Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 13, Number 1, pp. 73–90 (2018)
http://campus.mst.edu/adsa

Sufficient Conditions for Oscillation and Nonoscillation of a Class of Second-Order Neutral Differential Equations

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Abstract

In this work, we establish the necessary and sufficient conditions for oscillation of a class of second order nonlinear neutral differential equations for various ranges of neutral coefficient.

AMS Subject Classifications: 34K, 34C10.
Keywords: Oscillation, nonoscillation, neutral, delay, nonlinear.

1 Introduction

Consider the nonlinear neutral delay differential equations of the form:

$$(r(t)(x(t)+p(t)x(\tau(t)))')' + q(t)G(x(\sigma(t))) + v(t)H(x(\eta(t))) = 0, \tag{1.1}$$

where r, q, v, τ, σ, $\eta \in C(\mathbb{R}_+, \mathbb{R}_+)$, $p \in C(\mathbb{R}_+, \mathbb{R})$ such that $\tau(t) \leq t$, $\sigma(t) \leq t$, $\eta(t) \leq t$ with $\lim_{t\to\infty} \tau(t) = \infty = \lim_{t\to\infty} \sigma(t) = \infty = \lim_{t\to\infty} \eta(t)$ and G, $H \in C(\mathbb{R}, \mathbb{R})$ satisfying the property $yG(y) > 0$, $uH(u) > 0$ for y, $u \neq 0$. In this work, our objective is to establish the sufficient conditions for oscillation and nonoscillation of solutions of (1.1) under the assumption

(A_0) $$R(t) = \int_0^t \frac{ds}{r(s)} < +\infty \text{ as } t \to \infty$$

for various range of $p(t)$.

Received April 27, 2017; Accepted March 13, 2018
Communicated by Martin Bohner

Baculikova et al. [4] have studied the linear counterpart of (1.1), that is,

$$(r(t)(x(t)+p(t)x(\tau(t)))')' + q(t)x(\sigma(t)) + v(t)x(\eta(t)) = 0 \tag{1.2}$$

when $0 \leq p(t) \leq p_0 < \infty$. The authors have obtained sufficient conditions for oscillation of solutions of (1.2) through some comparison results. Here, an attempt is made to study (1.1) without comparison results. Indeed, (1.2) is a special case of (1.1). It is interesting to see that our method provide a better understanding than [4] as long as oscillatory behaviour of solutions of (1.1)/(1.2) is concerned for any $|p(t)| < \infty$. Tripathy et al. [11] have studied (1.1) along with

$$(r(t)(x(t)+p(t)x(\tau(t)))')' + q(t)G(x(\sigma(t))) = 0 \tag{1.3}$$

and obtained the sufficient conditions for oscillation, nonoscillation and asymptotic behavior of solutions of (1.1) and (1.3) provided G, H could be linear or nonlinear. In another work [12], the authors have established oscillation criteria for (1.1) under the assumption

$$R(t) = \int_0^t \frac{ds}{r(s)} \to +\infty \text{ as } t \to \infty,$$

where G and H could be strictly sublinear or superlinear. In this work, we continue to study (1.1) under the assumption (A_0). We note that not only the present work generalizes the work of [4], but also it generalizes the works of [2, 3].

Neutral differential equations find numerous applications in natural sciences and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines (see for e.g., [7]). In this paper, we restrict our attention to study (1.1) which includes a class of nonlinear functional differential equations of neutral as well as nonneutral type. In this direction we refer the reader to the monographs [5,6] and some of the works [1,9,10,13,14] and the references cited therein.

Definition 1.1. By a solution of (1.1), we mean a continuously differentiable function $x(t)$ which is defined for $t \geq T^* = \min\{\tau(t_0),\ \sigma(t_0),\ \eta(t_0)\}$ such that $x(t)$ satisfies (1.1) for all $t \geq t_0$. In the sequel, it will always be assumed that the solutions of (1.1) exist on some half line $[t_1, \infty)$, $t_1 \geq t_0$. A solution of (1.1) is said to be oscillatory, if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory, if all its solutions are oscillatory.

2 Main Results

This section deals with the necessary and sufficient conditions for oscillation of all solutions of (1.1). Throughout our discussion, we use the notation

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{2.1}$$

Lemma 2.1. *Assume that* (A_0) *holds. Let* $x(t)$ *be a positive solution of* (1.1) *defined on* $[t_0, \infty)$ *such that* $z(t) > 0$ *and* $(r(t)z'(t))' \leq 0$ *for* $t \geq t_0$*. If* $z'(t) < 0$ *for* $t \geq t_0$*, then* $z(t) \geq -R_1(t)r(t)z'(t)$*,* $R_1(t) = \displaystyle\int_t^\infty \frac{ds}{r(s)}$.

Proof. For $s \geq t,\ r(s)z'(s) \leq r(t)z'(t)$ implies that $z'(s) \leq \dfrac{r(t)z'(t)}{r(s)}$ which on integration from t to s, we get

$$\int_t^s z'(\theta)d\theta \leq r(t)z'(t)\int_t^s \frac{d\theta}{r(\theta)},$$

that is,

$$z(t) + r(t)z'(t)R_1(t) \geq z(s) \geq 0 \text{ as } s \to \infty.$$

This completes the proof. □

Theorem 2.2. *Let* $-1 < -a \leq p(t) \leq 0$*,* $a > 0$*,* $t \in \mathbb{R}_+$*. Assume that* (A_0) *holds.*
(a) If
(A_1) $G(-u) = -G(u),\ H(-u) = -H(u),\ u \in \mathbb{R},$
(A_2) $\tau^n(t) = \tau^{n-1}(\tau(t))$ *and* $\lim\limits_{n\to\infty} \tau^n(t) < \infty,\ t \in \mathbb{R}_+$
and
(A_3) $\displaystyle\int_0^\infty \frac{ds}{r(s)} < +\infty$
hold, then every unbounded solution of (1.1) *oscillates.*
(b) If (A_3) *does not hold and if*
(A_4) $\displaystyle\int_T^\infty [q(s)G(CR(\sigma(s)) + v(s)H(CR(\eta(s))]ds < \infty,\ T > 0\ for\ every\ C > 0,$
then (1.1) *admits a positive bounded solution.*

Proof. (a) On the contrary, let's assume that $x(t)$ is an unbounded nonoscillatory solution of (1.1) such that $x(t) > 0$ for $t \geq t_0$. Hence, there exists $t_1 > t_0$ such that

$$x(t) > 0,\ x(\tau(t)) > 0,\ x(\sigma(t)) > 0,\ x(\eta(t)) > 0 \text{ for } t \geq t_1.$$

Using (2.1), (1.1) becomes

$$(r(t)z'(t))' = -q(t)G(x(\sigma(t)) - v(t)H(x(\eta(t)) \leq 0,\ \not\equiv 0 \text{ for } t \geq t_1. \tag{2.2}$$

Therefore, there exists $t_2 > t_1$ such that $r(t)z'(t)$ and $z(t)$ are monotonic on $[t_2, \infty)$. Let $t_3 > t_2$ be such that $z(t) > 0$ for $t \geq t_3$. Indeed, $z(t) < 0$ for $t \geq t_3$ is not possible. Because, in this case $x(t) < x(\tau(t))$ implies that

$$x(t) \leq x(\tau(t)) \leq x(\tau^2(t)) \leq x(\tau^3(t)) \leq \cdots \leq x(t_3),$$

that is, $x(t)$ is bounded. If $z(t) > 0, r(t)z'(t) > 0$ for $t \geq t_3$, then $r(t)z'(t)$ is nonincreasing on $[t_3, \infty)$. So there exist a constant $C > 0$ and $t_4 > t_3$ such that $r(t)z'(t) \leq C$ for $t \geq t_4$. Consequently

$$\begin{aligned} z(t) &\leq z(t_3) + C\int_{t_3}^{t} \frac{ds}{r(s)} \\ &< \infty \text{ as } t \to \infty \end{aligned}$$

due to (A_0). On the other hand, $x(t)$ is unbounded implies that there exists an increasing sequence $\{\sigma_n\}$ such that $\sigma_n \to \infty$ and $x(\sigma_n) \to \infty$ as $n \to \infty$ and $x(\sigma_n) = \max\{x(s) : t_1 \leq s \leq \sigma_n\}$. Therefore,

$$\begin{aligned} z(\sigma_n) &= x(\sigma_n) + p(\sigma_n)x(\tau(\sigma_n)) \\ &\geq x(\sigma_n) - ax(\tau(\sigma_n)) \\ &\geq x(\sigma_n) - ax(\sigma_n) \\ &= (1-a)x(\sigma_n) (\because 1 - a > 0) \\ &\to +\infty \text{ as } n \to \infty \end{aligned}$$

gives a contradiction. The case $r(t)z'(t) < 0, z(t) > 0$ for $t \geq t_3$ is not possible due to unbounded $z(t)$.

(b) Suppose that (A_3) does not hold. For $C > 0$, let

$$\int_T^{\infty} [q(t)G(CR(\sigma(t))) + v(t)H(CR(\eta(t)))]\, dt \leq \frac{C}{5}.$$

Consider

$$M = \Big\{ x : x \in C([t_0, \infty), \mathbb{R}),\ x(t) = 0,\ t \in [t_0,\ T], \\ \frac{C}{5}R(T, t) \leq x(t) \leq CR(T, t), t \geq T \Big\},$$

where $R(T, t) = R(t) - R(T)$. Define

$$\Psi x(t) = \begin{cases} 0, \quad t \in [t_0, T], \\ -p(t)x(\tau(t)) + \int_T^t \frac{1}{r(u)}\Big[\frac{C}{5} + \int_u^{\infty} q(s)G(x(\sigma(s)))ds \\ + \int_u^{\infty} v(s)H(x(\eta(s))ds\Big],\ t \geq T. \end{cases}$$

For every $x \in M$,

$$\Psi x(t) \geq \int_T^t \frac{1}{r(u)} \left[\frac{C}{5} + \int_u^{\infty} \{q(s)G(x(\sigma(s))) + v(s)H(x(\eta(s)))\}ds\right] du$$

$$\geq \frac{C}{5}\int_T^t \frac{du}{r(u)} = \frac{C}{5}R(T,t)$$

and $x(t) \leq CR(T,t)$ implies that

$$\begin{aligned}\Psi x(t) &\leq -p(t)x(\tau(t)) + \frac{2C}{5}\int_T^t \frac{du}{r(u)} \\ &\leq aCR(T,\tau(t)) + \frac{2C}{5}R(T,t) \\ &\leq aCR(T,t) + \frac{2C}{5}R(T,t) \\ &= \left(a + \frac{2}{5}\right) CR(T,t) \leq CR(T,t)\end{aligned}$$

implies that $\Psi x(t) \in M$. Define $u_n : [t_0, +\infty) \to \mathbb{R}$ by the recursive formula

$$u_n(t) = (\Psi u_{n-1})(t),\ n \geq 1$$

with the initial condition

$$u_0(t) = \begin{cases} 0, & t \in [t_0, T] \\ \frac{C}{5}R(T,t), & t \geq T. \end{cases}$$

Inductively it is easy to verify that

$$\frac{C}{5}R(T,t) \leq u_{n-1}(t) \leq u_n(t) \leq CR(T,t),\ t \geq T.$$

Therefore, $\lim_{n\to\infty} u_n(t)$ exists. By the Lebesgue's dominated convergence theorem, $u \in M$ and $\Psi u(t) = u(t)$, where $u(t)$ is a solution of (1.1) on $[t_0, \infty)$ such that $u(t) > 0$. This completes the proof. □

Theorem 2.3. *Let* $-1 < -a \leq p(t) \leq 0$, $a > 0$, $\eta(t) \geq \sigma(t)$, $r(t) \geq r(\sigma(t))$ *and* $\tau(t)$ *is bijective for* $t \in \mathbb{R}_+$. *Assume that* $(A_0) - (A_4)$ *hold. Furthermore, assume that*

(A_5) $\quad G, H$ *are superlinear such that* $\dfrac{G(u)}{u^\beta} \geq \dfrac{G(v)}{v^\beta},\ \dfrac{H(u)}{u^\beta} \geq \dfrac{H(v)}{v^\beta},\ u \geq v > 0,\ \beta > 1,$

(A_6) $\quad \displaystyle\int_T^\infty \frac{1}{r(\theta)}\int_T^\theta [q(s) + Lv(s)]\, ds d\theta = \infty,\ L > 0,\ T > 0$

and

(A_7) $\quad \displaystyle\int_0^\infty \frac{1}{r(t)}\int_t^\infty [q(s) + Lv(s)dsdt = +\infty, L > 0,$ *is a constant hold. Then every solution of* (1.1) *either oscillates or converges to zero as* $t \to \infty$. *If* (A_7) *fails to hold, then* (1.1) *admits a positive bounded solution which does not tend to zero as* $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Proceeding as in the proof of Theorem 2.2, we have (2.2) for $t \geq t_1$. Hence, there exists $t_2 > t_1$ such that $r(t)z'(t)$ is nonincreasing on $[t_2, \infty)$. If $z(t) < 0$ for $t \geq t_2$, then $x(t)$ is bounded. Consequently, $\lim_{t\to\infty} z(t)$ exists. As a result,

$$\begin{aligned} 0 \geq \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty} z(t) \\ &\geq \limsup_{t\to\infty}(x(t) - ax(\tau(t))) \\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}(-ax(\tau(t))) \\ &= (1-a)\limsup_{t\to\infty} x(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ $(\because 1-a > 0)$ and hence $\lim_{t\to\infty} x(t) = 0$. Let $z(t) > 0$ for $t \geq t_2$. Consider $r(t)z'(t) < 0$ for $t \geq t_2$. Therefore, $\lim_{t\to\infty} z(t)$ exists. We claim that $x(t)$ is bounded. If not, there exists an increasing sequence $\{\gamma_n\}$ such that $\gamma_n \to \infty$ as $n \to \infty$, $x(\gamma_n) \to \infty$ and $x(\gamma_n) = \max\{x(s) : t_3 \leq s \leq \gamma_n\}$. Therefore

$$\begin{aligned} z(\gamma_n) &= x(\gamma_n) + p(\gamma_n)x(\tau(\gamma_n)) \\ &\geq (1-a)x(\gamma_n) \\ &\to +\infty \text{ as } \to \infty \end{aligned}$$

gives a contradiction. To show $\lim_{t\to\infty} x(t) = 0$, it is sufficient to show that $\liminf_{t\to\infty} x(t) = 0$. If not, there exist a constant $\alpha > 0$ and $t_3 > t_2$ such that $x(\sigma(t)) \geq \alpha > 0$ for $t \geq t_3$. Integrating (2.2) from t_3 to $t(\geq t_3)$, we obtain

$$[r(s)z'(s)]_{t_3}^t + \int_{t_3}^t [q(s)G(x(\sigma(s))) + v(s)H(x(\eta(s)))]\,ds \leq 0,$$

that is,

$$\int_{t_3}^t [q(s)G(\alpha) + v(s)H(\alpha)]\,ds \leq -[r(s)z'(s)]_{t_3}^t$$

implies that

$$\int_{t_3}^t [q(s)G(\alpha) + v(s)H(\alpha)]\,ds \leq -r(t)z'(t).$$

Consequently,

$$\frac{1}{r(t)}\left[G(\alpha)\int_{t_3}^t q(s)ds + H(\alpha)\int_{t_3}^t v(s)ds\right] \leq -z'(t).$$

Integrating the preceding inequality from t_4 to t, we get

$$G(\alpha)\left[\int_{t_4}^t \frac{1}{r(\theta)}\int_{t_3}^\theta q(s)dsd\theta + H(\alpha)\int_{t_4}^t \frac{1}{r(\theta)}\int_{t_3}^\theta v(s)dsd\theta\right] \leq -z(t) + z(t_4),$$

that is,

$$\int_{t_4}^{\infty} \frac{1}{r(\theta)} \int_{t_3}^{\theta} \Big[q(s) + Lv(s)\Big] ds d\theta < \infty, L = \frac{H(\alpha)}{G(\alpha)},$$

a contradiction to (A_6). Therefore, our assertion hold. Hence, there exists $\{\delta_n\}_{n=1}^{\infty} \subset [t_4, \infty)$ such that $\delta_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} x(\delta_n) = 0$. Let $\lim_{t\to\infty} z(t) = l, l \in (-\infty, 0]$. For $t \geq t_4$, we have

$$z(\tau^{-1}(t)) - z(t) = x(\tau^{-1}(t)) + [p(\tau^{-1}(t)) - 1]x(t) - p(t)x(\tau(t))$$

implies that

$$\lim_{t\to\infty} [x(\tau^{-1}(t)) + \{p(\tau^{-1}(t)) - 1\}x(t) - p(t)x(\tau(t))] = 0.$$

Equivalently,

$$\lim_{n\to\infty} [x(\tau^{-1}(\delta_n)) + \{p(\tau^{-1}(\delta_n)) - 1\}x(\delta_n) - p(\delta_n)x(\tau(\delta_n))] = 0$$

implies that

$$\lim_{n\to\infty} [x(\tau^{-1}(\delta_n)) - p(\delta_n)x(\tau(\delta_n))] = 0.$$

Using the fact that

$$x(\tau^{-1}(\delta_n)) - p(\delta_n)x(\tau(\delta_n)) \geq -p(\delta_n)x(\tau(\delta_n)),$$

then it follows that

$$\limsup_{n\to\infty} [-p(\delta_n)x(\tau(\delta_n))] = 0,$$

that is, $\lim_{n\to\infty} [-p(\delta_n)x(\tau(\delta_n))] = 0$. Ultimately,

$$l = \lim_{n\to\infty} z(\delta_n) = \lim_{n\to\infty} [x(\delta_n) + p(\delta_n)x(\tau(\delta_n))] = 0.$$

As a result

$$\begin{aligned}
0 &= \lim_{t\to\infty} z(t) \\
&= \limsup_{t\to\infty} (x(t) + p(t)x(\tau(t))) \\
&\geq \limsup_{t\to\infty} (x(t) - ax(\tau(t))) \\
&\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty} (-ax(\tau(t))) \\
&= (1 - a) \limsup_{t\to\infty} x(t)
\end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ and thus $\lim_{t\to\infty} x(t) = 0$. Suppose that $r(t)z'(t) > 0$ for $t \geq t_2$, that is, $\lim_{t\to\infty} r(t)z'(t)$ exists. Since $z(t)$ is nondecreasing, then there exist a constant $C > 0$ and $t_3 > t_2$ such that $z(\sigma(t)) \geq C$ and $z(\eta(t)) \geq C$ for $t \geq t_3$. Consequently,

$$\begin{aligned} G(z(\sigma(t))) &= \frac{G(z(\sigma(t)))}{z^\beta(\sigma(t)} z^\beta(\sigma(t)) \\ &\geq \frac{G(C)}{C^\beta} z^\beta(\sigma(t)) \end{aligned}$$

and $H(z(\eta(t))) \geq \dfrac{H(C)}{C^\beta} z^\beta(\eta(t))$ for $t \geq t_3$. Integrating (2.2) from $t(> t_3)$ to $(+\infty)$, we get

$$[r(s)z'(s)]_t^\infty + \int_t^\infty \Big[q(s)G(x(\sigma(s))) + v(s)H(x(\eta(s)))\Big] ds \leq 0,$$

that is,

$$\int_t^\infty \Big[q(s)G(x(\sigma(s))) + v(s)H(x(\eta(s)))\Big] ds \leq r(t)z'(t)$$

and hence

$$\begin{aligned} r(t)z'(t) &\geq \frac{G(C)}{C^\beta} \int_t^\infty q(s) z^\beta(\sigma(s)) ds + \frac{H(C)}{C^\beta} \int_t^\infty v(s) z^\beta(\eta(s)) ds \\ &\geq \left[\frac{G(C)}{C^\beta} \int_t^\infty q(s) ds\right] z^\beta(\sigma(t)) + \left[\frac{H(C)}{C^\beta} \int_t^\infty v(s) ds\right] z^\beta(\eta(t)). \end{aligned}$$

Using $\sigma(t) \leq t$ and $\eta(t) \geq \sigma(t)$, the above inequality yields

$$r(\sigma(t))z'(\sigma(t)) \geq \left[\frac{G(C)}{C^\beta} \int_t^\infty q(s) ds + \frac{H(C)}{C^\beta} \int_t^\infty v(s) ds\right] z^\beta(\sigma(t))$$

for $t \geq t_2$. Hence

$$z'(\sigma(t)) \geq \frac{G(C)}{C^\beta} \frac{z^\beta(\sigma(t))}{r(t)} \int_t^\infty \Big[q(s) + Lv(s)\Big] ds,$$

where $L = \dfrac{H(C)}{G(C)} > 0$. Integrating the preceding inequality from t_3 to $+\infty$, we get

$$\frac{G(C)}{C^\beta} \int_{t_3}^\infty \frac{1}{r(t)} \int_t^\infty \Big[q(s) + Lv(s)\Big] ds dt \leq \int_{t_3}^\infty \frac{z'(\sigma(t))}{z^\beta(\sigma(t))} dt < \infty$$

which is a contradiction to (A_7).

Next, we assume that (A_7) fails to hold. Let

$$\int_T^t \frac{1}{r(s)} \int_s^\infty [q(\theta) + Lv(\theta)] d\theta ds \leq \frac{R(t)}{3G(C^*)},\ T \geq T^*,$$

where $C^* = \max_{t \geq T} \{R(\sigma(t)), R(\eta(t))\}$. Consider

$$M = \left\{ x \in C([t_0, \infty), \mathbb{R}) : x(t) = \frac{R(t)}{3}, t \in [t_0, T]; \right.$$
$$\left. \frac{R(t)}{3} \leq x(t) \leq R(t), \text{for } t \geq T \right\}$$

and define

$$\Psi x(t) = \begin{cases} \Psi, \quad t \in [t_0, T] \\ -p(t)x(\tau(t)) + \dfrac{R(t)}{3} + \displaystyle\int_T^t \frac{1}{r(s)} \int_s^\infty \Big[q(\theta)G(x(\sigma(\theta))) + \\ v(\theta)H(x(\eta(\theta))) \Big] d\theta ds,\ t \geq T. \end{cases}$$

Indeed, for every $x \in M$, $\Psi x(t) \geq \dfrac{R(t)}{3}$ and

$$\begin{aligned} \Psi x(t) &\leq aR(t) + \frac{R(t)}{3} + \int_T^t \frac{1}{r(s)} \int_s^\infty [q(\theta)G(R(\sigma(\theta))) + v(\theta)H(R(\eta(\theta)))] d\theta ds \\ &= aR(t) + \frac{R(t)}{3} + G(C^*) \int_T^t \frac{1}{r(s)} \int_s^\infty [q(\theta) + Lv(\theta)] d\theta ds \\ &\leq aR(t) + \frac{R(t)}{3} + \frac{R(t)}{3} = \left(a + \frac{2}{3} \right) R(t) \\ &\leq R(t) \end{aligned}$$

implies that $\Psi x \in M$. Proceeding as in the proof of Theorem 2.2, we obtain that T has a fixed point $u \in M$, that is $u(t) = (Tu)(t)$. Therefore $u(t)$ is a solution of (1.1). This completes the proof. □

Theorem 2.4. *Let* $-1 < -a \leq p(t) \leq 0$, $a > 0$, $\eta(t) \geq \sigma(t)$, $r(t) \geq r(\sigma(t))$ *and* $\tau(t)$ *is bijective for* $t \in \mathbb{R}_+$. *Assume that* $(A_0) - (A_3)$, (A_6), (A_7) *and*

(A_8) $\quad G, H$ *are strictly sublinear such that* $\dfrac{G(u)}{u^\beta} \geq \dfrac{G(v)}{v^\beta}, \dfrac{H(u)}{u^\beta} \geq \dfrac{H(v)}{v^\beta},\ 0 < u \leq v,\ \beta < 1$

hold. Then every solution of (1.1) *either oscillates or converges to zero as* $t \to \infty$. *If* (A_7) *fails to hold, then* (1.1) *admits a positive bounded solution which does not tend to zero as* $t \to \infty$.

Proof. The proof follows from the proof of Theorem 2.3. For the case $r(t)z'(t) > 0, z(t) > 0$, we integrate $(r(t)z'(t))' \le 0$ from t_2 to t and obtained that $z'(t) \le \dfrac{r(t_2)z'(t_2)}{r(t)} = \dfrac{C}{r(t)}$ for $t \ge t_2$. Again integrating from t_2 to t we find $z(t) \le z(t_2) + C\displaystyle\int_{t_2}^{t} \frac{ds}{r(s)} < \infty$ as $t \to \infty$. Hence, there exist $C_1 > 0$ and $t_3 > t_2$ such that $z(t) \le C_1$ for $t \ge t_3$. Due to (A_8)

$$\begin{aligned} G(z(\sigma(t))) &= \frac{G(z(\sigma(t)))}{z^{\beta}(\sigma(t)} z^{\beta}(\sigma(t)) \\ &\ge \frac{G(C_1)}{C_1^{\beta}} z^{\beta}(\sigma(t)) \end{aligned}$$

and $H(z(\eta(t))) \ge \dfrac{H(C_1)}{C_1^{\beta}} z^{\beta}(\eta(t))$ for $t \ge t_3$. The rest of the proof follows from Theorem 2.3. Thus the proof is complete. □

Remark 2.5. In Theorem 2.3, the argument used to make $l = 0$ is true when $|p(t)| < \infty$ such that $p(t) \not\equiv 1$.

Theorem 2.6. *Let $-\infty < -a_1 \le p(t) \le -a_2 < -1$, $\eta(t) \ge \sigma(t)$, $r(t) \ge r(\sigma(t))$ and $\tau(t)$ is bijective for $t \in \mathbb{R}_+$, where a_1, $a_2 > 0$ such that $3a_2 > a_1$. Assume that (A_0), (A_1), (A_3) and $(A_6) - (A_8)$ hold. Then every bounded solution of* (1.1) *either oscillates or tends to zero as $t \to \infty$. If G and H are Lipschitzian on the intervals of the form $[c, d]$, $0 < c < d < \infty$ and (A_7) fails to hold, then* (1.1) *admits a positive bounded solution which does not tend to zero as $t \to \infty$.*

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1.1). Proceeding as in Theorem 2.2, it follows that $z(t), r(t)z'(t)$ are monotonic on $[t_2, \infty)$. Since $x(t)$ is bounded, we have $z(t)$ is bounded due to (2.1) and hence $\lim_{t\to\infty} z(t)$ exists. The case $z(t) > 0$ is similar to Theorem 2.4. In case $z(t) < 0$ for $t \ge t_2$, let $r(t)z'(t) > 0$. Using Remark 2.5 we conclude that $L = 0$. As a result

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty}(x(t) + p(t)x(\tau(t))) \\ &\le \liminf_{t\to\infty}(x(t) - a_2 x(\tau(t))) \\ &\le \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}(-a_2 x(\tau(t))) \\ &= (1 - a_2)\limsup_{t\to\infty} x(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ [$\because 1 - a_2 < 0$]. Hence, $\lim_{t\to\infty} x(t) = 0$. Consider $r(t)z'(t) < 0$. From (2.2), we have $(r(t)z'(t))' \le 0$. Using the same type of argument as in Theorem 2.4, we can find $C_2 > 0$ and $t_3 > t_2$ such that $z(\tau^{-1}(\sigma(t))) < -C_2$ and $z(\tau^{-1}(\eta(t))) <$

$-C_2$ for $t \geq t_3$. Hence, $z(t) \geq -a_1 x(\tau(t))$ implies that $x(t) \geq -a_1^{-1} z(\tau^{-1}(t))$, that is, $x(\sigma(t)) \geq -a_1^{-1} z(\tau^{-1}(\sigma(t))) \geq -a_1^{-1} C_2$ and $z(\eta(t)) \geq -a_1^{-1} C_2$ for $t \geq t_3$. Consequently, (1.1) reduces to

$$(r(t)z'(t))' + G(-a_1^{-1}C_2)q(t) + H(-a_1^{-1}C_2)v(t) \leq 0$$

for $t \geq t_3$. Twice integration of the last inequality from t_3 to t we obtain a contradiction to (A_6).

For the necessary part, it is possible to find $T \geq T^*$ such that

$$\int_T^\infty \frac{1}{r(s)} \int_s^\infty [q(t) + Lv(t)]dtds < \frac{a_2 - 1}{3K},$$

where $K = \max\left\{K_1,\ \frac{K_2}{L},\ G(1)\right\}$, K_1 and K_2 are Lipschitz constants of G and H on $[a, 1]$ respectively, where $a = \frac{(a_2 - 1)(3a_2 - a_1)}{3a_1 a_2}$.

Let $X = BC([t_0, \infty), \mathbb{R})$ be the space of real valued continuous functions defined on $[t_0, \infty)$. Indeed, X is a Banach space with the supremum norm defined by

$$\| x \| = \sup\{|x(t)| : t \ \geq t_0\}.$$

Define

$$S = \{u \in X : a \leq u(t) \leq 1,\ t \geq t_0\}$$

and we note that S is a closed convex subspace of X. Let $\Psi : S \to S$ be such that

$$\Psi x(t) = \begin{cases} \Psi x(T), \quad t \in [t_0, T] \\ -\dfrac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \dfrac{a_2 - 1}{p(\tau^{-1}(t))} \\ +\dfrac{1}{p(\tau^{-1}(t))} \displaystyle\int_T^{\tau^{-1}(t)} \frac{1}{r(s)} \left[\int_s^\infty q(\theta)G(x(\sigma(\theta)))d\theta \right. \\ \left. + \displaystyle\int_s^\infty v(\theta)H(x(\eta(\theta)))d\theta \right] ds,\ t \geq T. \end{cases}$$

For every $x \in S$,

$$\Psi x(t) \leq -\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{a_2 - 1}{p(\tau^{-1}(t))} \leq \frac{1}{a_2} + \frac{a_2 - 1}{a_2} = 1$$

and

$$\begin{aligned} \Psi x(t) \ &\geq \ -\frac{a_2 - 1}{p(\tau^{-1}(t))} + \frac{1}{p(\tau^{-1}(t))} \int_T^{\tau^{-1}(t)} \frac{1}{r(s)} \left[\int_s^\infty q(\theta)G(x(\sigma(\theta)))d\theta \right. \\ &+ \left. \int_s^\infty v(\theta)H(x(\eta(\theta)))d\theta \right] ds \end{aligned}$$

$$\begin{aligned}
&\geq -\frac{a_2-1}{a_1}+\frac{G(1)}{p(\tau^{-1}(t))}\int_T^{\tau^{-1}(t)}\frac{1}{r(s)}\left[\int_s^\infty q(\theta)d\theta+\frac{H(1)}{G(1)}\int_s^\infty v(\theta)d\theta\right]ds\\
&\geq -\frac{a_2-1}{a_1}-\frac{G(1)}{a_2}\int_T^\infty \frac{1}{r(s)}\left[\int_s^\infty q(\theta)d\theta+L\int_s^\infty v(\theta)d\theta\right]ds\\
&\geq -\frac{a_2-1}{a_1}-\frac{a_2-1}{3a_2}=a
\end{aligned}$$

implies that $\Psi x\in S$. Now for $x_1,x_2\in S$, we have

$$\begin{aligned}
|\Psi x_1(t)-\Psi x_2(t)| &\leq \frac{1}{|p(\tau^{-1}(t))|}|x_1(\tau^{-1}(t))-x_2(\tau^{-1}(t))|\\
&+\frac{K_1}{|p(\tau^{-1}(t))|}\int_T^{\tau^{-1}(t)}\frac{1}{r(s)}\int_s^\theta |x_1(\sigma(\theta))-x_2(\sigma(\theta))|q(\theta)d\theta ds\\
&+\frac{K_2}{|p(\tau^{-1}(t))|}\int_T^{\tau^{-1}(t)}\frac{1}{r(s)}\int_s^\infty |x_1(\eta(\theta))-x_2(\eta(\theta))|v(\theta)d\theta ds\\
&\leq \frac{1}{a_2}\parallel x_1-x_2\parallel+\frac{a_2-1}{3a_2}\parallel x_1-x_2\parallel\\
&= \gamma\parallel x_1-x_2\parallel
\end{aligned}$$

implies that

$$\parallel \Psi x_1-\Psi x_2\parallel\leq\gamma\parallel x_1-x_2\parallel,$$

where $\gamma=\frac{1}{a_2}\left(1+\frac{a_2-1}{3}\right)<1$. Therefore, Ψ is a contraction. Hence by Banach's fixed point theorem Ψ has a unique fixed point $x\in S$. It is easy to see that $\lim_{t\to\infty}x(t)\neq 0$. This completes the proof. □

Theorem 2.7. *Let $-\infty<-a_1\leq p(t)\leq -a_2<-1$, $\eta(t)\geq\sigma(t)$, $r(t)\geq r(\sigma(t))$ and $\tau(t)$ is bijective for $t\in\mathbb{R}_+$, where a_1, $a_2>0$ such that $3a_2>a_1$. Assume that (A_0), (A_1), (A_3) and $(A_5)-(A_7)$ hold. Then every bounded solution of* (1.1) *either oscillates or tends to zero as $t\to\infty$. If G and H are Lipschitzian on the intervals of the form $[c,d]$, $0<c<d<\infty$ and (A_7) fails to hold, then* (1.1) *admits a positive bounded solution which does not tend to zero as $t\to\infty$.*

Proof. The proof follows from the proof of Theorem 2.6 except the cases, $z(t)>0, r(t)z'(t)>0$ and $z(t)>0, r(t)z'(t)<0$. Since $x(t)$ is bounded, we have these two cases follow from the proof of Theorem 2.3 and Remark 2.5. Proceeding as in Theorem 2.6, we find that $\lim_{t\to\infty}x(t)=0$. This completes the proof. □

Theorem 2.8. *Let $0\leq p(t)\leq a<1$, $r(t)\geq r(\sigma(t))$, $\eta(t)\geq\sigma(t)$ and $\tau(t)$ is bijective, for $t\in\mathbb{R}_+$. Assume that (A_0), (A_1) and $(A_5)-(A_7)$ hold. Then every solution of* (1.1) *either oscillates or tends to zero as $t\to\infty$. If G and H are Lipschitzian on the intervals of the form $[c,d]$, $0<c<d<\infty$ and (A_7) fails to hold, then* (1.1) *admits a positive bounded solution which does not tend to zero as $t\to\infty$.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Then proceeding as in Theorem 2.2, we have two cases viz. $z(t) > 0, r(t)z'(t) < 0$ and $z(t) > 0, r(t)z'(t) > 0$ for $t \in [t_2, \infty)$. For the former case $z(t)$ is bounded and hence $\lim_{t\to\infty} z(t)$ exists. Since $z(t) \geq x(t)$, we have $x(t)$ is bounded. Now, we claim that $\liminf_{t\to\infty} x(t) = 0$. If not, there exist a constant $\alpha > 0$ and $t_3 > t_2$ such that $x(\sigma(t)) \geq \alpha > 0$ for $t \geq t_3$. Integrating (2.2) from t_3 to $t(\geq t_3)$ and then using the same type of argument as in Theorem 2.3 we obtain a contradiction to (A_6). So, our claim holds. Consequently, Remark 2.5 implies that $\lim_{t\to\infty} z(t) = 0$. Ultimately, $0 = \lim_{t\to\infty} z(t) \geq \lim_{t\to\infty} x(t)$. Consider the latter case. Then there exist a constant $C > 0$ and $t_3 > t_2$ such that $z(\sigma(t)) \geq C$ and $z(\eta(t)) \geq C$ for $t \geq t_3$. Therefore,

$$
\begin{aligned}
G(z(\sigma(t))) &= \frac{G(z(\sigma(t)))}{z^{\beta}(\sigma(t)}z^{\beta}(\sigma(t)) \\
&\geq \frac{G(C)}{C^{\beta}}z^{\beta}(\sigma(t))
\end{aligned}
$$

and $H(z(\eta(t))) \geq \dfrac{H(C)}{C^{\beta}}z^{\beta}(\eta(t))$ for $t \geq t_3$. Since

$$
\begin{aligned}
z(t) - p(t)z(\tau(t)) &= x(t) + p(t)x(\tau(t)) - p(t)x(\tau(t)) \\
&- p(t)p(\tau(t))p(\tau(\tau(t))) \\
&= x(t) - p(t)p(\tau(t))p(\tau(\tau(t))) \\
&\leq x(t),
\end{aligned}
$$

we have $x(t) \geq (1-a)z(t)$ and hence (1.1) becomes

$$
(r(t)z'(t))' + q(t)G((1-a)z(\sigma(t))) + v(t)H((1-a)z(\eta(t))) \leq 0.
$$

With the preceding inequality, we proceed as in Theorem 2.3 to obtain a contradiction to (A_7). The necessary part can similarly be dealt with Theorem 2.3. This completes the proof. □

Theorem 2.9. *Let* $0 \leq p(t) \leq a < 1$, $r(t) \geq r(\sigma(t))$, $\eta(t) \geq \sigma(t)$ *and* $\tau(t)$ *is bijective, for* $t \in \mathbb{R}_+$. *Assume that* (A_0), (A_1) *and* $(A_6) - (A_8)$ *hold. Then every solution of* (1.1) *either oscillates or tends to zero as* $t \to \infty$. *If G and H are Lipschitzian on the intervals of the form* $[c, d]$, $0 < c < d < \infty$ *and* (A_7) *fails to hold, then* (1.1) *admits a positive bounded solution which does not tend to zero as* $t \to \infty$.

Proof. The proof follows from the proof of Theorem 2.8. Due to (A_8), we use the same type of argument as in Theorem 2.4 for the case $z(t) > 0, r(t)z'(t) > 0$. Hence the details are omitted. Thus the proof is completed. □

Theorem 2.10. *Let* $1 \leq p(t) \leq a < \infty$ *for* $t \in \mathbb{R}_+$ *and* $G(a) \geq H(a)$. *Assume that* (A_0), (A_1) *and* (A_3) *hold. Furthermore, assume that there exist* λ, $\mu > 0$ *such that*

(A_9) $G(u)+G(s)\geq\lambda G(u+s),\ H(u)+H(s)\geq\mu H(u+s)\ for\ u,\ s\in\mathbb{R}_+$, *(see e.g.,* [9])
(A_{10}) $G(us)\leq G(u)G(s),\ H(us)\leq H(u)H(s),\ u,s\in\mathbb{R}_+$,
(A_{11}) $\tau o\sigma=\sigma o\tau, \tau o\eta=\eta o\tau$ *for all* $t\in\mathbb{R}_+$,
(A_{12}) $\displaystyle\int_T^\infty\frac{1}{r(t)}\Big[\int_T^t Q(s)G(CR_1(\sigma(s))+\frac{\mu}{\lambda}\int_T^t V(s)H(CR_1(\eta(s))\Big]dsdt=\infty;$
$T>0,\ C>0,$
and
(A_{13}) $\displaystyle\int_T^\infty[Q(t)+L_1V(t)]dt=\infty;\ T>0,\ L_1=\frac{\mu H(C)}{\lambda G(C)}>0,\ C>0$
hold, where $Q(t)=\min\{q(t),q(\tau(t))\}, V(t)=\min\{v(t),v(\tau(t))\}$. *Then every solution of* (1.1) *oscillates.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Proceeding as in Theorem 2.3, we have two cases viz. $r(t)z'(t)<0$ and $r(t)z'(t)>0$ for $t\in[t_2,\infty)$. If $r(t)z'(t)<0$ for $t\in[t_2,\infty)$, then $z(t)$ is bounded and $\lim_{t\to\infty}z(t)$ exists. Using Lemma 2.1, we have $z(t)\geq -R_1(t)r(t)z'(t)$ for $t\geq t_2$. From (1.1), it is easy to see that

$$\begin{aligned}0&=(r(t)z'(t))'+q(t)G(x(\sigma(t)))+v(t)H(x(\eta(t)))\\&\quad+G(a)\big[r(\tau(t))z'(\tau(t)))'+q(\tau(t)))G(x(\sigma(\tau(t)))+v(\tau(t)))H(x(\eta(\tau(t)))\big]\end{aligned}$$

in which we use (A_9), (A_{10}) and (A_{11}) to obtain

$$\begin{aligned}0&\geq(r(t)z'(t))'+G(a)(r(\tau(t))z'(\tau(t)))'+\lambda Q(t)G(z(\sigma(t)))+v(t)H(x(\eta(t)))\\&\quad+G(a)v(\tau(t))H(x(\eta(\tau(t)))\\&\geq(r(t)z'(t))'+G(a)(r(\tau(t))z'(\tau(t)))'+\lambda Q(t)G(z(\sigma(t)))\\&\quad+v(t)H(x(\eta(t)))+H(a)v(\tau(t))H(x(\eta(\tau(t))),\end{aligned}$$

that is,

$$(r(t)z'(t))'+G(a)(r(\tau(t))z'(\tau(t)))'+\lambda Q(t)G(z(\sigma(t)))+\mu V(t)H(z(\eta(t)))\leq 0,\quad (2.3)$$

where $z(t)\leq x(\sigma(t))+ax(\sigma(\tau(t)))$ for $t\geq t_3>t_2$. Using the fact that $r(t)z'(t)$ is nonincreasing, we can find a constant $C>0$ such that $r(t)z'(t)\leq -C$ and $z(t)\geq CR_1(t)$ (due to Lemma 2.1) for $t\geq t_3$ and hence (2.3) further implies that

$$\begin{aligned}(r(t)z'(t))'+G(a)(r(\tau(t))z'(\tau(t)))'+\lambda Q(t)G(CR_1(\sigma(s)))\\+\mu V(t)H(CR_1(\eta(s)))\leq 0.\end{aligned}$$

Integrating the preceding inequality form t_3 to $t(>t_3)$, we obtain

$$\lambda\int_{t_3}^t Q(s)G(CR_1(\sigma(s)))ds+\mu\int_{t_3}^t V(s)H(CR_1(\eta(s)))ds$$

$$\leq -r(t)z'(t) - G(a)(r(\tau(t))z'(\tau(t))$$
$$\leq -(1+G(a))r(t))z'(t)).$$

$$\frac{1}{(1+G(a))}\frac{1}{r(t)}\left[\lambda\int_{t_3}^{t} Q(s)G(CR_1(\sigma(s))ds + \mu\int_{t_3}^{t} V(s)H(CR_1(\eta(s))ds\right] \leq -z'(t). \tag{2.4}$$

Integrating (2.4) from $t(> t_3)$ to $+\infty$, we obtain a contradiction to (A_{12}).

Let $r(t)z'(t) > 0$ for $t \geq t_2$. Then there exist a constant $C > 0$ and $t_3 > t_2$ such that $z(t) \geq C$ for $t \geq t_3$. Now, (2.3) yields that

$$G(a)(r(\tau(t))z'(\tau(t)))' + \lambda Q(t)G(C) + \mu V(t)H(C) \leq -(r(s)z'(s))'ds.$$

Integrating the above inequality from t_3 to $+\infty$, we obtain a contradiction to (A_{13}). This completes the proof. □

Theorem 2.11. *Let* $1 < a_1 \leq p(t) \leq a_2 < \infty$ *for* $t \in \mathbb{R}_+$ *such that* $a_1^2 \geq a_2$. *Assume that* (A_0) *holds and* (A_7) *fails to hold. Furthermore, assume that* G *and* H *are Lipschitzian on the intervals of the form* $[c, d]$, $0 < c < d < \infty$. *Then* (1.1) *admits a positive bounded solution.*

Proof. If possible, let there exist $T \geq T^*$ such that

$$\int_T^\infty \frac{1}{r(t)}\left[\int_t^\infty q(s)ds + L\int_t^\infty v(s)ds\right]dt < \frac{a_1-1}{3K},$$

where $K = \max\left\{K_1,\ \frac{K_2}{L},\ G(d)\right\}$, K_1 is the Lipschitz constant of G and K_2 is the Lipschitz constant of H on $[c, d]$ with

$$c = \frac{3\mu(a_1^2 - a_2) - a_2(a_1-1)}{3a_1^2 a_2},\ d = \frac{a_1 - 1 + 3\mu}{3a_1},\ \mu > \frac{a_2(a_1-1)}{3(a_1^2 - a_2)} > 0.$$

Let $X = BC([t_0,\infty),\ \mathbb{R})$ be the space of real valued continuous functions on $[t_0,\ \infty)$. Indeed, X is a Banach space with respect to the sup norm defined by

$$\| x \| = \sup\{|x(t)| :\ t \geq t_0\}.$$

Define

$$S = \{u \in X : c \leq u(t) \leq d,\ t \geq t_0\}.$$

We may note that S is a closed convex subspace of X. Let $\Psi : S \to S$ be such that

$$\Psi x(t) = \begin{cases} \Psi x(T), \quad t \in [T_0, T] \\ -\dfrac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \dfrac{\mu}{p(\tau^{-1}(t))} \\ +\dfrac{1}{p(\tau^{-1}(t))}\displaystyle\int_T^{\tau^{-1}(t))}\frac{1}{r(s)}\left[\int_s^\infty q(\theta)G(x(\sigma(\theta)))d\theta\right. \\ \left. +\displaystyle\int_s^\infty v(\theta)H(x(\eta(\theta)))d\theta\right]ds,\ t \geq T. \end{cases}$$

For every $x \in S$,

$$\begin{aligned}\Psi x(t) &\leq \frac{\mu}{p(\tau^{-1}(t))} + \frac{1}{p(\tau^{-1}(t))}\int_T^{\tau^{-1}(t))} \frac{G(d)}{r(s)}\left[\int_s^\infty q(\theta)d\theta + \frac{H(d)}{G(d)}\int_s^\infty v(\theta)d\theta\right]ds \\ &\leq \frac{\mu}{p(\tau^{-1}(t))} + \frac{1}{a_1}\int_T^\infty \frac{G(d)}{r(s)}\left[\int_s^\infty q(\theta)d\theta + L\int_s^\infty v(\theta)d\theta\right]ds \\ &\leq \frac{\mu}{a_1} + \frac{G(d)}{a_1}\int_T^\infty \frac{1}{r(s)}\left[\int_s^\infty q(\theta)d\theta + L\int_s^\infty v(\theta)d\theta\right]ds \\ &\leq \frac{1}{a_1}\left[\frac{a_1-1}{3} + \mu\right] = b\end{aligned}$$

and

$$\begin{aligned}\Psi x(t) &\geq -\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{\mu}{p(\tau^{-1}(t))} \\ &> -\frac{d}{a_1} + \frac{\mu}{a_2} = c\end{aligned}$$

implies that $\Psi x \in S$. Again for $x_1,\ x_2 \in S$

$$\begin{aligned}|\Psi x_1(t) - \Psi x_2(t)| &\leq \frac{1}{|p(\tau^{-1}(t))|}|x_1(\tau^{-1}(t)) - x_2(\tau^{-1}(t))| \\ &+ \frac{K_1}{|p(\tau^{-1}(t))|}\int_T^{\tau^{-1}(t))} \frac{1}{r(s)}\int_s^\infty q(\theta)|x_1(\sigma(\theta)) - x_2(\sigma(\theta))|d\theta ds \\ &+ \frac{K_2}{|p(\tau^{-1}(t))|}\int_T^{\tau^{-1}(t))} \frac{1}{r(s)}\int_s^\infty v(\theta)|x_1(\eta(\theta)) - x_2(\eta(\theta))|d\theta ds \\ &\leq \frac{1}{a_1} \parallel x_1 - x_2 \parallel + \frac{K_1}{a_1} \parallel x_1 - x_2 \parallel \int_T^\infty \frac{1}{r(s)}\int_s^\infty q(\theta)d\theta ds \\ &+ \frac{K_2}{a_1} \parallel x_1 - x_2 \parallel \int_T^\infty \frac{1}{r(s)}\int_s^\infty v(\theta)d\theta ds \\ &\leq \frac{1}{a_1}\left(1 + \frac{a_1-1}{3}\right) \parallel x_1 - x_2 \parallel\end{aligned}$$

implies that

$$\parallel \Psi x_1 - \Psi x_2 \parallel \leq \left(\frac{1}{a_1} + \frac{a_1-1}{3a_1}\right) \parallel x_1 - x_2 \parallel .$$

Since $\left(\frac{1}{a_1} + \frac{a_1-1}{3a_1}\right) < 1$, then $\Psi : S \to S$ is a contraction. By Banach's fixed point theorem, Ψ has a unique fixed point on $[c, d]$. It is easy to verify that

$$x(t) = \begin{cases} \Psi x(T), \quad t \in [t_0, T] \\ -\dfrac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \dfrac{\mu}{p(\tau^{-1}(t))} + \dfrac{1}{p(\tau^{-1}(t))} \times \\ \displaystyle\int_T^{\tau^{-1}(t))} \frac{1}{r(s)} \left[\int_s^\infty q(\theta) G(x(\sigma(\theta))) d\theta \right. \\ \left. + \displaystyle\int_s^\infty v(\theta) H(x(\sigma(\theta))) d\theta \right] ds, t \geq T \end{cases}$$

is a positive bounded solution of (1.1) on $[c, d]$. Hence, the proof is complete. □

Example 2.12. Consider the differential equations

$$(e^t(x(t) + e^{3\pi}x(t - 3\pi))')' + e^{3t+2\pi}G(x(t - 2\pi)) + e^{3t+3\pi}H(x(t - 3\pi)) = 0, \quad (2.5)$$

where $t \geq 2\pi$, $G(x) = H(x) = x$. All conditions of Theorem 2.10 are satisfied for (2.5). Hence, every solution of (2.5) oscillates. In particular, $x(t) = e^t \sin t$ is one of such solution of (2.5).

Acknowledgement

This work is supported by UGC RGNF, Government of India, through the letter with the number No F1-17.1/2013-14/RGNF-2013-14-SC-ORI-42425, dated Feb. 6, 2014

References

[1] R. P. Agarwal, M. Bohner, T. Li, C. Zhang; *Oscillation of second order differential equations with a sublinear neutral term,* Carpathian J. Math., 30(2014), 1–6.

[2] B. Baculikova, J. Dzurina; *Oscillation theorems for second order neutral differential equations,* Comput. Math. Appl., 61(2011), 94–99.

[3] B. Baculikova, J. Dzurina; *Oscillation theorems for second order nonlinear neutral differential equations,* Comput. Math. Appl., 62(2011), 4472–4478.

[4] B. Baculikova, T. Li, J. Dzurina; *Oscillation theorems for second order neutral differential equations,* Elect. J. Qual. Theo. Diff. Eqn., 74(2011), 1–13.

[5] L. H. Erbe, Q. Kong, B. G. Zhang; *Oscillation Theory for Functional Differential Equations,* Marcel Dekker, Inc., (1995).

[6] I. Gyori, G. Ladas; *Oscillation Theory of Delay Differential Equations with Applications,* Clarendon Press, Oxford, (1991).

[7] J. K. Hale; *Theory of functional differential equations,* Springer-Verlag, New York, (1977).

[8] T. H. Hildebrandt; *Introduction to the Theory of Integration,* Academic press, New York, (1963).

[9] T. Li , Y. V. Rogovchenko; *Oscillation theorems for second order nonlinear neutral delay differential equations,* Abst. Appl. Anal., 2014(2014), ID 594190, 1–5.

[10] Y. Qian, R. Xu; *Some new osciilation criteria for higher order quasi-linear neutral delay differential equations,* Diff. Equ. Appl., 3(2011), 323–335.

[11] A. K. Tripathy, B. Panda, A. K. Sethi; *On oscillatory nonlinear second order neutral delay differential equations,* Diff. Equ. Appl., 8(2016), 247–258.

[12] A. K. Tripathy, A. K. Sethi; *Oscillation of sublinear and superlinear second order neutral differential equations,* Int. J. Pure and Appl. Math., 113(2017), 73–91.

[13] Q. Yang, Z. Xu; *Oscillation criteria for second order quasi-linear neutral delay differential equations on time scales,* Comp. Math. Appl., 62(2011), 3682–3691.

[14] L. Ye, Z. Xu; *Oscillation criteria for second order quasilinear neutral delay differential equations,* Appl. Math. Comp., 207(2009), 388–396.