathfrak{B}_h\times X\to X$, $F:[0,T]\times\mathfrak{B}_h\times X\to X$, $\varrho:D_1\times\mathfrak{B}_h\to X$, $H:[0,T]\times\mathfrak{B}_h\times X\to\mathcal{L}_2^0$ where $\mathcal{L}_2^0$ to be specified later and $I_i:X\to X$, $i=1,\cdots,m$ are appropriate mappings satisfying certain conditions to be specified later, where $D_1=\{(t,s)\in[0,T]\times[0,T]:s\leqslant t\}$, $\mathfrak{B}_h$ denote the phase space defined axiomatically later. Let K be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t):t\geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q>0$ defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by the Wiener process w. We are also employing the same notation $\|\cdot\|$ for the norm $L(K;X)$, where $L(K;X)$ denotes the space of all bounded linear operators from K into X. The history $u_t:(-\infty,0]\to X$, $u_t(\theta)=u(t+\theta)$, belongs to some abstract phase space $\mathfrak{B}_h$ defined axiomatically; the initial data $\{\varphi(t):-\infty<t\leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathfrak{B}$-valued random variable independent of the Wiener process w with finite second moment. Stochastic differential equations arise in many areas of science and engineering, wherein, quite often the future state of such systems depends not only on present state but also on its history leading to stochastic functional differential equations with delays rather than SDEs. However, many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays. Neutral stochastic differential equations with delays are often used to describe such systems (see, e.g., [6, 15]).

Likewise the theory of impulsive differential equations plays a major role in investigation of qualitative theory. Impulsive differential equations, are differential equations

involving impulse effect, appear as a natural description of observed evolution phenomena of several real world problems, for detail refer [20]. In other way, many dynamical systems (Physical, Social, Biological, Engineering etc.) can be conveniently expressed in the form of differential equations. In case of physical systems such as air crafts, some external forces act which are not continuous with respect to time and the duration of their effect is near negligible as compared with total duration of original process. Same phenomena are observed in case of biological systems (e.g. heart beat, blood-flow, pulse frequency), social system (e.g. price-index frequency, demand and supply of goods) and in many other dynamical systems also such effects are called impulsive effects.

On the other hand, there has been intense interest in the study of impulsive neutral stochastic partial differential equations with memory (e.g. delay) and integro-differential equations with resolvent operators. Since many control systems arising for realistic models depends heavily on histories (that is, effect of infinite delay on the state equations), there is a real need to discuss the existence results for impulsive partial stochastic neutral integro-differential equations with state-dependent delay. Recently, the problem of the existence of solutions for partial impulsive functional differential equations with infinite delay has been investigated in many publications such as [2, 3, 5, 7–11, 19] and the references therein. Motivated by the previously mentioned works, in this paper, we will extend some such results of mild solutions for (1.1).

Our main results concerning (1.1) rely essentially on techniques using strongly continuous family of operators $\{R(t), t \geq 0\}$, defined on the Hilbert space X and called their resolvent. The resolvent operator is similar to the semigroup operator for abstract differential equations in Banach spaces. There is a rich theory for analytic semigroups and we wish to develop theories for (1.1) which yield analytic resolvent. However, the resolvent operator does not satisfy semigroup properties (see, for instance [4, 17]) and our objective in the present paper is to apply the theory developed by Grimmer [12], because it is valid for generators of strongly continuous semigroup, not necessarily analytic. The first sufficient condition proving existence and uniqueness of the mild solution is derived by utilizing Banach fixed point theorem and resolvent operator under Lipschitz continuity of nonlinear term. The second existence result for existence of the mild solution is obtained via technique of Krasnoselskii–Schaefer fixed point theorem and compact resolvent operator under non-Lipschitz continuity of nonlinear term. The main contribution of this manuscript is that it proposes a framework for studying the mild solution to stochastic fractional differential equation with infinite delay and impulsive conditions.

The rest of the work is organized as follows. In Section 2, we recall some necessary preliminaries on stochastic integral and resolvent operator. In Section 3, we study the existence of the mild solutions of (1.1). Finally in Section 4, an example is presented which illustrates the main results for equation (1.1).

2 Preliminaries

2.1 Wiener Process

Throughout this paper, let X and K be two real separable Hilbert spaces. We denote by $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_K$ their inner products and by $\|\cdot\|_X$, $\|\cdot\|_K$ their vector norms, respectively. $\mathcal{L}(K,X)$ denote the space of all bounded linear operators from K into X , equipped with the usual operator norm $\|\cdot\|$ and we abbreviate this notation to $\mathcal{L}(X)$ when $X = K$.

In the sequel, we always use the same symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises.

Moreover, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e. it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

Let $\{w(t) : t \geq 0\}$ denote a K-valued Wiener process defined on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, with covariance operator Q; that is $\mathbb{E}\langle w(t),x\rangle_K\langle w(t),y\rangle_K = (t\wedge s)\langle Qx,y\rangle_K$, for all $x,y\in K$, where Q is a positive, self-adjoint, trace class operator on K. In particular,we denote W a K-valued Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. To define stochastic integrals with respect to the Q-Wiener process with W,we introduce the subspace $K_0 = Q^{1/2}K$ of K endowed with the inner product $\langle u,v\rangle_{K_0} = \langle Q^{1/2}u, Q^{1/2}v\rangle_K$ as a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}$ in K, a bounded sequence of positive real numbers λ_i such that $Qe_i = \lambda_i e_i$, $i = 1,2,\ldots$, and a sequence $\{\beta_i(t)\}_{i\geq 1}$ of independent standard Brownian motions such that $w(t) = \sum_{i=1}^{+\infty}\sqrt{\lambda_i}\beta_i(t)e_i$ for $t\geq 0$ and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0\geq s\geq t\}$. Let $\mathcal{L}_2^0 = \mathcal{L}_2(K_0,X)$ be the space of all Hilbert-Schmidt operators from K_0 to X. It turns out to be a separable Hilbert space equipped with the norm $\|v\|_{\mathcal{L}_2^0}^2 = tr((vQ^{1/2})(vQ^{1/2})^*)$ for any $v\in\mathcal{L}_2^0$. Obviously, for any bounded operator $v\in\mathcal{L}_2^0$, this norm reduces to $\|v\|_{\mathcal{L}_2^0}^2 = tr(vQv^*)$.

2.2 Deterministic Integro-Differential Equations

In the present section, we recall some definitions, notations and properties needed in the sequel.

In what follows, X will denote a Banach space, A and $B(t)$ are closed linear operators on X. Y represents the Banach space $D(A)$, the domain of operator A, equipped with the graph norm

$$\|y\|_Y = \|Ay\| + \|y\|, \qquad y\in Y.$$

The set $C([0,+\infty[;Y)$ is the space of all continuous functions from $[0,+\infty[$ into Y.

We then consider the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t \Gamma(t-s)u(s)ds, \text{ for } t \geq 0, \\ u(0) = u_0 \in X. \end{cases} \tag{2.1}$$

Definition 2.1 (See [12])**.** A resolvent operator of (2.1) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$, satisfying the following properties:

1. $R(0) = I$ and $\|R(t)\| \leq \eta e^{\delta t}$ for some constants η and δ.
2. For each $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.
3. For $x \in Y$, $R(\cdot)x \in C^1([0, +\infty[; X) \cap C([0, +\infty[; Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t \Gamma(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)\Gamma(s)xds, \quad \text{for, } t \geq 0. \end{aligned}$$

For additional detail on resolvent operators, we refer the reader to [18] and [12]. The resolvent operator plays an important role to study the existence of solutions and to establish a variation of constants formula for non-linear systems. For this reason, we need to know when the linear system (2.1) possesses a resolvent operator. Theorem 2.2 below provides a satisfactory answer to this problem.

In what follows, we suppose the following assumptions:

(H1) A is the infinitesimal generator of a C_0-semigroup which is compact on X.

(H2) For all $t \geq 0$, $t \mapsto \Gamma(t)$ is a continuous linear operator from $(Y, \|\cdot\|_Y)$ into $(X, \|\cdot\|_X)$. Moreover, there exists an integrable function $c : [0, +\infty[\to \mathbb{R}^+$ such that for any $y \in Y$, $t \mapsto B(t)y$ belongs to $W^{1,1}([0, +\infty[; X)$ and

$$\left\| \frac{d}{dt}\Gamma(t)y \right\|_X \leq c(t)\|y\|_Y, \quad \text{for} \quad y \in Y, \quad \text{and} \quad t \geq 0.$$

We recall that $W^{k,p}(\Omega) = \{\tilde{w} \in L^p(\Omega) : D^\alpha \tilde{w} \in L^p(\Omega), \quad \forall |\alpha| \leq k\}$, where $D^\alpha \tilde{w}$ is the weak α-th partial derivative of $\tilde{w}$.

Theorem 2.2 (See [12])**.** *Assume that **(H1)** and **(H2)** hold. Then there exists a unique resolvent operator for* (2.1)*.*

In the sequel, we recall some results on the existence of solutions for the following integro-differential equation:

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t \Gamma(t-s)u(s)ds + q(t), \text{ for } t \geq 0, \\ u(0) = u_0 \in X. \end{cases} \tag{2.2}$$

where $q : [0, +\infty[\to X$ is a continuous function.

Definition 2.3 (See [12]). A continuous function $v : [0, +\infty[\to X$ is said to be a strict solution of (2.2) if

1. $u \in C^1([0, +\infty[, X) \cap C([0, +\infty[, Y)$,
2. u satisfies (2.2) for $t \geq 0$.

Remark 2.4. From this definition we deduce that $u(t) \in D(A)$, and the function $s \mapsto \Gamma(t-s)u(s)$ is integrable, for all $t > 0$ and $s \in [0, +\infty[$.

Theorem 2.5 (See [12]). *Assume that **(H1)**, **(H2)** hold. If u is a strict solution of* (2.2)*, then the following variation of constants formula holds*

$$u(t) = R(t)u_0 + \int_0^t R(t-s)q(s)ds, \quad \textit{for} \quad t \geq 0. \tag{2.3}$$

Accordingly, we can establish the following definition.

Definition 2.6 (See [12]). A function $u : [0, +\infty[\to X$ is called a mild solution of (2.2) for $u_0 \in X$, if u satisfies the variation of constants formula (2.3).

Theorem 2.7 (See [12]). *Let $q \in C^1([0, +\infty[; X)$ and u be defined by* (2.3)*. If $u_0 \in D(A)$, then u is a strict solution of* (2.2)*.*

Theorem 2.8 (See [17]). *Assume that hypotheses **(H1)** and **(H2)** hold. Then, the corresponding resolvent operator $R(t)$ of* (2.1) *is continuous for $t \geq 0$ on the operator norm, namely, for all $t_0 > 0$, it holds that $\lim_{h \to 0} \|R(t_0 + h) - R(t_0)\| = 0$.*

To treat the impulsive neutral stochastic fractional differential equation, we present the abstract space phase $\mathfrak{B}_h$.

Definition 2.9. Let $h : (-\infty, 0] \to (0, \infty)$ be assumed to be a continuous function with $l = \int_{-\infty}^{0} h(t)dt < \infty$. For any $c > 0$, we define the phase space:

$$\begin{aligned} \mathfrak{B}_h \;=\; & \Big\{ \varphi : (-\infty, 0] \to X \text{ such that } \left(E\|\varphi(\cdot)\|^2\right)^{1/2} \text{ is bounded and measurable on} \\ & [-c, 0] \text{ with } \varphi(0) = 0 \text{ and } \quad \int_{-\infty}^{0} h(s) \sup_{s \leq \zeta \leq 0} \left(E\|\varphi(\zeta)\|^2\right)^{1/2} ds < \infty \Big\}. \end{aligned} \tag{2.4}$$

It is not difficult to verify that $\mathfrak{B}_h$ is Banach space endowed with the norm

$$\|\varphi\|_{\mathfrak{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \zeta \leq 0} \left(E\|\varphi(\zeta)\|^2\right)^{1/2} ds, \quad \text{for all } \varphi \in \mathfrak{B}_h \tag{2.5}$$

i.e., $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a Banach space [13]. Next, we consider the space

$$\mathfrak{B}_T \;=\; \{u : (-\infty, T] \to X \quad \text{such that} \quad u|_{J_k} \in C(J_k, X) \quad \text{and there exist}$$

$$u(t_k^+) = u(t_k)\text{and} \quad u(t_k^-), \quad u_0 = \phi \in \mathfrak{B}_h, k = 0, 1, 2, \cdots, m\Big\}. \quad (2.6)$$

Here $u|_{J_k}$ denotes the restriction of u to $J_k =]t_k, t_{k+1}]$, $k = 1, \cdots, m$ and the notion $C(J_k, X)$ stands for the space of all continuous X-valued stochastic process $\{\psi(t) : t \in J_k, k = 1, \cdots, m\}$. Let $\|\cdot\|_T$ be a seminorm in $\mathfrak{B}_T$ which is defined by

$$\|u\|_T = \|u_0\|_{\mathfrak{B}_h} + \sup_{s\in[0,T]} \left(E\|u(s)\|^2\right)^{1/2}, \quad u \in \mathfrak{B}_T. \quad (2.7)$$

Now, we give the following lemma [16].

Lemma 2.10 (See [14]). *If $u \in \mathfrak{B}_T$, then for $t \in [0, T]$, $u_t \in \mathfrak{B}_h$. Moreover,*

$$l\left(E\|u(t)\|^2\right)^{1/2} \leq \|u_t\|_{\mathfrak{B}_h} \leq l \sup_{s\in[0,T]} \left(E\|u(s)\|^2\right)^{1/2} + \|u_0\|_{\mathfrak{B}_h}, \quad (2.8)$$

here $l = \displaystyle\int_{-\infty}^{0} h(s)ds < \infty$.

Now, we state the statement of Krasnoselskii–Schaefer fixed point theorem which is our main tool to establish the second existence result.

Theorem 2.11 (See [1, 16]). *Let Φ_1, Φ_2 be two operators such that :*

(a) Φ_1 is a contraction, and

(b) Φ_2 is completely continuous.

Then either:

(i) the operator equation $x = \Phi_1 x + \Phi_2 x$ has a solution, or

(ii) the set $\Lambda = \{x \in X : \lambda\Phi_1(\frac{x}{\lambda}) + \lambda\Phi_2 x = x\}$ is unbounded for $\lambda \in (0, 1)$.

Before expressing and demonstrating the main result, we present the definition of the mild solution to problem (1.1).

Definition 2.12. A stochastic process $u(\cdot) : (-\infty, T] \times \Omega \to X$ is said to be a mild solution to the problem (1.1) if

(i) $u(\cdot)$ is measurable and $\mathcal{F}_t$-adapted for each $t \geq 0$.

(ii) $u(\cdot)$ has cádlág paths on $t \in [0,T]$ a.s. such that u satisfies the integral equation

$$u(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0] \\ -R(t)(G(0,\phi,0)) + G\left(t, u_t, \int_0^t g(t,s,u_s)ds\right) \\ \quad + \int_0^t R(t-s)F\left(s, u_s, \int_0^s f(s,\tau,u_\tau)d\tau\right)ds \\ \quad + \int_0^t R(t-s)H\left(s, u_s, \int_0^s \varrho(s,\tau,u_\tau)d\tau\right)dw(s), \quad t \in [0,t_1] \\ R(t-t_1)\left[u(t_1^-) + I_1(u(t_1^-)) - G\left(t_1, u_{t_1^+}, \int_0^{t_1} g(t_1,s,u_s)ds\right)\right] \\ \quad + G\left(t, u_t, \int_0^t g(t,s,u_s)ds\right) \\ \quad + \int_{t_1}^t R(t-s)F\left(s, u_s, \int_0^s f(s,\tau,u_\tau)d\tau\right)ds \\ \quad + \int_{t_1}^t R(t-s)H\left(s, u_s, \int_0^s \varrho(s,\tau,u_\tau)d\tau\right)dw(s), \quad t \in (t_1,t_2] \\ \vdots \quad \vdots \quad \vdots \\ R(t-t_m)\left[u(t_m^-) + I_m(u(t_m^-)) - G\left(t_m, u_{t_m^+}, \int_0^{t_m} g(t_m,s,u_s)ds\right)\right] \\ \quad + G\left(t, u_t, \int_0^t g(t,s,u_s)ds\right) \\ \quad + \int_{t_m}^t R(t-s)F\left(s, u_s, \int_0^s f(s,\tau,u_\tau)d\tau\right)ds \\ \quad + \int_{t_m}^t R(t-s)H\left(s, u_s, \int_0^s \varrho(s,\tau,u_\tau)d\tau\right)dw(s), \quad t \in (t_m,T]. \end{cases} \tag{2.9}$$

3 Main Results

In this section, the existence of the mild solution for (1.1) is studied. Our first existence result is based on the Banach fixed point theorem. So, we make the following assumptions.

(B1) The resolvent operator $R(t), \ t \geq 0$ is compact and there exists a constant M such that $\|R(t)\|^2 \leq M, \ \forall t \in [0,T]$.

(B2) (i) There exists a constant $L_G > 0$ such that

$$\mathbb{E}\|G(t,u_1,v_1) - G(t,u_2,v_2)\|^2 \leq L_G[\|u_1 - u_2\|^2_{\mathfrak{B}_h} + \mathbb{E}\|v_1 - v_2\|^2]$$

for all $t \in [0,T]$, $u_1, u_2 \in \mathfrak{B}_h$, $v_1, v_2 \in X$ and $C_1 = \sup_{t\in[0,T]} \|G(t,0,0)\|^2$.

(ii) There exists a constant $L_g > 0$ such that

$$\mathbb{E}\left\|\int_0^t [g(t,s,x) - g(t,s,y)]ds\right\|^2 \leq L_g\|x-y\|^2_{\mathfrak{B}_h},$$

for all $t \in [0,T]$, $x, y \in \mathfrak{B}_h$ and $C_2 = T \sup_{(t,s)\in D_1} g^2(t,s,0)$.

(B3) (i) The nonlinear map $H : [0,T]\times\mathfrak{B}_h \times X \to L(K,X)$ is continuous and there is a constant $L_H > 0$ such

$$\mathbb{E}\|H(t,u,v) - H(t,w,z)\|^2 \leq L_H[\|u-w\|^2_{\mathfrak{B}_h} + \mathbb{E}\|v-z\|^2],$$

for all $t \in [0,T]$ and $u, w \in \mathfrak{B}_h$, $v, z \in X$.

(ii) There exists a constant $L_\varrho > 0$ such that

$$\mathbb{E}\left\|\int_0^t [\varrho(t,s,x) - \varrho(t,s,y)]ds\right\|^2 \leq L_\varrho\|x-y\|^2_{\mathfrak{B}_h},$$

for all $t \in [0,T]$, $x, y \in \mathfrak{B}_h$.

(B4) (i) The nonlinear map $H : [0,T]\times\mathfrak{B}_h \times X \to L(K,X)$ is continuous and there is a constant $L_F > 0$ such

$$\mathbb{E}\|F(t,u,v) - F(t,w,z)\|^2 \leq L_F[\|u-w\|^2_{\mathfrak{B}_h} + \mathbb{E}\|v-z\|^2],$$

for all $t \in [0,T]$ and $u, w \in \mathfrak{B}_h$, $v, z \in X$.

(ii) There exists a constant $L_f > 0$ such that

$$\mathbb{E}\left\|\int_0^t [f(t,s,x) - f(t,s,y)]ds\right\|^2 \leq L_f\|x-y\|^2_{\mathfrak{B}_h},$$

for all $t \in [0,T]$, $x, y \in \mathfrak{B}_h$.

(B5) There exist constants $\mathcal{D}_i > 0$ such that

$$\mathbb{E}\|I_i(u) - I_i(v)\|^2 \leq \mathcal{D}_i\|u-v\|^2, \qquad u, v \in X,\ i = 1, \cdots, m.$$

Theorem 3.1. *Assume that assumptions* (***H1***), (***H2***) *and* $(B1)$–$(B5)$ *hold. Then there exists a unique mild solution for the problem* (1.1) *provided that*

$$\begin{aligned} &6 \sup_{1\leq i\leq m} \big[M^2\{1 + \mathcal{D}_i + L_G(1+L_g)\} + L_G(1+L_g) + M^2T^2L_F(1+L_f) \\ &+\mathrm{Tr}(Q)M^2TL_H(1+L_\varrho)\big] < 1. \end{aligned} \tag{3.1}$$

Proof. We consider the operator $\Phi : \mathfrak{B}_T \to \mathfrak{B}_T$ defined by $\Phi u(t) =$

$$\begin{cases} \phi(t), \quad t \in (-\infty, 0] \\ -R(t)(G(0,\phi,0)) + G\left(t, u_t, \int_0^t g(t,s,u_s)ds\right) \\ \quad + \int_0^t R(t-s)F\left(s, u_s, \int_0^s f(s,\tau,u_\tau)d\tau\right)ds \\ \quad + \int_0^t R(t-s)H\left(s, u_s, \int_0^s \varrho(s,\tau,u_\tau)d\tau\right)dw(s), \quad t \in [0,t_1] \\ R(t-t_1)\left[u(t_1^-) + I_1(u(t_1^-)) - G\left(t_1, u_{t_1^+}, \int_0^{t_1} g(t_1,s,u_s)ds\right)\right] \\ +G\left(t, u_t, \int_0^t g(t,s,u_s)ds\right) + \int_{t_1}^t R(t-s)F\left(s, u_s, \int_0^s f(s,\tau,u_\tau)d\tau\right)ds \\ \quad + \int_{t_1}^t R(t-s)H\left(s, u_s, \int_0^s \varrho(s,\tau,u_\tau)d\tau\right)dw(s), \quad t \in (t_1,t_2] \\ \vdots \quad \vdots \quad \vdots \\ R(t-t_m)\left[u(t_m^-) + I_m(u(t_m^-)) - G\left(t_m, u_{t_m^+}, \int_0^{t_m} g(t_m,s,u_s)ds\right)\right] \\ \quad +G\left(t, u_t, \int_0^t g(t,s,u_s)ds\right) + \int_{t_m}^t R(t-s)F\left(s, u_s, \int_0^s f(s,\tau,u_\tau)d\tau\right)ds \\ \quad + \int_{t_m}^t R(t-s)H\left(s, u_s, \int_0^s \varrho(s,\tau,u_\tau)d\tau\right)dw(s), \quad t \in (t_m,T]. \end{cases}$$

Let $y(\cdot) : (-\infty, T] \to X$ be the function defined by

$$y(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0] \\ 0, \quad t \in [0,T]. \end{cases}$$

Thus $y_0 = \phi$. We also define a function

$$\hat{z}(t) = \begin{cases} 0, \quad t \in (-\infty, 0] \\ z(t), \quad t \in [0,T] \end{cases}$$

for every $z \in C([0,T], X)$. We set $u(t) = y(t) + \hat{z}(t)$ for each $t \in [0,T]$. Clearly, u is

the solution for problem (1.1) if and only if z satisfies $z_0 = 0$, $t \in (-\infty, 0]$ and $z(t) =$

$$
\begin{cases}
-R(t)(G(0,\phi,0)) + G\left(t, y_t + \hat{z}_t, \int_0^t g(t,s,y_s+\hat{z}_s)ds\right) \\
\quad + \int_0^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\
\quad + \int_0^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t \in [0,t_1] \\
R(t-t_1)[y(t_1^-) + \hat{z}(t_1^-) + I_1(y(t_1^-)+\hat{z}(t_1^-)) \\
\quad - G\left(t_1, y_{t_1^+} + \hat{z}_{t_1^+}, \int_0^{t_1} g(t_1,s,y_s+\hat{z}_s)ds\right)] \\
\quad + G\left(t, y_t+\hat{z}_t, \int_0^t g(t,s,y_s+\hat{z}_s)ds\right) \\
\quad + \int_{t_1}^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\
\quad + \int_{t_1}^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t \in (t_1,t_2] \\
\vdots \quad \vdots \quad \vdots \\
R(t-t_m)[y(t_m^-) + \hat{z}(t_m^-) + I_m(y(t_m^-)+\hat{z}(t_m^-)) \\
\quad - G\left(t_m, y_{t_m^+} + \hat{z}_{t_m^+}, \int_0^{t_m} g(t_m,s,y_s+\hat{z}_s)ds\right)] \\
\quad + G\left(t, y_t+\hat{z}_t, \int_0^t g(t,s,y_s+\hat{z}_s)ds\right) \\
\quad + \int_{t_m}^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\
\quad + \int_{t_m}^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t \in (t_m,T].
\end{cases}
$$

Let $\mathfrak{B}_T^0 = \{z \in \mathfrak{B}_T : z_0 = 0 \in \mathfrak{B}_h\}$ and for any $z \in \mathfrak{B}_T^0$, we get

$$\|z\|_T = \|z_0\|_{\mathfrak{B}_h} + \sup_{t\in[0,T]} \left(E|z(t)|^2\right)^{1/2} = \sup_{t\in[0,T]} \left(E|z(t)|^2\right)^{1/2}. \tag{3.2}$$

It is easy to verify that $(\mathfrak{B}_T^0, \|\cdot\|_{\mathfrak{B}_T^0})$ is a Banach space. Now, we define the operator

$\mathbb{Q} : \mathfrak{B}_T^0 \to \mathfrak{B}_T^0$ by $\mathbb{Q}z(t) =$

$$
\begin{cases}
-R(t)(G(0,\phi,0)) + G\left(t, y_t + \hat{z}_t, \int_0^t g(t,s,y_s+\hat{z}_s)ds\right) \\
\quad + \int_0^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\
\quad + \int_0^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t \in [0,t_1] \\
R(t-t_1)[y(t_1^-) + \hat{z}(t_1^-) + I_1(y(t_1^-)+\hat{z}(t_1^-)) \\
\quad - G\left(t_1, y_{t_1^+} + \hat{z}_{t_1^+}, \int_0^{t_1} g(t_1,s,u_s)ds\right)] \\
\quad + G\left(t, y_t+\hat{z}_t, \int_0^t g(t,s,u_s)ds\right) \\
\quad + \int_{t_1}^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\
\quad + \int_{t_1}^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t \in (t_1,t_2] \\
\vdots \quad \vdots \quad \vdots \\
R(t-t_m)[y(t_m^-) + \hat{z}(t_m^-) + I_m(y(t_m^-)+\hat{z}(t_m^-)) \\
\quad - G\left(t_m, y_{t_m^+} + \hat{z}_{t_m^+}, \int_0^{t_m} g(t_m,s,u_s)ds\right)] \\
\quad + G\left(t, y_t+\hat{z}_t, \int_0^t g(t,s,u_s)ds\right) \\
\quad + \int_{t_m}^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\
\quad + \int_{t_m}^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t \in (t_m,T].
\end{cases}
$$

To prove the existence result, it is enough to prove that $\mathbb{Q}$ has a unique fixed point. Let $z, z^* \in \mathfrak{B}_T^0$, then for $t \in [0,t_1]$, we conclude that

$$
\begin{aligned}
&\mathbb{E}\|(\mathbb{Q}z)(t) - (\mathbb{Q}z^*)(t)\|^2 \\
&\leq \; 3\mathbb{E}\left\|G\left(t, y_t+\hat{z}_t, \int_0^t g(t,s,y_s+\hat{z}_s)ds\right) - G\left(t, y_t+\hat{z}_t^*, \int_0^t g(t,s,y_s+\hat{z}_s^*)ds\right)\right\|^2 \\
&\quad + 3\mathbb{E}\left\|\int_0^t R(t-s)\left[F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right.\right. \\
&\quad \left.\left. - F\left(s, y_s+\hat{z}_s^*, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau^*)d\tau\right)\right]ds\right\|^2 \\
&\quad + 3\mathbb{E}\left\|\int_0^t R(t-s)\left[H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right.\right.
\end{aligned}
$$

$$-H\left(s, y_s+\hat{z}_s^*, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau^*)d\tau\right)\Big] dw(s)\Big\|^2_{\mathcal{L}_2^0},$$

$$\begin{aligned}
&\mathbb{E}\|(\mathbb{Q}z)(t)-(\mathbb{Q}z^*)(t)\|^2\\
&\leq\ 3L_G(\mathbb{E}\|z_t-z_t^*\|_{\mathfrak{B}_h}+L_g\mathbb{E}\|z_t-z_t^*\|_{\mathfrak{B}_h})+3\int_0^t\|R(t-s)\|ds\\
&\quad\times\int_0^t\|R(t-s)\|\mathbb{E}\left\|F\left(s,y_s+\hat{z}_s,\int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right.\\
&\quad\left.-F\left(s,y_s+\hat{z}_s^*,\int_0^s f(s,\tau,y_\tau+\hat{z}_\tau^*)d\tau\right)\right\|^2_X ds\\
&\quad+3\int_0^t\|R(t-s)\|^2\mathbb{E}\left\|H\left(s,y_s+\hat{z}_s,\int_0^s\varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right.\\
&\quad\left.-H\left(s,y_s+\hat{z}_s^*,\int_0^s\varrho(s,\tau,y_\tau+\hat{z}_\tau^*)d\tau\right)\right\|^2_{\mathcal{L}_2^0}ds\\
&\leq\ 3L_G(1+L_g)\mathbb{E}\|z_t-z_t^*\|_{\mathfrak{B}_h}+3TM^2\int_0^t L_F(1+L_f)\mathbb{E}\|z_s-z_s^*\|^2_{\mathfrak{B}_h}ds\\
&\quad+3M^2\int_0^t L_H(1+L_\varrho)\mathbb{E}\|z_s-z_s^*\|^2_{\mathfrak{B}_h}ds\\
&\leq\ 3L_G(1+L_g)\sup_{t\in[0,T]}\mathbb{E}\|z(t)-\hat{z}(t)\|^2\\
&\quad+3T^2M^2L_F(1+L_f)\sup_{t\in[0,T]}\mathbb{E}\|z(t)-z^*(t)\|^2\\
&\quad+TM^2L_H(1+L_\varrho)\sup_{t\in[0,T]}\mathbb{E}\|z(t)-z^*(t)\|^2\\
&\leq\ 3\left(L_G(1+L_g)+M^2T^2L_F(1+L_f)+M^2TL_H(1+L_\varrho)\right)\|z-z^*\|^2_{\mathfrak{B}_T^0},
\end{aligned}$$

since $\|z_0\|^2_{\mathfrak{B}_h}=0$ and $\|z_0^*\|^2_{\mathfrak{B}_h}=0$. For $t\in(t_1,t_2]$, we get

$$\begin{aligned}
&\mathbb{E}\|(\mathbb{Q}z)(t)-(\mathbb{Q}z^*)(t)\|^2\\
&\leq\ 6\|R(t-t_1)\|^2\mathbb{E}\|I_1(\hat{z}(t_1^-))-I_1(\hat{z}^*(t_1^-))\|^2\\
&\quad+6\|R(t-t_1)\|^2\mathbb{E}\|\hat{z}(t_1^-)-\hat{z}^*(t_1^-)\|^2\\
&\quad+6\|R(t-t_1)\|^2\mathbb{E}\left\|G\left(t_1,y_{t_1^+}+\hat{z}_{t_1^+},\int_0^{t_1}g(t_1,s,y_s+\hat{z}_s)ds\right)\right.\\
&\quad\left.-G\left(t_1,y_{t_1^+}+\hat{z}^*_{t_1^+},\int_0^{t_1}g(t_1,s,y_s+\hat{z}_s^*)ds\right)\right\|^2\\
&\quad+6\mathbb{E}\left\|G\left(t,y_t+\hat{z}_t,\int_{t_1}^t g(t,s,y_s+\hat{z}_s)ds\right)\right.\\
&\quad\left.-G\left(t,y_t+\hat{z}_t^*,\int_{t_1}^t g(t,s,y_s+\hat{z}_s^*)ds\right)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+6\mathbb{E}\left\|\int_{t_1}^{t} R(t-s)\left[F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right.\right.\\
&\left.\left.-F\left(s, y_s+\hat{z}^*_s, \int_0^s f(s,\tau,y_\tau+\hat{z}^*_\tau)d\tau\right)\right]ds\right\|^2\\
&+6\mathbb{E}\left\|\int_{t_1}^{t} R(t-s)\left[H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right.\right.\\
&\left.\left.-H\left(s, y_s+\hat{z}^*_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}^*_\tau)d\tau\right)\right]dw(s)\right\|^2_{\mathcal{L}_2^0},
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\|(\mathbb{Q}z)(t)-(\mathbb{Q}z^*)(t)\|^2\\
&\leq\ 6M^2[\mathbb{E}\|z(t_1^-)-z^*(t_1^-)\|^2_X+\mathbb{E}\|I_1(z(t_1{}^-))-I_1(z^*(t_1{}^-))\|^2_X\\
&\quad +L_G(1+L_g)\|z_{t_1}-z^*_{t_1}\|^2_{\mathfrak{B}_h}]\\
&\quad +6L_G(\mathbb{E}\|z_t-\hat{z}_t\|_{\mathfrak{B}_h}+L_g\mathbb{E}\|z_t-\hat{z}_t\|_{\mathfrak{B}_h})+6\int_0^t\|R(t-s)\|ds\\
&\quad \times\int_{t_1}^{t}\|R(t-s)\|\mathbb{E}\left\|F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right.\\
&\quad \left.-F\left(s, y_s+\hat{z}^*_s, \int_0^s f(s,\tau,y^*_\tau+\hat{z}_\tau)d\tau\right)\right\|^2_X ds\\
&\quad +6\int_{t_1}^{t}\|R(t-s)\|^2\mathbb{E}\left\|H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right.\\
&\quad \left.-H\left(s, y_s+\hat{z}^*_s, \int_0^s \varrho(s,\tau,y^*_\tau+\hat{z}_\tau)d\tau\right)\right\|^2_{\mathcal{L}_2^0} ds\\
&\leq\ 6\left[M^2\{1+\mathcal{D}_1+L_G(1+L_g)\}+L_G(1+L_g)+M^2T^2L_F(1+L_f)\right.\\
&\quad \left.+M^2TL_H(1+L_\varrho)\right]\|z-z^*\|^2_{\mathfrak{B}^0_T}.
\end{aligned}
$$

Similarly, for $t\in(t_i,t_{i+1}]$, $i=1,\cdots,m$, we get

$$
\begin{aligned}
&\mathbb{E}\|(\mathbb{Q}z)(t)-(\mathbb{Q}z^*)(t)\|^2\\
&\leq\ 6\left[M^2\{1+\mathcal{D}_i+L_G(1+L_g)\}+L_G(1+L_g)+M^2T^2L_F(1+L_f)\right.\\
&\quad \left.+M^2TL_H(1+L_\varrho)\right]\|z-z^*\|^2_{\mathfrak{B}^0_T}.
\end{aligned}
$$

Thus for all $t\in[0,T]$, we obtain that

$$
\begin{aligned}
&\mathbb{E}\|(\mathbb{Q}z)(t)-(\mathbb{Q}z^*)(t)\|^2\\
&\leq\ 6\sup_{1\leq i\leq m}\left[M^2\{1+\mathcal{D}_i+L_G(1+L_g)\}+L_G(1+L_g)+M^2T^2L_F(1+L_f)\right.\\
&\quad \left.+M^2TL_H(1+L_\varrho)\right]\|z-z^*\|^2_{\mathfrak{B}^0_T}.
\end{aligned}
$$

By the condition (3.1), we conclude that $\mathbb{Q}$ is a contraction map and therefore $\mathbb{Q}$ has a unique fixed point $z \in \mathfrak{B}_T^0$ which is just a unique mild solution for system (1.1) on $(-\infty, T]$. This concludes the proof. □

Remark 3.2. The condition (3.1) requires that L_G be a small enough positive number. For example, if G is constant, L_G can take any positive value. Moreover if $G(t,u,v) = \alpha v$ with α be a small enough positive number, we can take $L_G = \alpha^2$.

Now, we establish our second existence result. Sufficient condition for existence of the mild solution for the system (1.1) is obtained by using Krasnoselskii–Schaefer fixed point theorem and compact resolvent operator. Particularly, the nonlinear functions F, f, H and ϱ are continuous functions without forcing Lipschitz condition. Consequently, our result have more useful applications in this area. For proving the result, we have to assume the following assumptions.

(B6) The map $f(t,s,\cdot) : \mathfrak{B}_h \to X$ is continuous for each $(t,s) \in D_1$ and the map $f(\cdot,\cdot,u) : D_1 \to X$ is measurable for each $u \in \mathfrak{B}_h$. There is a constant $L_f > 0$ such that

$$\mathbb{E}\|f(t,s,u)\|^2 \leq L_f \mathcal{W}\left(\mathbb{E}\|u\|^2_{\mathfrak{B}_h}\right),$$

for each $u \in \mathfrak{B}_h$ where $\mathcal{W} : [0,\infty) \to [0,\infty)$ and a continuous nondecreasing function.

(B7) The function $F : [0,T] \times \mathfrak{B}_h \times X \to X$ is a continuous function such that

(i) $t \mapsto F(t,u,v)$ is measurable for each $(u,v) \in \mathfrak{B}_h \times X$;

(ii) $(u,v) \mapsto F(t,u,v)$ is continuous function for almost all $t \in [0,T]$.

(iii) There exists a continuous function $m_F : [0,\infty) \to [0,\infty)$ and a continuous increasing function $\mathcal{K} : [0,\infty) \to [0,\infty)$ such that

$$\mathbb{E}\|F(t,u,v)\|^2_X \leq m_F(t)\mathcal{K}\left(\|u\|^2_{\mathfrak{B}_h} + \mathbb{E}\|v\|^2_X\right),$$

for almost all $t \in [0,T]$, $u \in \mathfrak{B}_h$, $v \in X$.

(B8) $I_i : X \to X$, $i = 1, \cdots, m$ are completely continuous functions such that

$$\Psi = \max_{1\leq i\leq m} \mathbb{E}\|I_i(u)\|^2_X, \qquad \forall u \in X, i = 1, \cdots, m.$$

(B9) There is a compact set $\Omega' \subseteq X$ such that

$$R(t-s)F(s,u,v), R(t-s)H(s,u,v), R(t-s)h(s) \in \Omega'$$

for all $u \in \mathfrak{B}_h$, $v \in X$ and $0 \leq s \leq t \leq T$.

(B10) (i) There is a continuous function $m_H(\cdot) \in L^1_{loc}([0,T],\mathbb{R}^+)$ and an increasing function $\mathcal{W}_H : \mathbb{R} \to (0,\infty)$ such that

$$\mathbb{E}\|H(t,u,v)\|_X^2 \leq m_H(t)\mathcal{W}_H\left(\|u\|_{\mathfrak{B}_h}^2 + \mathbb{E}\|v\|_X^2\right),$$

for almost all $t \in [0,T]$, $u \in \mathfrak{B}_h$, $v \in X$.

(ii) The map $\varrho(t,s,\cdot) : \mathfrak{B}_h \to X$ is continuous for each $(t,s) \in D_1$ and map $\varrho(\cdot,\cdot,u) : D_1 \to X$ is measurable for each $u \in \mathfrak{B}_h$. There is a constant $m_\varrho > 0$ such that

$$\mathbb{E}\|\varrho(t,s,u)\|^2 \leq m_\varrho \mathcal{W}_\varrho\left(\|u\|_{\mathfrak{B}_h}^2\right)$$

for each $u \in \mathfrak{B}_h$, where $\mathcal{W}_\varrho : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function.

(B11) $\displaystyle\int_0^T \bar{\kappa}(s)ds \leq \int_{\xi(0)}^{\infty} \frac{1}{\mathcal{K}(s) + \mathcal{W}_H(s) + \mathbb{W}(s)}ds < \infty$, where

$$\xi(0) = \frac{1}{1-\mathfrak{F}}\left\{2l^2\|\phi\|_{\mathfrak{B}_h}^2 + \frac{2l^2\mathcal{F}}{1-6M^2}\right\},$$
$$\mathfrak{L} = \max\{L_f, L_\varrho\}, \quad \mathbb{W}(x) = \max\{\mathcal{W}(x) + \mathcal{W}_\varrho(x)\},$$
$$\bar{\kappa}(t) = \max\{m^*(t), T\mathfrak{L}\},$$
$$\mathcal{F} = \max\{\mathcal{F}_1, \mathcal{F}_2\},$$
$$\mathcal{F}_1 = 8M^2(L_G\|\phi\|_{\mathfrak{B}_h}^2 + C_1) + 16L_GC_2 + 8C_1,$$
$$\mathcal{F}_2 = 6M^2\Psi + 24M^2L_GC_2 + 12M^2C_1 + 24L_GC_2 + 12C_1,$$
$$m^*(t) = \max\left\{\frac{1}{1-\mathfrak{F}}10l^2M^2Tm_F(t), \frac{1}{1-\mathfrak{F}}10l^2M^2\mathrm{Tr}(Q)m_H(t)\right\},$$
$$2(M^2+1)L_G(1+L_g) < 1,$$
$$\mathfrak{F} = \frac{24l^2}{1-6M^2}(M+1)L_G(1+2L_g) < 1,$$
$$(1-6M^2) < 1.$$

To this end, we introduce the decomposition of operator $\mathbb{Q}$ such that $(\Phi_1 z)(t) =$

$$\begin{cases} 0, \quad t \in (-\infty, 0] \\ R(t)(-G(0,\phi,0)) + G\left(t, y_t + \hat{z}_t, \int_0^t g(t,s,y_s+\hat{z}_s)ds\right), \quad t \in [0,t_1] \\ -R(t-t_1)G\left(t_1, y_{t_1^+} + \hat{z}_{t_1^+}, \int_0^{t_1} g(t_1,s,y_s+\hat{z}_s)ds\right) \\ +G\left(t, y_t + \hat{z}_t, \int_0^t g(t,s,y_s+\hat{z}_s)ds\right), \quad t \in (t_1,t_2] \\ \vdots \quad \vdots \quad \vdots \\ -R(t-t_m)G\left(t_m, y_{t_m^+} + \hat{z}_{t_m^+}, \int_0^{t_m} g(t_m,s,y_s+\hat{z}_s)ds\right) \\ +G\left(t, y_t + \hat{z}_t, \int_0^t g(t,s,y_s+\hat{z}_s)ds\right), \quad t \in (t_m,T] \end{cases}$$

and $(\Phi_2 z)(t) =$

$$\begin{cases} \int_0^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad + \int_0^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t \in [0,t_1] \\ R(t-t_1)[y(t_1^-)+\hat{z}(t_1^-)+I_1(y(t_1^-)+\hat{z}(t_1^-))] \\ + \int_{t_1}^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad + \int_{t_1}^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t \in (t_1,t_2] \\ \vdots \quad \vdots \quad \vdots \\ R(t-t_m)[y(t_m^-)+\hat{z}(t_m^-)+I_1(y(t_m^-)+\hat{z}(t_m^-))] \\ + \int_{t_m}^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad + \int_{t_m}^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t \in (t_m,T]. \end{cases}$$

Lemma 3.3. *Let us assume that **(H1)–(H2)**, $(B1)$–$(B2)$ and $(B6)$–$(B11)$ are fulfilled. Then, Φ_1 is a contraction and Φ_2 is completely continuous on $(-\infty, T]$.*

Proof. Let $q > 0$ and $\mathfrak{B}_q = \{y \in \mathfrak{B}_T^0 : \|y\|_{\mathfrak{B}_T^0} \leq q\}$. Clearly, $\mathfrak{B}_q$ a bounded convex set in $\mathfrak{B}_T^0$. For $\hat{z} \in \mathfrak{B}_q$ and by Lemma 2.10, we have that

$$\begin{aligned} \|y_t + \hat{z}_t\|_{\mathfrak{B}_h}^2 &\leq 2\left(\|y_t\|_{\mathfrak{B}_h}^2 + \|\hat{z}_t\|_{\mathfrak{B}_h}^2\right) \\ &\leq 4\left(l^2 \sup_{\tau\in[0,t]} \mathbb{E}\|y(\tau)\|_X^2 + \|y_0\|_{\mathfrak{B}_h}^2\right) + 4\left(l^2 \sup_{\tau\in[0,t]} \mathbb{E}\|\hat{z}(\tau)\|_X^2 + \|\hat{z}0\|_{\mathfrak{B}_h}^2\right) \end{aligned}$$

$$\leq \; 4\left(\|\phi\|^2_{\mathfrak{B}_h} + l^2 q\right).$$

To apply Theorem 2.11 for establishing the existence result, we have to show that Φ_1 is a contraction while Φ_2 is a compact operator. To this end, we divide the proof into a few steps.

Step 1. Φ_1 is a contraction on $\mathfrak{B}^0_T$. For $z_1, z_2 \in \mathfrak{B}^0_T$ and $t \in [0, t_1]$, we get

$$\begin{aligned}
&\mathbb{E}\| \left(\Phi_1 z_1\right)(t) - \left(\Phi_1 z_2\right)(t)\|^2_X \\
&\leq \; \mathbb{E}\left\| G\left(t, y_t + (\hat{z_1})_t, \int_0^t g(t,s,y_s + (\hat{z_1})_s)ds\right)\right. \\
&\qquad \left. - G\left(t, y_t + (\hat{z_2})_t, \int_0^t g(t,s,y_s + (\hat{z_2})_s)ds\right)\right\|^2 \\
&\leq \; 2L_G\left(\| (z_1)_t - (z_2)_t \|^2_{\mathfrak{B}_h} + L_g \| (z_1)_t - (z_2)_t \|^2_{\mathfrak{B}_h}\right) \\
&\leq \; \Theta_0 \sup_{t\in[0,T]} \mathbb{E}\| (z_1)(t) - (z_2)(t)\|^2,
\end{aligned}$$

where $\Theta_0 = 2L_G(1 + L_g) < 1$. For $t \in (t_i, t_{i+1}]$, $i = 1, \cdots, m$,

$$\begin{aligned}
&\mathbb{E}\| \left(\Phi_1 z_1\right)(t) - \left(\Phi_1 z_2\right)(t)\|^2_X \\
&\leq \; 2M^2\mathbb{E}\left\| G\left(t, y_t + (\hat{z_1})_t, \int_0^t g(t,s,y_s + (\hat{z_1})_s)ds\right)\right. \\
&\qquad \left. - G\left(t, y_t + (\hat{z_2})_t, \int_0^t g(t,s,y_s + (\hat{z_2})_s)ds\right)\right\|^2 \\
&\qquad + 2\mathbb{E}\left\| G\left(t_i, y_{t_i^+} + (\hat{z_1})_{t_i^+}, \int_0^{t_i} g(t_i,s,y_s + (\hat{z_1})_s)ds\right)\right. \\
&\qquad \left. - G\left(t_i, y_{t_i^+} + (\hat{z_2})_{t_i^+}, \int_0^{t_i} g(t_i,s,y_s + (\hat{z_2})_s)ds\right)\right\|^2 \\
&\leq \; \Theta_i \sup_{t\in[0,T]} \mathbb{E}\| (z_1)(t) - (z_2)(t)\|^2,
\end{aligned}$$

where $\Theta_i = 2(M^2 + 1)L_G(1 + L_g) < 1$. Taking the supremum over t, we obtain

$$\| \left(\Phi_1 z_1\right) - \left(\Phi_1 z_2\right) \|^2_T \leq \Theta \|z_1 - z_2\|^2_T,$$

where $\Theta = \max_{i=1,\cdot,m} \Theta_i < 1$. Then Φ_1 is a contraction on $\mathfrak{B}^0_T$. Next, we show that Φ_2 is completely continuous.

Step 2. We first prove that Φ_2 map bounded sets into bounded sets in $\mathfrak{B}^0_T$. To this end, it sufficient to show that there exists a positive constant U such that for each $z \in \mathfrak{B}_q$ one has $\mathbb{E}\| \left(\Phi_2 z\right)(t)\|^2_T \leq U$. Now, for each $\in \mathfrak{B}_q$ and for $t \in [0, t_1]$,

$$\mathbb{E}\| \left(\Phi_2 z\right)(t)\|^2_X$$

$$
\begin{aligned}
&\leq 2\mathbb{E}\left\|\int_0^t R(t-s)F\left(s,y_s+\hat{z}_s,\int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds\right\|^2\\
&\quad +2\mathbb{E}\left\|\int_0^t R(t-s)H\left(s,y_s+\hat{z}_s,\int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s)\right\|^2_{\mathcal{L}_2^0}\\
&\leq 2\int_0^t \|R(t-s)\|\,ds\int_0^t \|R(t-s)\|\times\\
&\quad \times\mathbb{E}\left\|F\left(s,y_s+\hat{z}_s,\int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right\|^2 ds\\
&\quad +2M^2\mathrm{Tr}(Q)\int_0^t \mathbb{E}\left\|H\left(s,y_s+\hat{z}_s,\int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right\|^2 ds\\
&\leq 2M^2T\int_0^t \mathbb{E}\left\|F\left(s,y_s+\hat{z}_s,\int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right\|^2 ds\\
&\quad +2M^2\mathrm{Tr}(Q)\int_0^t \mathbb{E}\left\|H\left(s,y_s+\hat{z}_s,\int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right\|^2 ds,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\|\left(\Phi_2 z\right)(t)\|_X^2\\
&\leq 2M^2T\int_0^t m_F(s)\mathcal{K}\left(4(\|\phi\|^2_{\mathfrak{B}_h}+ql^2)+\int_0^s TL_f\mathcal{W}(4(\|\phi\|^2_{\mathfrak{B}_h}+ql^2))d\tau\right)ds\\
&\quad +2M^2\mathrm{Tr}(Q)\times\\
&\quad \times\int_0^t m_H(s)\mathcal{W}_H\left(4(\|\phi\|^2_{\mathfrak{B}_h}+ql^2)+\int_0^s Tm_\varrho\mathcal{W}_\varrho(4(\|\phi\|^2_{\mathfrak{B}_h}+ql^2))d\tau\right)ds\\
&:= U_0.
\end{aligned}
$$

For $t\in(t_i,t_{i+1}]$, $i=1,\cdots,m$, we have

$$
\begin{aligned}
&\mathbb{E}\|\left(\Phi_2 z\right)(t)\|_X^2\\
&\leq 4\mathbb{E}\|R(t-t_i)I_i(\hat{z}(t_i^-))\|_X^2+4\mathbb{E}\|R(t-t_i)\hat{z}(t_i^-)\|_X^2\\
&\quad +4\mathbb{E}\left\|\int_{t_i}^t R(t-s)F\left(s,y_s+\hat{z}_s,\int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds\right\|^2\\
&\quad +4\mathbb{E}\left\|\int_{t_i}^t R(t-s)H\left(s,y_s+\hat{z}_s,\int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s)\right\|^2_{\mathcal{L}_2^0}\\
&\leq 4M^2\|z\|^2_{\mathfrak{B}_T^0}+4M^2\mathbb{E}\|I_i(\hat{z}(t_i^-))\|_X^2\\
&\quad +4\int_0^t \|R(t-s)\|\,ds\int_{t_i}^t \|R(t-s)\|\times\\
&\quad \times\mathbb{E}\left\|F\left(s,y_s+\hat{z}_s,\int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right\|^2 ds
\end{aligned}
$$

$$+4M^2\mathrm{Tr}(Q)\int_{t_i}^{t}\mathbb{E}\left\|H\left(s,y_s+\hat{z}_s,\int_0^s\varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right\|^2ds,$$

$$\begin{aligned}
&\mathbb{E}\|\left(\Phi_2 z\right)(t)\|_X^2\\
&\leq\ 3M^2(\Psi+q)+4M^2T\int_{t_i}^{t}\mathbb{E}\left\|F\left(s,y_s+\hat{z}_s,\int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right\|^2ds\\
&\quad+4M^2\mathrm{Tr}(Q)\int_{t_i}^{t}\mathbb{E}\left\|H\left(s,y_s+\hat{z}_s,\int_0^s\varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)\right\|^2ds\\
&\leq\ 4M^2(\Psi+q)\\
&\quad+4M^2T\int_{t_i}^{t}m_F(s)\mathcal{K}\left(4(\|\phi\|^2_{\mathfrak{B}_h}+ql^2)+\int_0^s TL_f\mathcal{W}(4(\|\phi\|^2_{\mathfrak{B}_h}+ql^2))d\tau\right)ds\\
&\quad+4M^2\mathrm{Tr}(Q)\times\\
&\quad\times\int_{t_i}^{t}m_H(s)\mathcal{W}_H\left(4(\|\phi\|^2_{\mathfrak{B}_h}+ql^2)+\int_0^s Tm_\varrho\mathcal{W}_\varrho(4(\|\phi\|^2_{\mathfrak{B}_h}+ql^2))d\tau\right)ds\\
&:=\ U_i.
\end{aligned}$$

Set $U=\max\limits_{i=0,1,\cdots,m}U_i$, for all $t\in[0,T]$. Thus, we have $\mathbb{E}\|\left(\Phi_2 z\right)(t)\|_X^2\leq U$.

Step 3. Φ_2 is continuous. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $\mathfrak{B}_q$ with $z_n\to z\in\mathfrak{B}_q$. By continuity of F, H and I_k, we have

$$F\left(s,y_s+\left(\hat{z_n}\right)_s,\int_0^s f(s,y_\tau+\left(\hat{z_n}\right)_\tau)d\tau\right)\to F\left(s,y_s+\left(\hat{z}\right)_s,\int_0^s f(s,y_\tau+\left(\hat{z}\right)_\tau)d\tau\right),$$

$$H\left(s,y_s+\left(\hat{z_n}\right)_s,\int_0^s \varrho(s,y_\tau+\left(\hat{z_n}\right)_\tau)d\tau\right)\to H\left(s,y_s+\left(\hat{z}\right)_s,\int_0^s \varrho(s,y_\tau+\left(\hat{z}\right)_\tau)d\tau\right),$$

when $n\to\infty$. For $t\in[0,t_1]$, we get

$$\begin{aligned}
&\mathbb{E}\|\left(\Phi_2 z_n\right)(t)-\left(\Phi_2 z\right)(t)\|_X^2\\
&\leq\ 2\mathbb{E}\left\|\int_0^t R(t-s)\left[F\left(s,y_s+\left(\hat{z_n}\right)_s,\int_0^s f(s,y_\tau+\left(\hat{z_n}\right)_\tau)d\tau\right)\right.\right.\\
&\quad\left.\left.-F\left(s,y_s+\left(\hat{z}\right)_s,\int_0^s f(s,\tau,y_\tau+\left(\hat{z}\right)_\tau)d\tau\right)\right]ds\right\|^2\\
&\quad+2\mathbb{E}\left\|\int_0^t R(t-s)\left[H\left(s,y_s+\left(\hat{z_n}\right)_s,\int_0^s \varrho(s,\tau,y_\tau+\left(\hat{z_n}\right)_\tau)d\tau\right)\right.\right.\\
&\quad\left.\left.-H\left(s,y_s+\left(\hat{z}\right)_s,\int_0^s \varrho(s,\tau,y_\tau+\left(\hat{z}\right)_\tau)d\tau\right)\right]dw(s)\right\|^2_{\mathcal{L}_2^0},
\end{aligned}$$

$$\mathbb{E}\|\left(\Phi_2 z_n\right)(t)-\left(\Phi_2 z\right)(t)\|_X^2$$

$$
\begin{aligned}
&\leq 2M^2T\int_0^t \mathbb{E}\left\|\left[F\left(s,y_s+(\hat{z_n})_s,\int_0^s f(s,y_\tau+(\hat{z_n})_\tau)d\tau\right)\right.\right.\\
&\quad\left.\left.-F\left(s,y_s+(\hat{z})_s,\int_0^s f(s,\tau,y_\tau+(\hat{z})_\tau)d\tau\right)\right]\right\|^2 ds\\
&\quad+2\mathrm{Tr}(Q)\int_0^t \|R(t-s)\|^2\mathbb{E}\left\|\left[H\left(s,y_s+(\hat{z_n})_s,\int_0^s \varrho(s,\tau,y_\tau+(\hat{z_n})_\tau)d\tau\right)\right.\right.\\
&\quad\left.\left.-H\left(s,y_s+(\hat{z})_s,\int_0^s \varrho(s,\tau,y_\tau+(\hat{z})_\tau)d\tau\right)\right]\right\|^2 ds\\
&\to 0, \qquad \text{as} \quad n\to\infty.
\end{aligned}
$$

For $t\in(t_i,t_{i+1}]$, $i=1,\cdots,m$,

$$
\begin{aligned}
&\mathbb{E}\|\left(\Phi_2 z_n\right)(t)-\left(\Phi_2 z\right)(t)\|_X^2\\
&\leq 4\mathbb{E}\|R(t-t_i)[I_i(\hat{z_n}(t_i^-))-I_i(\hat{z}(t_i^-))]\|_X^2+4\mathbb{E}\|R(t-t_i)[\hat{z_n}(t_i^-)-\hat{z}(t_i^-)]\|_X^2\\
&\quad+4\mathbb{E}\left\|\int_{t_i}^t R(t-s)\left[F\left(s,y_s+(\hat{z_n})_s,\int_0^s f(s,y_\tau+(\hat{z_n})_\tau)d\tau\right)\right.\right.\\
&\quad\left.\left.-F\left(s,y_s+(\hat{z})_s,\int_0^s f(s,\tau,y_\tau+(\hat{z})_\tau)d\tau\right)\right]ds\right\|^2\\
&\quad+4\mathbb{E}\left\|\int_{t_i}^t R(t-s)\left[H\left(s,y_s+(\hat{z_n})_s,\int_0^s \varrho(s,\tau,y_\tau+(\hat{z_n})_\tau)d\tau\right)\right.\right.\\
&\quad\left.\left.-H\left(s,y_s+(\hat{z})_s,\int_0^s \varrho(s,\tau,y_\tau+(\hat{z})_\tau)d\tau\right)\right]dw(s)\right\|_{\mathcal{L}_2^0}^2\\
&\leq 4\mathbb{E}\|R(t-t_i)[\hat{z_n}(t_i^-)-\hat{z}(t_i^-)]\|_X^2\\
&\quad+4M^2\mathbb{E}\|I_i(y(t_i^-)+\hat{z_n}(t_i^-))-I_i(y(t_i^-)+\hat{z}(t_i^-))\|_X^2\\
&\quad+4M^2T\int_{t_i}^t \mathbb{E}\left\|\left[F\left(s,y_s+(\hat{z_n})_s,\int_0^s f(s,y_\tau+(\hat{z_n})_\tau)d\tau\right)\right.\right.\\
&\quad\left.\left.-F\left(s,y_s+(\hat{z})_s,\int_0^s f(s,\tau,y_\tau+(\hat{z})_\tau)d\tau\right)\right]\right\|^2 ds\\
&\leq 4M^2\mathbb{E}\|\hat{z_n}(t_i^-)-\hat{z}(t_i^-)\|_X^2\\
&\quad+4M^2\mathbb{E}\|I_i(y(t_i^-)+\hat{z_n}(t_i^-))-I_i(y(t_i^-)+\hat{z}(t_i^-))\|_X^2\\
&\quad+4\mathrm{Tr}(Q)\int_{t_i}^t \|R(t-s)\|^2\mathbb{E}\left\|\left[H\left(s,y_s+(\hat{z_n})_s,\int_0^s \varrho(s,\tau,y_\tau+(\hat{z_n})_\tau)d\tau\right)\right.\right.\\
&\quad\left.\left.-H\left(s,y_s+(\hat{z})_s,\int_0^s \varrho(s,\tau,y_\tau+(\hat{z})_\tau)d\tau\right)\right]\right\|^2 ds\\
&\to 0 \qquad \text{as} \quad n\to\infty.
\end{aligned}
$$

Thus, we deduce that

$$
\mathbb{E}\|\left(\Phi_2 z_n\right)(t)-\left(\Phi_2 z\right)(t)\|_X^2\to 0 \qquad \text{as} \quad n\to\infty
$$

for all $t \in [0,T]$. Hence, Φ_2 is continuous on $\mathfrak{B}_q$.

Step 4. Φ_2 maps bounded sets into equicontinuous sets of $\mathfrak{B}_q$. Let $\tau_1, \tau_2 \in (t_i, t_{i+1}]$, $i = 1, \cdots, m$ with $\tau_2 > \tau_1$. For $z \in \mathfrak{B}_q$,

$$
\begin{aligned}
&\mathbb{E}\| \left(\Phi_2 z\right)(\tau_2) - \left(\Phi_2 z\right)(\tau_1)\|_X^2 \\
&\leq \; 8\mathbb{E}\|[R(\tau_2 - t_i) - R(\tau_t - t_i)]\hat{z}(t_i^-))\|_X^2 \\
&+ \; 8\mathbb{E}\|[R(\tau_2 - t_i) - R(\tau_t - t_i)]I_i(y(t_i^-) + \hat{z}(t_i^-))\|_X^2 \\
&\quad +8\mathbb{E}\left\| \int_{t_i}^{\tau_1-\epsilon} [R(\tau_2 - s) - R(\tau_1 - s)] \times \right. \\
&\qquad \left. \times F\left(s, y_s + (\hat{z})_s, \int_0^s f(s, y_\tau + (\hat{z})_\tau)d\tau\right) ds \right\|^2 \\
&\quad +8\mathbb{E}\left\| \int_{\tau_1-\epsilon}^{\tau_1} [R(\tau_2 - s) - R(\tau_1 - s)] \times \right. \\
&\qquad \left. \times F\left(s, y_s + (\hat{z})_s, \int_0^s f(s, y_\tau + (\hat{z})_\tau)d\tau\right) ds \right\|^2 \\
&\quad +8\mathbb{E}\left\| \int_{\tau_1}^{\tau_2} R(\tau_2 - s) F\left(s, y_s + (\hat{z})_s, \int_0^s f(s, y_\tau + (\hat{z})_\tau)d\tau\right) ds \right\|^2 \\
&\quad +8\mathbb{E}\left\| \int_{t_i}^{\tau_1-\epsilon} [R(\tau_2 - s) - R(\tau_1 - s)] \times \right. \\
&\qquad \left. \times H\left(s, y_s + (\hat{z})_s, \int_0^s \varrho(s, y_\tau + (\hat{z})_\tau)d\tau\right) dw(s) \right\|^2 \\
&\quad +8\mathbb{E}\left\| \int_{\tau_1-\epsilon}^{\tau_1} [R(\tau_2 - s) - R(\tau_1 - s)] \times \right. \\
&\qquad \left. \times H\left(s, y_s + (\hat{z})_s, \int_0^s \varrho(s, y_\tau + (\hat{z})_\tau)d\tau\right) dw(s) \right\|^2 \\
&\quad +8\mathbb{E}\left\| \int_{\tau_1}^{\tau_2} R(\tau_2 - s) H\left(s, y_s + (\hat{z})_s, \int_0^s \varrho(s, y_\tau + (\hat{z})_\tau)d\tau\right) ds \right\|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\| \left(\Phi_2 z\right)(\tau_2) - \left(\Phi_2 z\right)(\tau_1)\|_X^2 \\
&\leq \; 8\|R(\tau_2 - t_i) - R(\tau_t - t_i)\|^2\mathbb{E}\|\hat{z}(t_i^-))\|_X^2 \\
&\quad +8\|R(\tau_2 - t_i) - R(\tau_t - t_i)\|^2\mathbb{E}\|I_i(\hat{z}(t_i^-))\|_X^2 \\
&\quad +8\int_{t_i}^{\tau_1-\epsilon} \|R(\tau_2 - s) - R(\tau_1 - s)\| ds \times \\
&\qquad \times \int_{t_i}^{\tau_1-\epsilon} \|R(\tau_2 - s) - R(\tau_1 - s)\| \times \\
&\qquad \times \mathbb{E}\left\| F\left(s, y_s + (\hat{z})_s, \int_0^s f(s, y_\tau + (\hat{z})_\tau)d\tau\right) \right\|^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&+8\int_{\tau_1-\epsilon}^{\tau_1}\|R(\tau_2-s)-R(\tau_1-s)\|\,ds\times\\
&\quad\times\int_{\tau_1-\epsilon}^{\tau_1}\|R(\tau_2-s)-R(\tau_1-s)\|\times\\
&\quad\times\mathbb{E}\left\|F\left(s,y_s+(\hat{z})_s,\int_0^s f(s,y_\tau+(\hat{z})_\tau)d\tau\right)\right\|^2 ds\\
&+8\int_{\tau_1}^{\tau_2}\|R(\tau_2-s)\|ds\int_{\tau_1}^{\tau_2}\|R(\tau_2-s)\|\times\\
&\quad\times\mathbb{E}\left\|F\left(s,y_s+(\hat{z})_s,\int_0^s f(s,y_\tau+(\hat{z})_\tau)d\tau\right)\right\|^2 ds\\
&+8\mathrm{Tr}(Q)\int_{t_i}^{\tau_1-\epsilon}\|R(\tau_2-s)-R(\tau_1-s)\|^2\times\\
&\quad\times\mathbb{E}\left\|H\left(s,y_s+(\hat{z})_s,\int_0^s \varrho(s,y_\tau+(\hat{z})_\tau)d\tau\right)\right\|^2 ds\\
&+8\mathrm{Tr}(Q)\int_{t_1-\epsilon}^{\tau_1}\|R(\tau_2-s)-R(\tau_1-s)\|^2\times\\
&\quad\times\mathbb{E}\left\|H\left(s,y_s+(\hat{z})_s,\int_0^s \varrho(s,y_\tau+(\hat{z})_\tau)d\tau\right)\right\|^2 ds\\
&+8\mathrm{Tr}(Q)\int_{\tau_1}^{\tau_2}\|R(\tau_2-s)\|^2\times\\
&\quad\times\mathbb{E}\left\|H\left(s,y_s+(\hat{z})_s,\int_0^s \varrho(s,y_\tau+(\hat{z})_\tau)d\tau\right)\right\|^2 ds.
\end{aligned}
$$

From the above inequalities, we see that the right-hand side of

$$\mathbb{E}\|\left(\Phi_2 z\right)(\tau_2)-\left(\Phi_2 z\right)(\tau_1)\|_X^2$$

tends to zero independent of $z\in\mathfrak{B}_q$ as $\tau_2-\tau_1\to 0$ with ε sufficiently small, since the compactness of $R(t)$ for $t>0$ implies the continuity in the uniform operator topology. Thus, the set $\{\Psi_1 z : z\in\mathfrak{B}_q\}$ is equicontinuous.

Step 5. Φ_2 maps $\mathfrak{B}_q$ into precompact subset of $\mathfrak{B}_T^0$. It is obvious that

$$\mathcal{V}(0)=\{\Phi_2(0)\}$$

is relatively compact in X. For $t\in(0,T]$, we decompose Φ_2 as

$$\Phi_2=\Psi_1+\Psi_2,$$

where $(\Psi_1 z)(t)$ is given by

$$\begin{cases} \int_0^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad + \int_0^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t\in[0,t_1] \\ \int_{t_1}^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad + \int_{t_1}^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t\in(t_1,t_2] \\ \vdots \quad \vdots \quad \vdots \\ \int_{t_m}^t R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad + \int_{t_m}^t R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t\in(t_m,T] \end{cases}$$

and $(\Psi_2 z)(t)$ is given by

$$\begin{cases} 0, \quad t\in[0,t_1] \\ R(t-t_1)[\hat{z}(t_1^-)+I_1(\hat{z}(t_1^-))], \quad t\in(t_1,t_2] \\ \vdots \quad \vdots \quad \vdots \\ R(t-t_m)[\hat{z}(t_m^-)+I_m(\hat{z}(t_m^-))], \quad t\in(t_m,T]. \end{cases}$$

Let $0<t\le s\le T$ fixed and let ε be a real number satisfying $\varepsilon\in(0,t)$. For $z\in B_q$, we define the operators $(\Psi_1^{*\varepsilon} z)(t) =$

$$\begin{cases} R(\varepsilon)\int_0^{t-\varepsilon} R(t-s-\varepsilon)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad + R(\varepsilon)\int_0^{t-\varepsilon} R(t-s-\varepsilon)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \\ \quad t\in[0,t_1] \\ R(\varepsilon)\int_{t_1}^{t-\varepsilon} R(t-s-\varepsilon)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad + R(\varepsilon)\int_{t_1}^{t-\varepsilon} R(t-s-\varepsilon)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \\ \quad t\in(t_1,t_2] \\ \vdots \quad \vdots \quad \vdots \\ R(\varepsilon)\int_{t_m}^{t-\varepsilon} R(t-s-\varepsilon)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad + R(\varepsilon)\int_{t_m}^{t-\varepsilon} R(t-s-\varepsilon)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \\ \quad t\in(t_m,T] \end{cases}$$

and $(\Psi_1^\varepsilon z)(t) =$

$$\begin{cases} \displaystyle\int_0^{t-\varepsilon} R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad \displaystyle+\int_0^{t-\varepsilon} R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t\in[0,t_1] \\ \displaystyle\int_{t_1}^{t-\varepsilon} R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad \displaystyle+\int_{t_1}^{t-\varepsilon} R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t\in(t_1,t_2] \\ \vdots \quad \vdots \quad \vdots \\ \displaystyle\int_{t_m}^{t-\varepsilon} R(t-s)F\left(s, y_s+\hat{z}_s, \int_0^s f(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)ds \\ \quad \displaystyle+\int_{t_m}^{t-\varepsilon} R(t-s)H\left(s, y_s+\hat{z}_s, \int_0^s \varrho(s,\tau,y_\tau+\hat{z}_\tau)d\tau\right)dw(s), \quad t\in(t_m,T]. \end{cases}$$

By the compactness of the operators $R(t)$, the set $U_\varepsilon^*(t) = \{(\Psi_1^{*\varepsilon}z)(t); z \in B_q\}$ is relatively compact in X, for every ε, $\varepsilon \in (0,t)$. One has for $t\in[0,t_1]$,

$$\begin{aligned} &\mathbb{E}\|(\Psi_1^\varepsilon z)(t)-(\Psi_1^{*\varepsilon}z)(t)\|^2 \\ &\leq\ 2\mathbb{E}\left\|\int_0^{t-\varepsilon}[R(t-s)-R(\varepsilon)R(t-s-\varepsilon)]\times\right. \\ &\qquad \left.\times F\left(s, y_s+(\hat{z})_s, \int_0^s f(s,\tau,y_\tau+(\hat{z})_\tau)d\tau\right)ds\right\|^2 \\ &\qquad +2\mathbb{E}\left\|\int_0^{t-\varepsilon}[R(t-s)-R(\varepsilon)R(t-s-\varepsilon)]\times\right. \\ &\qquad \left.\times H\left(s, y_s+(\hat{z})_s, \int_0^s \varrho(s,\tau,y_\tau+(\hat{z})_\tau)d\tau\right)dw(s)\right\|^2_{\mathcal{L}_2^0} \\ &\leq\ 2\int_0^{t-\varepsilon}\|R(t-s)-R(\varepsilon)R(t-s-\varepsilon)\|^2\times \\ &\qquad \times m_F(s)\mathcal{K}\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)+\int_0^s TL_f\mathcal{W}\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)\right)d\tau\right)ds \\ &\qquad +2\mathrm{Tr}(Q)\int_0^{t-\varepsilon}\|R(t-s)-R(\varepsilon)R(t-s-\varepsilon)\|^2\times \\ &\qquad \times m_H(s)\mathcal{W}_H\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)+\int_0^s TL_\varrho\mathcal{W}_\varrho\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)\right)d\tau\right)ds, \end{aligned}$$

$$\begin{aligned} &\mathbb{E}\|(\Psi_1^\varepsilon z)(t)-(\Psi_1^{*\varepsilon}z)(t)\|^2 \\ &\leq\ \left[2\sup_{t\in[0,T]} m_F(t)\mathcal{K}\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)+T^2L_f\mathcal{W}\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)\right)\right)\right]\times \end{aligned}$$

$$
\begin{aligned}
&\times \int_0^{t-\varepsilon} \|R(t-s) - R(\varepsilon)R(t-s-\varepsilon)\|^2 ds \\
&+ \left[2\mathrm{Tr}(Q) \sup_{t\in[0,T]} m_H(t) \mathcal{W}_H \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) + T^2 L_\varrho \mathcal{W}_\varrho \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) \right) \right) \right] \times \\
&\times \int_0^{t-\varepsilon} \|R(t-s) - R(\varepsilon)R(t-s-\varepsilon)\|^2 ds \\
\leq\ & P \int_0^{t-\varepsilon} \|R(t-s) - R(\varepsilon)R(t-s-\varepsilon)\|^2 ds, \qquad (3.3)
\end{aligned}
$$

where

$$
\begin{aligned}
P \ =\ & 2 \sup_{t\in[0,T]} m_F(t) \mathcal{K} \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) + T^2 L_f \mathcal{W} \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) \right) \right) \\
& + 2\mathrm{Tr}(Q) \sup_{t\in[0,T]} m_H(t) \mathcal{W}_H \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) + T^2 L_\varrho \mathcal{W}_\varrho \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) \right) \right).
\end{aligned}
$$

For $t \in (t_i, t_{i+1}]$, $i = 1, \cdots, m$,

$$
\begin{aligned}
&\mathbb{E}\|(\Psi_1^\varepsilon z)(t) - (\Psi_1^{*\varepsilon} z)(t)\|^2 \\
&\leq\ 2\mathbb{E} \left\| \int_{t_i}^{t-\varepsilon} [R(t-s) - R(\varepsilon)R(t-s-\varepsilon)] \times \right. \\
&\qquad \left. \times F\left(s, y_s + (\hat{z})_s, \int_0^s f(s,\tau, y_\tau + (\hat{z})_\tau) d\tau \right) ds \right\|^2 \\
&\quad + 2\mathbb{E} \left\| \int_{t_i}^{t-\varepsilon} [R(t-s) - R(\varepsilon)R(t-s-\varepsilon)] \times \right. \\
&\qquad \left. \times H\left(s, y_s + (\hat{z})_s, \int_0^s \varrho(s,\tau, y_\tau + (\hat{z})_\tau) d\tau \right) dw(s) \right\|^2_{\mathcal{L}_2^0} \\
&\leq\ 2 \int_{t_i}^{t-\varepsilon} \|R(t-s) - R(\varepsilon)R(t-s-\varepsilon)\|^2 \times \\
&\qquad \times m_F(s) \mathcal{K} \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) + \int_0^s T L_f \mathcal{W} \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) \right) d\tau \right) ds \\
&\quad + 2\mathrm{Tr}(Q) \int_{t_i}^{t-\varepsilon} \|R(t-s) - R(\varepsilon)R(t-s-\varepsilon)\|^2 \times \\
&\qquad \times m_H(s) \mathcal{W}_H \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) + \int_0^s T L_\varrho \mathcal{W}_\varrho \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) \right) d\tau \right) ds \\
&\leq\ \left[2 \sup_{t\in[0,T]} m_F(t) \mathcal{K} \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) + T^2 L_f \mathcal{W} \left(4\left(\|\phi\|^2_{\mathfrak{B}_h} + ql^2 \right) \right) \right) \right] \times \\
&\qquad \times \int_{t_i}^{t-\varepsilon} \|R(t-s) - R(\varepsilon)R(t-s-\varepsilon)\|^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&+\left[2\mathrm{Tr}(Q)\sup_{t\in[0,T]} m_H(t)\mathcal{W}_H\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)+T^2L_\varrho\mathcal{W}\varrho\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)\right)\right)\right]\\
&\quad\times\int_{t_i}^{t-\varepsilon}\|R(t-s)-R(\varepsilon)R(t-s-\varepsilon)\|^2ds\\
\leq\ & P\int_{t_i}^{t-\varepsilon}\|R(t-s)-R(\varepsilon)R(t-s-\varepsilon)\|^2ds. \qquad (3.4)
\end{aligned}
$$

By Lemma 2.8, the right-hand sides of the above inequalities (3.3) and (3.4) tend to zero as $\varepsilon\to 0$. So the set $U_\varepsilon(t)=\{(\Psi_1^\varepsilon z)(t); z\in B_q\}$ is precompact in X by using the total boundedness. We have

$$
\begin{aligned}
&\mathbb{E}\|(\Psi_1 z)(t)-(\Psi_1^\varepsilon z)(t)\|^2\\
&\leq\ 2\mathbb{E}\left\|\int_{t-\varepsilon}^{t}R(t-s)F\left(s,y_s+z_s,\int_0^s f(s,y_\tau+z_\tau)d\tau\right)ds\right\|^2\\
&\quad+\mathbb{E}\left\|\int_{t-\varepsilon}^{t}R(t-s)H\left(s,y_s+z_s,\int_0^s\varrho(s,\tau,y_\tau+z_\tau)d\tau\right)dw(s)\right\|^2\\
&\leq\ 2M^2\varepsilon\int_{t-\varepsilon}^{t}m_F(s)\mathcal{K}\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)+\int_0^s TL_f\mathcal{W}\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)\right)d\tau\right)ds\\
&\quad+2M^2\mathrm{Tr}(Q)\times\\
&\quad\times\int_{t-\varepsilon}^{t}m_H(s)\mathcal{W}_H\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)+\int_0^s TL_\varrho\mathcal{W}_\varrho\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)\right)d\tau\right)ds. \qquad (3.5)
\end{aligned}
$$

For $t\in(t_i,t_{i+1}]$, $i=1,\cdots,m$,

$$
\begin{aligned}
&\mathbb{E}\|(\Psi_1 z)(t)-(\Psi_1^\varepsilon z)(t)\|^2\\
&\leq\ 2\mathbb{E}\left\|\int_{t-\varepsilon}^{t}R(t-s)F\left(s,y_s+z_s,\int_0^s f(s,y_\tau+z_\tau)d\tau\right)\right\|^2\\
&\quad+\mathbb{E}\left\|\int_{t-\varepsilon}^{t}R(t-s)H\left(s,y_s+z_s,\int_0^s\varrho(s,\tau,y_\tau+z_\tau)d\tau\right)dw(s)\right\|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\|(\Psi_1 z)(t)-(\Psi_1^\varepsilon z)(t)\|^2\\
&\leq\ 2M^2\varepsilon\int_{t-\varepsilon}^{t}m_F(s)\mathcal{K}\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)+\int_0^s TL_f\mathcal{W}\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)\right)d\tau\right)ds\\
&\quad+2M^2\mathrm{Tr}(Q)\times\\
&\quad\times\int_{t-\varepsilon}^{t}m_H(s)\mathcal{W}_H\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)+\int_0^s TL_\varrho\mathcal{W}_\varrho\left(4\left(\|\phi\|^2_{\mathfrak{B}_h}+ql^2\right)\right)d\tau\right)ds. \qquad (3.6)
\end{aligned}
$$

The right-hand sides of the above inequalities (3.5) and (3.6) tend to zero as $\varepsilon \to 0$. Thus, there are relatively compact sets arbitrarily close to the set $U(t) = \{(\Psi_1 z)(t); z \in B_q\}$ and $U(t)$ is relatively compact in X. It is not difficult to show that $\Psi_1(B_q)$ is uniformly bounded. Since Φ_2 is equicontinuous. Thus, by the Arzelà–Ascoli theorem, we deduce that Ψ_1 is compact. Next, we show that $\Psi_2(\mathfrak{B}_q)(t)$ is relatively compact for every $t \in [0,T]$. For $t \in [0,t_1]$, it is obvious. Now for $t \in (t_i, t_{i+1}]$, $i = 1, ..., m$ and $z \in B_q$, we need to show that $U(t) = \{R(t-t_i)I_i(y(t_i^-) - \hat{z}(t_i^-)) :, , z \in \mathfrak{B}_q\}, t \in (t_i, t_{i+1}]$ is relatively compact in $C([t_i, t_{i+1}]; X)$. By the compactness and assumption on I_i , we conclude that the set $\{R(t-t_i)I_i(y(t_i^-) - \hat{z}(t_i^-)), z \in \mathfrak{B}_q\}$ is relatively compact in X. It can be easily shown that the functions in U are equicontinuous. Thus, from the Arzelà–Ascoli theorem, it follows that Ψ_2 is a compact operator. Hence,$\Phi_2 = \Psi_1 + \Psi_2$ is a completely continuous operator. □

Lemma 3.4. *Let us assume that **(H1)–(H2)**, $(B1)$–$(B2)$ and $(B6)$–$(B11)$ are satisfied. Then, there exists a priori bounds $\mathcal{C}$ such that*

$$\|u_t\|^2_{\mathfrak{B}_h} \leq \mathcal{C}, \qquad t \in J,$$

here $\mathcal{C}$ depends only on F, H and the function $\mathcal{K}, \mathcal{W}, \mathcal{W}_H, \mathcal{W}_\varrho$.

Proof. From (2.9), we obtain for $t \in [0, t_1]$,

$$\begin{aligned}
\mathbb{E}\|u(t)\|^2_X &\leq 4M^2\mathbb{E}\|G(0,\phi,0\|^2_X \\
&\quad +4\mathbb{E}\left\|G\left(t,u_t,\int_0^t g(t,s,u_s)ds\right)\right\|^2_X \\
&\quad +4\mathbb{E}\left\|\int_0^t R(t-s)F\left(s,u_s,\int_0^s f(s,\tau,u_\tau)d\tau\right)ds\right\|^2_X \\
&\quad +4\mathbb{E}\left\|\int_0^t R(t-s)H\left(s,u_s,\int_0^s \varrho(s,\tau,u_\tau)d\tau\right)dw(s)\right\|^2_X \\
&\leq 8M^2(L_G\|\phi\|^2_{\mathfrak{B}_h} + C_1) + 8[L_G(\|u_t\|^2_{\mathfrak{B}_h} + 2L_g\|u_t\|^2_{\mathfrak{B}_h} + 2C_2) + C_1] \\
&\quad +4\int_0^t \|R(t-s)\|ds\int_0^t \|R(t-s)\|\mathbb{E}\left\|F\left(s,u_s,\int_0^s f(s,\tau,u_\tau)d\tau\right)\right\|^2_X ds \\
&\quad +4\int_0^t \|R(t-s)\|^2\mathbb{E}\left\|H\left(s,u_s,\int_0^s \varrho(s,\tau,u_\tau)d\tau\right)\right\|^2 ds \\
&\leq 8M^2(L_G\|\phi\|^2_{\mathfrak{B}_h} + C_1) + 8[L_G(\|u_t\|^2_{\mathfrak{B}_h} + 2L_g\|u_t\|^2_{\mathfrak{B}_h} + 2C_2) + C_1] \\
&\quad +4M^2T\int_0^t m_F(s)\mathcal{K}\left[\|u_s\|^2_{\mathfrak{B}_h} + \int_0^s TL_f\mathcal{W}(\|u_s\|^2_{\mathfrak{B}_h})d\tau\right]ds \\
&\quad 4M^2\mathrm{Tr}(Q)\int_0^t m_H(s)\mathcal{W}_H\left[\|u_s\|^2_{\mathfrak{B}_h} + \int_0^s TL_\varrho\mathcal{W}_\varrho(\|u_s\|^2_{\mathfrak{B}_h})d\tau\right]ds.
\end{aligned}$$

Similarly, for $t \in (t_i, t_{i+1}]$, $i = 1, \cdots, m$, we get

$$\mathbb{E}\|u(t)\|^2_X \leq 6M^2\mathbb{E}\|u(t_i^-)\|^2_X + 6M^2\mathbb{E}\|R(t-t_i)I_i(u(t_i^-))\|^2_X$$

$$
\begin{aligned}
&+6\mathbb{E}\left\|R(t-t_i)G\left(t_i,u_{t_i^+},\int_0^{t_i}g(t_i,s,u_s)ds\right)\right\|_X^2\\
&+6\mathbb{E}\left\|G\left(t,u_t,\int_0^{t}g(t,s,u_s)ds\right)\right\|_X^2\\
&+6\mathbb{E}\left\|\int_{t_i}^{t}R(t-s)F\left(s,u_s,\int_0^{s}f(s,\tau,u_\tau)d\tau\right)ds\right\|_X^2\\
&+6\mathbb{E}\left\|\int_{t_i}^{t}R(t-s)H\left(s,u_s,\int_0^{s}\varrho(s,\tau,u_\tau)d\tau\right)dw(s)\right\|_X^2\\
\leq\ & 6M^2\mathbb{E}\|u(t_i^-)\|_X^2+6M^2\Psi+12[L_G(\|u_t\|_{\mathfrak{B}_h}^2+2L_g\|u_t\|_{\mathfrak{B}_h}^2+2C_2)+C_1]\\
&+12M^2[L_G(\|u_t\|_{\mathfrak{B}_h}^2+2L_g\|u_t\|_{\mathfrak{B}_h}^2+2C_2)+C1]\\
&+6\int_{t_i}^{t}\|R(t-s)\|ds\int_{t_i}^{t}\|R(t-s)\|\mathbb{E}\left\|F\left(s,u_s,\int_0^{s}f(s,\tau,u_\tau)d\tau\right)\right\|_X^2ds\\
&+6\int_{t_i}^{t}\|R(t-s)\|^2\mathbb{E}\left\|H\left(s,u_s,\int_0^{s}\varrho(s,\tau,u_\tau)d\tau\right)\right\|^2ds\\
\leq\ & 6M^2\mathbb{E}\|u(t)\|_X^2+6M^2\Psi+12[L_G(\|u_t\|_{\mathfrak{B}_h}^2+2L_g\|u_t\|_{\mathfrak{B}_h}^2+2C_2)+C_1]\\
&+12M^2[L_G(\|u_t\|_{\mathfrak{B}_h}^2+2L_g\|u_t\|_{\mathfrak{B}_h}^2+2C_2)+C1]\\
&+6M^2T\int_{t_i}^{t}m_F(s)\mathcal{K}\left[\|u_s\|_{\mathfrak{B}_h}^2+\int_0^{s}TL_f\mathcal{W}(\|u_s\|_{\mathfrak{B}_h}^2)d\tau\right]ds\\
&6M^2\mathrm{Tr}(Q)\int_{t_i}^{t}m_H(s)\mathcal{W}_H\left[\|u_s\|_{\mathfrak{B}_h}^2+\int_0^{s}TL_\varrho\mathcal{W}_\varrho(\|u_s\|_{\mathfrak{B}_h}^2)d\tau\right]ds.
\end{aligned}
$$

Thus, for all $t\in[0,T]$, we have

$$
\begin{aligned}
\mathbb{E}\|u(t)\|_X^2 \leq\ & \frac{\mathcal{F}}{1-6M^2}+12(M+1)L_G(1+2L_g)\frac{\|u_t\|_{\mathfrak{B}_h}^2}{1-6M^2}\\
&+\frac{6}{1-6M^2}M^2T\int_0^{t}m_F(s)\mathcal{K}\left[\|u_s\|_{\mathfrak{B}_h}^2+\int_0^{s}TL_f\mathcal{W}(\|u_s\|_{\mathfrak{B}_h}^2)d\tau\right]ds\\
&+\frac{6}{1-6M^2}M^2\mathrm{Tr}(Q)\int_0^{t}m_H(s)\mathcal{W}_H\left[\|u_s\|_{\mathfrak{B}_h}^2+\int_0^{s}TL_\varrho\mathcal{W}_\varrho(\|u_s\|_{\mathfrak{B}_h}^2)d\tau\right]ds,
\end{aligned}
$$

where $\mathcal{F}=\max\{\mathcal{F}_1,\mathcal{F}_2\}$ with

$$
\begin{aligned}
\mathcal{F}_1 &= 8M^2(L_G\|\phi\|_{\mathfrak{B}_h}^2+C_1)+16L_GC_2+8C_1\\
\mathcal{F}_2 &= 6M^2\Psi+24M^2L_GC_2+12M^2C_1+24L_GC_2+12C_1.
\end{aligned}
$$

Thus, by Lemma 2.10, we have for every $t\in[0,T]$,

$$
\|u_s\|_{\mathfrak{B}_h}^2 \leq 2l^2\|\phi\|_{\mathfrak{B}_h}^2+2l^2\sup_{s\in[0,t]}\mathbb{E}\|u(s)\|_{\mathfrak{B}_h}^2
$$

$$\begin{aligned}
&\leq 2l^2\|\phi\|^2_{\mathfrak{B}_h}+\frac{2l^2\mathcal{F}}{1-6M^2}+12(M+1)L_G(1+2L_g)\frac{2l^2\|u_t\|^2_{\mathfrak{B}_h}}{1-6M^2}\\
&\quad+\frac{12l^2M^2T}{1-6M^2}\int_0^t m_F(s)\mathcal{K}\left[\|u_s\|^2_{\mathfrak{B}_h}+\int_0^s TL_f\mathcal{W}(\|u_s\|^2_{\mathfrak{B}_h})d\tau\right]ds\\
&\quad+\frac{12l^2M^2\mathrm{Tr}(Q)}{1-6M^2}\int_0^t m_H(s)\mathcal{W}_H\left[\|u_s\|^2_{\mathfrak{B}_h}+\int_0^s TL_\varrho\mathcal{W}_\varrho(\|u_s\|^2_{\mathfrak{B}_h})d\tau\right]ds.
\end{aligned}$$

Let $\nu(t)=\sup_{s\in[0,t]}\|u_s\|^2_{\mathfrak{B}_h}$. Thus the function ν is increasing in $[0,T]$ and we conclude

$$\begin{aligned}
\nu(t) &\leq 2l^2\|\phi\|^2_{\mathfrak{B}_h}+\frac{2l^2\mathcal{F}}{1-6M^2}+12(M+1)L_G(1+2L_g)\frac{2l^2\|u_t\|^2_{\mathfrak{B}_h}}{1-6M^2}\\
&\quad+\frac{12l^2M^2T}{1-6M^2}\int_0^t m_F(s)\mathcal{K}\left[\nu(s)+\int_0^s TL_f\mathcal{W}(\nu(\tau))d\tau\right]ds\\
&\quad+\frac{12l^2M^2\mathrm{Tr}(Q)}{1-6M^2}\int_0^t m_H(s)\mathcal{W}_H\left[\nu(s)+\int_0^s TL_\varrho\mathcal{W}_\varrho(\nu(\tau))d\tau\right]ds,
\end{aligned}$$

$$\begin{aligned}
\nu(t) &\leq \frac{1}{1-\mathfrak{F}}\left\{2l^2\|\phi\|^2_{\mathfrak{B}_h}+\frac{2l^2\mathcal{F}}{1-6M^2}\right.\\
&\quad+\frac{12l^2M^2T}{1-6M^2}\int_0^t m_F(s)\mathcal{K}\left[\nu(s)+\int_0^s TL_f\mathcal{W}(\nu(\tau))d\tau\right]ds\\
&\quad\left.+\frac{12l^2M^2\mathrm{Tr}(Q)}{1-6M^2}\int_0^t m_H(s)\mathcal{W}_H\left[\nu(s)+\int_0^s TL_\varrho\mathcal{W}_\varrho(\nu(\tau))d\tau\right]ds\right\},
\end{aligned}\tag{3.7}$$

where $\mathfrak{F}=\dfrac{24l^2}{1-6M^2}(M+1)L_G(1+2L_g)$. Now, we denote the right-hand side of the inequality (3.7) by $\xi(t)$ and get

$$\nu(t)\leq\xi(t)\qquad\text{for all}\quad t\in[0,T],$$

and $\xi(0)=\dfrac{1}{1-\mathfrak{F}}\left\{2l^2\|\phi\|^2_{\mathfrak{B}_h}+\dfrac{2l^2\mathcal{F}}{1-6M^2}\right\}$. Thus, we have

$$\begin{aligned}
\xi'(t) &= \frac{1}{1-\mathfrak{F}}\left\{2l^2\|\phi\|^2_{\mathfrak{B}_h}+\frac{2l^2\mathcal{F}}{1-6M^2}\right.\\
&\quad+\frac{12l^2M^2T}{1-6M^2}m_F(t)\mathcal{K}\left[\nu(t)+\int_0^t TL_f\mathcal{W}(\nu(\tau))d\tau\right]\\
&\quad\left.+\frac{12l^2M^2\mathrm{Tr}(Q)}{1-6M^2}m_H(t)\mathcal{W}_H\left[\nu(t)+\int_0^t TL_\varrho\mathcal{W}_\varrho(\nu(\tau))d\tau\right]\right\}
\end{aligned}$$

$$\begin{aligned}\leq\ & \frac{1}{1-\mathfrak{F}}\left\{2l^2\|\phi\|^2_{\mathfrak{B}_h}+\frac{2l^2\mathcal{F}}{1-6M^2}\right.\\ &+\frac{12l^2M^2T}{1-6M^2}m_F(t)\mathcal{K}\left[\xi(s)+\int_0^t TL_f\mathcal{W}(\xi(\tau))d\tau\right]\\ &\left.+\frac{12l^2M^2\mathrm{Tr}(Q)}{1-6M^2}m_H(t)\mathcal{W}_H\left[\xi(t)+\int_0^t TL_\varrho\mathcal{W}_\varrho(\xi(\tau))d\tau\right]\right\},\end{aligned}$$

where $m^*(t)=\max\left\{\frac{1}{1-\mathfrak{F}}\frac{12l^2M^2T}{1-6M^2}m_F(t),\frac{1}{1-\mathfrak{F}}\frac{12l^2M^2\mathrm{Tr}(Q)}{1-6M^2}m_H(t)\right\}$. Let

$$\kappa(t)=\xi(t)+\int_0^t T\mathfrak{L}\mathbb{W}(\xi(\tau))d\tau,$$

where $\mathfrak{L}=\max\{L_f,L_\varrho\}$ and $\mathbb{W}(x)=\max\{\mathcal{W}(x)+\mathcal{W}_\varrho(x)\}$. Thus $\kappa(0)=\xi(0)$, $\xi(t)\leq\kappa(t)$ and

$$\begin{aligned}\kappa'(t) &\leq \xi'(t)+T\mathfrak{L}\mathbb{W}(\xi(t))\\ &\leq m^*(t)[\mathcal{K}(\kappa(t))+\mathcal{W}_H(\kappa(t))]+T\mathfrak{L}\mathbb{W}(\kappa(t))\\ &\leq \bar{\kappa}(t)[\mathcal{K}(\kappa(t))+\mathcal{W}_H(\kappa(t))+\mathbb{W}(\kappa(t)],\end{aligned}$$

where $\bar{\kappa}(t)=\max\{m^*(t),T\mathfrak{L}\}$. This gives that

$$\int_{\kappa(0)}^{\kappa(t)}\frac{ds}{\mathcal{K}(\kappa(t))+\mathcal{W}_H(\kappa(t))+\mathbb{W}(\kappa(t)}\leq\int_0^T\bar{\kappa}(s)ds\leq\int_\infty^{\xi(0)}\frac{ds}{\mathcal{K}(\kappa(t))+\mathcal{W}_H(\kappa(t))+\mathbb{W}(\kappa(t)},$$

which demonstrates that the function $\xi(t)<\infty$. Therefore, there exists a constant $\mathcal{C}$ such that $\xi(t)\leq\mathcal{C}$, where $\mathcal{C}$ depends only on F,H and $\mathcal{K},\mathcal{W}_H,\mathcal{W},\mathcal{W}_\varrho,\mathbb{W}$ and $\bar{\kappa}$. □

Theorem 3.5. *Let us assume that hypothesis **(H1)–(H2)**, $(B1)$–$(B2)$ and $(B6)$–$(B11)$ are fulfilled. Then, there exists at least one mild solution for the problem* (1.1) *on* $[0,T]$.

Proof. Let us consider the set

$$\bar{\mathcal{G}}(\mathbb{Q})=\{z\in\mathfrak{B}^0_T:z=\lambda\Phi_1\left(z/\lambda\right)+\lambda\Phi_2(z)\quad\text{for some}\quad\lambda\in(0,1)\}.$$

Thus, for any $z\in\bar{\mathcal{G}}(\mathbb{Q})$ we have $\|u_t\|^2_{\mathfrak{B}_h}\leq\mathcal{C}$, for all $t\in[0,T]$ and hence

$$\begin{aligned}\|z\|^2_T &\leq l^{-2}\mathcal{C}+M\mathbb{E}\|\phi(0)\|^2\\ &= l^{-2}\mathcal{C}.\end{aligned}$$

It implies that $\bar{\mathcal{G}}$ is bounded on $[0,T]$. Hence, by the Krasnoselskii–Schaefer type fixed point theorem, there exists a fixed point $z(\cdot)$ for $\mathbb{Q}$ on $\mathfrak{B}_q$ such that $\mathbb{Q}z(t)=z(t)$. Since $u(t)=y(t)+\hat{z}(t)$, u is a mild solution of the problem (1.1) on $[0,T]$. □

4 An Example

Consider the following impulsive neutral stochastic partial integro-differential equations of the form

$$\begin{cases} \dfrac{\partial}{\partial t}\left[u(t,x)+G_1\left(t,u(t-k,x),\displaystyle\int_0^t g_1(t,s,u(s-k,x))ds\right)\right] \\ =\dfrac{\partial^2}{\partial x^2}\left[u(t,x)-G_1\left(t,u(t-k,x),\displaystyle\int_0^t g_1(t,s,u(s-k,x))ds\right)\right] \\ +\displaystyle\int_0^t b(t-s)\frac{\partial^2}{\partial \xi^2}\left[u(s,x)+G_1\left(s,u(s-k,x),\int_0^s g_1(t,\tau,u(\tau-k,x))d\tau\right)\right]ds \\ +F_1\left(t,u(t-k,x),\displaystyle\int_0^t f_1(t,s,u(s-k,x))ds\right) \\ +H_1\left(t,u(t-k,x),\displaystyle\int_0^t \vartheta_1(t,s,u(s-k,x))ds\right)dw(t), \\ \qquad 0\le t\le b;\quad \tau>0,\quad 0\le x\le\pi \\ u(t,0)=u(t,\pi)=0,\quad t\in[0,T], \\ \Delta u(t,x)|_{t=t_i=}u(t_i^+,x)-u(t_i^-,x)=I_i(u(t_i,x))=\displaystyle\int_{-\infty}^{t_i} d_i(t_i-\tau)u(\tau,x)d\tau, \\ u(t,x)=\rho(t,x),\quad -\infty\le t\le 0,\quad 0\le x\le\pi, \end{cases} \tag{4.1}$$

where ρ is continuous and $I_k \in C(\mathbb{R},\mathbb{R}), w(t)$ denotes a standard cylindrical Wiener process in X defined on a stochastic space $(\Omega,\mathcal{F},P)$ and $X=L^2([0,\pi])$ with the norm $\|\cdot\|$. We define the operators $A:X\to X$ by $Au=u''$ with the domain

$$D(A):=\{u\in X: u,u' \text{ are absolutely continuous}, u''\in X, u(0)=u(\pi)=0\}.$$

Then

$$Au=\sum_{n=1}^{\infty} n^2\langle u,u_n\rangle u_n,\quad u\in D(A),$$

where

$$\left\{u_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx), n=1,2,\dots\right\}$$

is an orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $T(t), t\ge 0$ in X, where

$$T(t)u=\sum_{n=1}^{\infty}\exp(-n^2t)(u,u_n)u_n,\quad u\in X.$$

Let $B:D(A)\subset X\to X$ be the operator defined by

$$B(t)(y)=b(t)Ay,\quad t\ge 0,\quad y\in D(A).$$

Now, we consider $h(t) = e^{2t}$, $t < 0$. Then we get $l = \int_{-\infty}^{0} h(s)ds = 1/2$ and define

$$\|\phi\|_{\mathfrak{B}_h}^2 \int_{-\infty}^{0} h(s) \sup_{s\le\zeta\le 0} \left(E|\phi(\zeta)|^2\right)^{1/2} ds.$$

It is easy to verify that $(\mathfrak{B}_h, \|\|_{\mathfrak{B}_h})$ is a Banach space. Hence for $(t,\phi) \in [0,T] \times \mathfrak{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$, let $u(t)(x) = u(t, x)$. The function $G : [0,T] \times \mathfrak{B}_h \times X \to X$, $F : [0,T] \times \mathfrak{B}_h \times X \to X$, $H : [0,T] \times \mathfrak{B}_h \times \mathcal{L}_2^0 \to X$ are defined by

$$\begin{aligned}
G\left(t,\phi,\int_0^s g(t,s,\phi)ds\right)(x) &= G_1\left(t,\phi(\theta,x),\int_0^s g_1(t,s,\phi)ds\right)\\
&= \int_{-\infty}^{0} \nu_1(\theta)\phi(\theta)(x)d\theta\\
&\quad + \int_0^t \int_{-\infty}^{0} b_2(s)b_3(\tau)\phi(\tau,x)d\tau ds,\\
F\left(t,\phi,\int_0^s f(t,s,\phi)ds\right)(x) &= F_1\left(t,\phi(\theta,x),\int_0^s f_1(t,s,\phi)ds\right)\\
&= \int_{-\infty}^{0} \nu_1(\theta)\tilde{a}_1(t,s,x,\phi(s,x))ds\\
&\quad + \int_0^t \int_{-\infty}^{0} \tilde{b}_2(s)\tilde{a}_3(t,s,x,\phi(s,x))\phi(\tau,x)d\tau ds,\\
H\left(t,\phi,\int_0^s \varrho(t,s,\phi)ds\right)(x) &= H_1\left(t,\phi(\theta,x),\int_0^s \varrho_1(t,s,\phi)ds\right)\\
&= \int_{-\infty}^{0} \nu_1(\theta)\tilde{a}_1(t,s,x,\phi(s,x))ds\\
&\quad + \int_0^t \int_{-\infty}^{0} \tilde{b}_2(s)\tilde{b}_3(t,s,x,\phi(s,x))\phi(\tau,x)d\tau ds,
\end{aligned}$$

$$I_i(t,\phi)(x) = \int_{-\infty}^{0} d_i(-s)\phi(\theta)x ds,$$

where

(1) The function $\nu_1(\theta) \ge 0$ is continuous in $(-\infty, 0]$ satisfying

$$\int_{-\infty}^{0} \nu_1^2(\theta)d\theta < \infty, \qquad \gamma_G^1 = \left(\int_{-\infty}^{0} \frac{(\nu_1(s))^2}{h(s)}\right)^{1/2} < \infty.$$

(2) $b_2, b_3 : \mathbb{R} \to \mathbb{R}$ are continuous, and

$$\gamma_G^2 = \left(\int_{-\infty}^{0} \frac{(b_1(s))^2}{h(s)}\right)^{1/2} < \infty.$$

(3) The function $\tilde{b}_2, \tilde{a}_2 : \mathbb{R} \to \mathbb{R}$ are continuous and there exist continuous functions $r_j, q_j : \mathbb{R} \to \mathbb{R}$ $j = 1, 2, 3, 4$ such that

$$\begin{aligned}
|\tilde{b}_1(t,s,x,y)| &\leq r_1(t)r_2(s)|y|, \quad (t,s,x,y) \in \mathbb{R}^4,\\
|\tilde{b}_3(t,s,x,y)| &\leq r_3(t)r_24(s)|y|, \quad (t,s,x,y) \in \mathbb{R}^4,\\
|\tilde{a}_1(t,s,x,y)| &\leq q_1(t)q_2(s)|y|, \quad (t,s,x,y) \in \mathbb{R}^4,\\
|\tilde{a}_3(t,s,x,y)| &\leq q_3(t)q_4(s)|y|, \quad (t,s,x,y) \in \mathbb{R}^4,
\end{aligned}$$

with $L_1^{\tilde{b}} = \left(\int_{-\infty}^{0} \frac{(r_2(s))^2}{h(s)}\right)^{1/2} < \infty$, $L_2^{\tilde{b}} = \left(\int_{-\infty}^{0} \frac{(r_4(s))^2}{h(s)}\right)^{1/2} < \infty$, $L_1^{\tilde{a}} = \left(\int_{-\infty}^{0} \frac{(q_2(s))^2}{h(s)}\right)^{1/2} < \infty$ and $L_2^{\tilde{a}} = \left(\int_{-\infty}^{0} \frac{(q_4(s))^2}{h(s)}\right)^{1/2} < \infty$.

(4) The functions $d_i \in C(\mathbb{R}, \mathbb{R})$ and $L_{I_i} = \left(\int_{-\infty}^{0} \frac{(d_i^2(s))^2}{h(s)}\right)^{1/2}$, where $i = 1, \cdots, m$ are finite.

Thus the system (4.1) can be reformulated as (1.1). Moreover, G, F, and H are bounded linear operators with $\mathbb{E}\|G\|_X^2 \leq L_G$, $\mathbb{E}\|H\|_{L(K,X)}^2 \leq L_H$, where

$$\begin{aligned}
L_G &= [\gamma_G^1 + T\|b_2\|_\infty \gamma_G^2],\\
L_F &= [\|r_1\|_\infty L_1^{\tilde{b}} + \|\tilde{b}_2\|_\infty \|r_3\|_{L^1} L_2^{\tilde{b}}]^2,\\
L_H &= [\|q_1\|_\infty L_1^{\tilde{a}} + \|\tilde{a}_2\|_\infty \|q_3\|_{L^1} L_2^{\tilde{a}}]^2.
\end{aligned}$$

Moreover, if b is a bounded C^1-function such that b' is bounded and uniformly continuous, then $(H1)$ and $(H2)$ are satisfied, and hence, by Theorem 2.2, has a resolvent operator $(R(t))_{t\geq 0}$ on X. Therefore, we may easily verify all the assumptions of Theorem 3.5 and hence, there exists a mild solution for system (1.1).

5 Conclusion

In this paper, we investigate the existence of mild solutions for a class of stochastic functional differential impulsive equations with infinite delay on Hilbert space. The results are obtained by using the Banach fixed point theorem and Krasnoselskii–Schaefer type fixed point theorem combined with theories of resolvent operators. Our future work will be focused on investigating the existence of mild solutions for a class of stochastic functional differential impulsive equations driven by a fractional Brownian motion with infinite delay.

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References

[1] Agarwal, R., Meehan, M., O'Regan, D., Fixed point theory and applications, in: Cambridge Tracts in Mathematics, Cambridge University Press, New York, pp. 178–179 (2001).

[2] Anguraj, A., Arjunan, M.M., Hernández, E., Existence results for an impulsive neutral functional differential eqnarray with state-dependent delay. Appl. Anal. 86, 861–872 (2007).

[3] Caraballo, T., Real, J.: Partial differential equations with delay and random perturbations: existence, uniqueness and stability of solutions. Stochastic Anal. Appl. 11(5), 497–511 (1993).

[4] Chang, Y.K., Arjunan, M.M., Kavitha, V., Existence results for neutral functional integro-differential equations with infinite delay via fractional operators, J.Appl.Math.Comput. 36, 201–208 (2011).

[5] Chadha, A., Pandey, D.N.: Existence results for an impulsive problem neutral of stochastic fractional with mixed data integro-differential equation with infinite delay. Nonlinear Anal. 128, 149–175 (2012).

[6] Da Prato, G. and Zabczyk, J., Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge (1992).

[7] Diop, M.A. and Zene, M.M., On the asymptotic stability of impulsive neutral stochastic partial integro-differential equations with variable delays and Poisson jumps. Afrika Matematika 27 (1-2), 215–228 (2016).

[8] Diop, M.A., Ezzinbi, K., Lo, M., Existence and Uniqueness of Mild Solutions to Some Neutral Stochastic Partial Functional integro-differential Equations with Non-Lipschitz Coefficients. Int. J. Math. Math. Sci. doi:10.1155/2012/748590, Article ID 748590 (2012).

[9] Diop, M.A., Ezzinbi, K., Lo, M.:Mild Solutions to Neutral Stochastic Partial Functional integro-differential Equations with non-Lipschitz Coefficients. Afr. Mat. doi:10.1007/s13370-012-0089-3 (2012).

[10] Diop, M.A., Gbaguidi Amoussou, A.T., Ogouyandjou, C. and Mohamed, M.S., Existence results for an neutral integrodifferential stochastic functional differential impulsive equations with state-dependent delay, Transactions of A. Razmadze Mathematical Institute, Vol. 173, 17–31 (2019).

[11] Diop, M.A., Gbaguidi Amoussou, A.T., Ogouyandjou, C. and Sakthivel, R., Asymptotic behaviour of mild solution of nonlinear stochastic partial functional equations driven by Rosenblatt process. Nonlinear analysis: Modeling and control, Vol. 24, 523–544 (2019).

[12] Grimmer, R. C. and Pritchard, A. J., Analytic resolvent operators for integral equations in Banach space, Journal of Differential Equations, 50(2), 234–259 (1983).

[13] Hino Y., Murakami, S., Naito T., Functional Differential Equations with Infinite Delay, in: Lecture Notes in Math., vol. 1473, Springer-Verlag, Berlin (1991).

[14] Hu, L. , Ren Y., Existence results for impulsive neutral stochastic functional integro-differential equations with infinite delays, Acta Appl. Math. 111, 303–317 (2010).

[15] Kolmanovskii, V. B. and Myshkis, A. D.: Applied Theory of Functional Differential Equations, Kluwer Academic, Dordrecht (1992).

[16] Li Y., Liu B., Existence of solutions of nonlinear neutral functional differential inclusions with infinte delay, Stoch. Anal. Appl. 25, 397–415 (2017).

[17] Liang J., Liu J.H., Xiao T.J., Nonlocal problems for integro-differential equations. Dyn. Cont. Discrete Impuls. Syst. Ser. A 15, 815–824 (2008).

[18] Prüss, J.: Evolutionary Integral Equations and Applications. Birkhäuser, Basel (1993).

[19] Sakthivel, R., Anandhi, E.R., Approximate controllability of impulsive differential eqnarrays with statedependent delay. Int. J. Control. 83(2), 387–393 (2010).

[20] Lakshmikantham, V., Bainov, D.D. and Simeonov, P.S., Theory of impulsive differential equations, World Scientific (1989).