Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 13, Number 1, pp. 59–71 (2018)
http://campus.mst.edu/adsa

The Invariant Curve in a Planar System of Difference Equations

Senada Kalabušić, **Naida Mujić** and **Esmir Pilav**
University of Sarajevo
Department of Mathematics
Sarajevo, 71000, Bosnia and Herzegovina
senadak@pmf.unsa.ba, mujicnaida@gmail.com
and esmir.pilav@pmf.unsa.ba

Abstract

We find the asymptotic approximation of the invariant curve in Neimark–Sacker theorem circling the positive equilibrium of a certain planar discrete system considered in [1].

AMS Subject Classifications: 39A10, 39A28, 39A30.
Keywords: Discrete systems, invariant curve, Neimark–Sacker bifurcation.

1 Introduction

The planar system of difference equations

$$\left\{\begin{array}{lcl} x_{n+1} & = & x_n\left(\dfrac{r}{1+x_n}-y_n\right) \\ y_{n+1} & = & y_n\left(Bx_n-\mu\right) \end{array}\right. , \quad n=0,1,2,\dots, \tag{1.1}$$

where the parameters B, λ are positive numbers, $r > 1$ and

$$B > \frac{\mu+1}{r-1}, \tag{1.2}$$

was considered in [1]. System (1.1) is obtained from the planar system of difference equations

$$\left\{\begin{array}{lcl} x_{n+1} & = & \dfrac{rkx_n}{k+(r-1)x_n}-\alpha x_n y_n \\ y_{n+1} & = & \beta x_n y_n-\mu y_n, \end{array}\right. \quad n=0,1,2,\dots, \tag{1.3}$$

Received December 15, 2017; Accepted March 16, 2018
Communicated by Mustafa R. S. Kulenović

by reducing the number of parameters. System (1.3) can be used as a certain predator–prey model, see [1]. The authors established the boundedness and persistence of all solutions of (1.1).

The local asymptotic stability of the following three equilibrium points of (1.1)

$$E_0 = (0,0),\ E_1 = (r-1, 0),$$

and

$$E_* = (\bar{x}, \bar{y}) = \left(\frac{1+\mu}{B}, \frac{Br - B - \mu - 1}{B + \mu + 1}\right) = (\bar{x}, \frac{r}{\bar{x}+1} - 1) \tag{1.4}$$

was discussed in [1]. The authors in [1] denoted E_0 and E_1 as the boundary equilibria and E_* as the positive one. They proved that E_0 is the saddle point when $0 < \mu < 1$ and the repeller when $\mu > 1$, while the equilibrium E_1 is the saddle point when $|\tilde{B} - \mu| > 1$ and is locally asymptotically stable when $|\tilde{B} - \mu| < 1$.

The behaviour of the positive equilibrium E_* is described by the following lemmas, see [1].

Lemma 1.1 (See [1]). *Let*

$$C = B\bar{x}\bar{y}, \quad D = \frac{r\bar{x}}{(1+\bar{x})^2}. \tag{1.5}$$

The eigenvalues are both within the unit circle if and only if (C, D) *is within the triangle with vertices* $(0,0), (0,2)$ *and* $(4,4)$*. If* $D > 2 + \frac{1}{2}C$*, then* $|\lambda_2| < 1$ *and* $|\lambda_1| > 1$.

Lemma 1.2 ([1]). *Each of the following holds true for* (1.1)

i) If the parameters B, r *and* μ *satisfy the inequalities*

$$\begin{cases} \left(B > \dfrac{1+\mu}{r-1} \text{ and } C > 0\right) \text{ or} \\ ([\mu(r-1) + (r-5)]\, B < (\mu+5)(\mu+1) \text{ and } C < 4) \text{ or} \\ \left(B^2(r-1) + (\mu r - 2\mu - 2)B < (\mu+1)^2 \text{ and } C < D\right) \text{ or} \\ \left([r + 3 + \mu(r-1)]\, B^2 + (1+\mu)\,[4 + (1-\mu)(2-r)]\, B\right. \\ +(3-\mu)(1+\mu)^2 > 0 \text{ and } D < 2 + \dfrac{C}{2} \end{cases}$$

then E_* *is locally asymptotically stable.*

ii) If the parameters B, r *and* μ *satisfy*

$$[r + 3 + \mu(r-1)]B^2 + (1+\mu)[4 + (1-\mu)(2-r)]B + (3-\mu)(1+\mu)^2 < 0,$$

then E_* *is a saddle point.*

iii) Both eigenvalues of the Jacobian matrix at E_* are nonreal of modulus one when

$$\left.\begin{aligned} r &= \frac{(\mu+t+1)^2}{(\mu+1)(1-\mu(t-1))} \\ B &= \frac{(\mu+2)(\mu+1)(r-1)+t}{(\mu+1)(r-1)} \end{aligned}\right., 0 < t < \min\left\{4, 1+\frac{1}{\mu}\right\},$$

where either $0 < t \leq 4, \mu < \dfrac{1}{3}$, *or* $0 < t < 1 + \dfrac{1}{\mu}, \mu \geq \dfrac{1}{3}$.

In [1, Section 5], the authors focused on the positive equilibrium E_* when the two eigenvalues of the Jacobian matrix are nonreal and located on the unit circle. Under certain conditions on the parameters, they proved that (1.1) undergoes Neimark–Sacker bifurcation and the obtained curve is supercritical. More precisely, the following result has been proved.

Theorem 1.3 (See [1]). *Consider* (1.1)*, where*

$$r = \frac{(\mu+t+1)^2}{(\mu+1)(1-\mu(t-1))}, \quad B = \frac{(\mu+2)(\mu+1)(r-1)+t}{(\mu+1)(r-1)}, \ 0 < t < 4.$$

If $(\mu+1)(1-2\mu)r \neq (\mu+4)^2$ *and* $(1-\mu^2)r \neq (\mu+3)^2$*, then Neimark–Sacker bifurcation occurs and the obtained curve is supercritical.*

In [4] and [5], certain rational difference equations with quadratic terms were considered. The authors computed the direction of the Neimark–Sacker bifurcation and gave the asymptotic approximation of the invariant curve. Their computational method is based on the computational algorithm developed in [12]. The advantage of the computational algorithm of [12] lies in the fact that this algorithm computes also the approximate equation of the invariant curve in Neimark–Sacker theorem, [15, Theorem 3.2.3], which is not provided by Wans algorithm in [14] which is used in [1].

In this note, we use the same approach as in [4] and [5] to find the asymptotic approximation of the invariant curve circling the positive equilibrium E_* of (1.1). Also, for some numerical values of parameters we give visual evidence that the approximate equation of the invariant curve is accurate.

2 Preliminaries

In this section, for the sake of completeness we give the basic facts about the Neimark–Sacker bifurcation.

Hopf bifurcation is a well-known phenomenon for a system of ordinary differential equations in two or more dimension, whereby, when some parameter is varied, a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point crosses the imaginary axis, so that the fixed point changes its behavior from stable to unstable and a limit cycle appears.

In the discrete setting, the Neimark–Sacker bifurcation is the discrete analogue of the Hopf bifurcation. The Neimark–Sacker bifurcation occurs for a discrete system depending on a parameter, h, with a fixed point whose Jacobian has a pair of complex conjugate $\lambda(h)$, $\bar{\lambda}(h)$ which cross the unit transversally at critical value $h = h_0$. See [2, 7, 11, 13, 15].

Consider a general map $F(h, x)$ that has a fixed point at the origin with complex eigenvalues $\lambda(h) = \alpha(h)+i\beta(h)$ and $\bar{\lambda}(h) = \alpha(h)-i\beta(h)$ satisfying $\alpha(h)^2+\beta(h)^2 = 1$ and $\beta(h) \neq 0$. By putting the linear part of such a map into Jordan Canonical form, we may assume F to have the following form near the origin

$$F(h, x) = A(h)x + G(h, x),$$

where

$$A(h) = \begin{pmatrix} \alpha(h) & -\beta(h) \\ \beta(h) & \alpha(h) \end{pmatrix}, G(h, x) = \begin{pmatrix} g_1(h, x_1, x_2) \\ g_2(h, x_1, x_2) \end{pmatrix}. \tag{2.1}$$

Let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of A associated with λ satisfying

$$A\mathbf{q} = \lambda\mathbf{q},\ \mathbf{p}A = \lambda\mathbf{p},\ ,\mathbf{pq} = 1$$

and

$$\Phi = (\mathbf{q}, \overline{\mathbf{q}}).$$

Assume that

$$G\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right) = \frac{1}{2}(g_{20}z^2 + 2g_{11}z\overline{z} + g_{02}\overline{z}^2) + O(|z|^3)$$

and

$$\begin{aligned} K_{20} &= (\lambda^2 I - A)^{-1} g_{20} \\ K_{11} &= (I - A)^{-1} g_{11} \\ K_{02} &= (\overline{\lambda}^2 I - A)^{-1} g_{02}, \end{aligned} \tag{2.2}$$

where $A = A(h_0)$. Let

$$G\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix} + \frac{1}{2}(K_{20}\xi^2 + 2K_{11}\xi\overline{\xi} + K_{02}\overline{\xi}^2)\right) = \frac{1}{2}(g_{20}\xi^2 + 2g_{11}\xi\overline{\xi} + g_{02}\overline{\xi}^2)$$
$$+ \frac{1}{6}(g_{30}\xi^3 + 3g_{21}\xi^2\overline{\xi} + 3g_{12}\xi\overline{\xi}^2 + g_{03}\overline{\xi}^3) + O(|\xi|^4), \tag{2.3}$$

then

$$a(h_0) = \frac{1}{2}\mathrm{Re}(\mathbf{p}q_{21}\overline{\lambda}).$$

The invariant curve from the previous computational algorithm can be approximated by using the following corollary (see [12]).

Corollary 2.1. *Assume $a(h_0) \neq 0$ and $h = h_0 + \eta$ where η is a sufficient small parameter. If $\overline{x}$ is a fixed point of F then the invariant curve $\Gamma(h)$ from Neimark–Sacker theorem can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \overline{x} + 2\rho_0 \mathrm{Re}(qe^{i\theta}) + \rho^2(\mathrm{Re}(K_{20}e^{2i\theta}) + K_{11}),$$

where

$$d = \frac{d}{d\eta}|\lambda(h)|\Big|_{h=h_0}, \ \rho_0 = \sqrt{-\frac{d}{a}\eta}, \ \theta \in \mathbb{R}.$$

3 Approximation of the Invariant Curve

In this section, we approximate the invariant curve circling the positive equilibrium E_* of (1.1). The bifurcation parameter is B and its critical value is equal to B_0.

Theorem 3.1. *Consider* (1.1) *and its positive equilibrium E_*. Assume that $\mu > 0, r > 1, B > \dfrac{m+1}{r-1}$ and for critical value B_0 the following conditions are satisfied*

i) $B_0 > \dfrac{m+1}{r-1}$,

ii) $B_0^2(r-1) + (\mu r - 2\mu - 2)B_0 - (\mu+1)^2 = 0$,

iii) $B_0(\mu+1)r - 4(B_0+\mu+1)^2 < 0$,

iv) $(-\mu-3)(B_0+\mu+1) + B_0(\mu+1)r \neq 0$ *and*

v) $(-\mu-4)(B_0+\mu+1) + B_0(\mu+1)r \neq 0$.

Then there is a neighborhood U of the equilibrium point $(\bar{x}, \bar{y})$ and a $\rho > 0$ such that for $|B - B_0| < \rho$, the ω-limit set of solution of (1.1) *is equilibrium point $(\bar{x}, \bar{y})$ if $B < B_0$ and belongs to a closed invariant C^1 curve $\Gamma(B)$ encircling the $(\bar{x}, \bar{y})$ if $B > B_0$. Furthermore, $\Gamma(B_0) = 0$ and invariant curve can be approximated by*

$$\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} \bar{x} - \dfrac{\rho_0\left(B_0\bar{x}\bar{y}\cos(\theta) + \Delta\sin(\theta)\right)}{B_0\bar{y}} + \dfrac{(B-B_0)(\bar{x}+1)\Upsilon_1}{\Upsilon} \\ \bar{y} + 2\rho_0\cos(\theta) + \dfrac{(B_0-B)(\bar{x}+1)\Upsilon_2}{\Upsilon} \end{pmatrix},$$

where

$$\begin{aligned} \Upsilon &= B_0\bar{x}\left(B_0\bar{x}\bar{y} - 3\right) \\ &\quad \left(B_0\bar{x}\bar{y}\left(B_0\bar{x} + B_0 - 1\right)\left(\bar{x}\left(B_0\bar{x} + B_0 - 1\right) + 1\right) - B_0\bar{x}^2 + \bar{x} + B_0 - 2\right) \\ \Upsilon_1 &= (\bar{x}+1)\,\bar{y}\left(B_0\bar{x}\left(B_0\bar{x}\bar{y} - 3\right) + \left(B_0\bar{x}\left(\bar{y}\left(B_0\bar{x} - 1\right) - 3\right) + 4\right)\cos(2\theta)\right) \end{aligned}$$

$$
\begin{aligned}
&+\left(\Delta\left(-\bar{y}\left(\bar{x}\left(B_0\bar{x}+B_0-1\right)+1\right)+\bar{x}+1\right)\right)\sin(2\theta),\\
\Upsilon_2 \;=\;& B_0\left(\bar{x}\left(\bar{y}\left(B_0\bar{x}\bar{y}\left(B_0\bar{x}+B_0-1\right)-4B_0\left(\bar{x}+1\right)+3\right)+4\right)+\bar{y}+4\right)\bar{x}\cos(2\theta)\\
&+B_0\bar{x}\bar{y}\left(\bar{x}\left(B_0\bar{x}+B_0-1\right)+1\right)\left(B_0\bar{x}\bar{y}-3\right)-\left(\Delta\left(\bar{x}\left(B_0\bar{x}\bar{y}\left(B_0\bar{x}+B_0-1\right)\right.\right.\right.\\
&\left.\left.\left.-2B_0\left(\bar{x}+1\right)+1\right)+1\right)\right)\sin(2\theta).
\end{aligned}
$$

Proof. In order to apply Corollary 2.1 we shift the positive equilibrium E_* to the origin by taking the substitution

$$
\begin{aligned}
u_n &= x_n-\bar{x},\\
v_n &= y_n-\bar{y}.
\end{aligned}
$$

Now, (1.1) has the (u,v) form

$$
\left\{
\begin{aligned}
u_{n+1} &= \left(\bar{x}+u_n\right)\left(\frac{r}{\bar{x}+u_n+1}-\bar{y}-v_n\right)-\bar{x}\\
v_{n+1} &= \left(\bar{y}+v_n\right)\left(B\left(\bar{x}+u_n\right)-\mu\right)-\bar{y}
\end{aligned}
\right. ,\quad n=0,1,2,\ldots. \tag{3.1}
$$

The corresponding map F_B associated to (3.1) is given by

$$
F_B\begin{pmatrix}u\\u\end{pmatrix}=\begin{pmatrix}\left(\bar{x}+u\right)\left(\dfrac{r}{\bar{x}+u+1}-\bar{y}-v\right)-\bar{x}\\ \left(\bar{y}+v\right)\left(B\left(\bar{x}+u\right)-\mu\right)-\bar{y}\end{pmatrix}.
$$

The Jacobian matrix of the map F_B is given by

$$
J_{F_B}(x,y)=\begin{pmatrix}\dfrac{r}{\left(u+\bar{x}+1\right)^2}-v-\bar{y} & -u-\bar{x}\\ B\left(v+\bar{y}\right) & B\left(u+\bar{x}\right)-m\end{pmatrix}.
$$

The Jacobian matrix of the map F_B at the shifted equilibrium $(0,0)$ is given by

$$
J_{F_B}(0,0)=\begin{pmatrix}\dfrac{r}{\left(\bar{x}+1\right)^2}-\bar{y} & -\bar{x}\\ B\bar{y} & B\bar{x}-\mu\end{pmatrix}. \tag{3.2}
$$

By using (1.4) and (1.5), we can rewrite (3.2) as

$$
J_{F_B}(0,0)=\begin{pmatrix}1-D & -\bar{x}\\ B\bar{y} & 1\end{pmatrix}. \tag{3.3}
$$

Thus, we have

$$
F_B\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}1-D & -\bar{x}\\ B\bar{y} & 1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}f_1(B,u,v)\\f_2(B,u,v)\end{pmatrix}, \tag{3.4}
$$

where

$$\begin{aligned} f_1(B,u,v) &= \frac{u\left(D\left(\bar{x}+1\right)-v\left(\bar{x}+1\right)-\left(2\bar{x}+1\right)\left(\bar{y}+1\right)+r\right)}{\bar{x}+u+1} \\ &\quad +\frac{u^2\left(-\bar{y}+D-v-1\right)-\bar{x}\left(\bar{x}\bar{y}+\bar{x}+\bar{y}-r+1\right)}{\bar{x}+u+1} \\ f_2(B,u,v) &= Buv. \end{aligned}$$

For $B = B_0$, we obtain

$$E = (\bar{x},\bar{y}) = \left(\frac{1+\mu}{B_0}, \frac{B_0 r - B_0 - \mu - 1}{B_0+\mu+1}\right). \tag{3.5}$$

According to Lemma 1.1, for bifurcation to occur we need

$$B_0\bar{x}\bar{y} = \frac{r\bar{x}}{(1+\bar{x})^2} \text{ and } 0 < B_0\bar{x}\bar{y} < 4.$$

This implies

$$B_0 = \frac{r}{\bar{y}(1+\bar{x})^2} \text{ and } 0 < \frac{r\bar{x}}{(1+\bar{x})^2} < 4, \tag{3.6}$$

and

$$\begin{gathered} B_0^2(r-1) + (\mu r - 2\mu - 2)B_0 + (3-\mu) - (1+\mu)^2 = 0, \\ B_0(\mu+1)r - 4(B_0+\mu+1)^2 < 0. \end{gathered}$$

For $B = B_0$, the Jacobian in (3.3) becomes

$$J_{F_{B_0}}(0,0) = \begin{pmatrix} 1 - B_0\bar{x}\bar{y} & -\bar{x} \\ B_0\bar{y} & 1 \end{pmatrix}.$$

The eigenvalues of $J_{F_{B_0}}(0,0)$ are $\lambda(B_0)$ and $\bar{\lambda}(B_0)$, where

$$\lambda(B_0) = \frac{2 - B_0\bar{x}\bar{y} + i\sqrt{(4-B_0\bar{x}\bar{y})B_0\bar{x}\bar{y}}}{2}.$$

Now, (1.4) and (1.2) imply

$$\begin{aligned} &(4-B_0\bar{x}\bar{y})B_0\bar{x}\bar{y} \\ &\qquad = \frac{(\mu+1)(Br-B-\mu-1)((\mu+5)(B+\mu+1)-B(\mu+1)r)}{(B+\mu+1)^2} > 0. \end{aligned}$$

The eigenvectors corresponding to $\lambda(B_0)$ are $\mathbf{q}(B_0)$ and $\bar{\mathbf{q}}(B_0)$, where

$$\mathbf{q}(B_0) = \left(\frac{-B_0\bar{x}\bar{y} + i\sqrt{(4-B_0\bar{x}\bar{y})B_0\bar{x}\bar{y}}}{2B_0\bar{y}}, 1\right).$$

Set $\Delta = \sqrt{(4 - B_0\bar{x}\bar{y})B_0\bar{x}\bar{y}}$. It is easy to see that $|\lambda(B_0)| = 1$ and

$$\begin{aligned}
\lambda^2(B_0) &= \frac{1}{2}\left((2 - B_0\bar{x}\bar{y})^2 + i\,(2 - B_0\bar{x}\bar{y})\,\Delta - 2\right), \\
\lambda^3(B_0) &= \frac{1}{2}\left((2 - B_0\bar{x}\bar{y})\left(B_0\bar{x}\bar{y}^2 - 4B_0\bar{x}\bar{y} + 1\right) + i\,(B_0\bar{x}\bar{y} - 3)\,(B_0\bar{x}\bar{y} - 1)\,\Delta\right) \\
\lambda^4(B_0) &= \frac{1}{2}\left((B_0\bar{x}\bar{y} - 4)\,B_0\bar{x}\bar{y}\,(B_0\bar{x}\bar{y} - 2)^2\right. \\
&\quad \left. -i\left(B_0\bar{x}\bar{y}^2 - 4B_0\bar{x}\bar{y} + 2\right)(B_0\bar{x}\bar{y} - 2)\,\Delta + 2\right).
\end{aligned}$$

One can see that $\lambda^k \neq 1$ for $k = 1, 2, 3, 4$. If

$$B_0\bar{x}\bar{y} \neq 2 \quad \text{and} \quad B_0\bar{x}\bar{y} \neq 3,$$

then

$$(-\mu-3)(B_0+\mu+1)+B_0(\mu+1)r \neq 0 \quad \text{and} \quad (-\mu-4)(B_0+\mu+1)+B_0(\mu+1)r \neq 0.$$

Furthermore, we get

$$d(B_0) = \frac{d}{dB}|\lambda(B)|\bigg|_{B=B_0} = \frac{\bar{x}\bar{y}}{2\sqrt{B\bar{x}\bar{y} - \frac{r\bar{x}}{(\bar{x}+1)^2} + 1}}\Bigg|_{B=B_0} = \frac{\bar{x}\bar{y}}{2} > 0.$$

From the above considerations, we conclude that all conditions of Corollary 2.1 are satisfied. Let us now approximate the invariant curve. Substituting $B = B_0$ into (3.4), we get

$$F\begin{pmatrix} u \\ v \end{pmatrix} = J_{F_{B_0}}\begin{pmatrix} u \\ v \end{pmatrix} + G\begin{pmatrix} u \\ v \end{pmatrix},$$

where

$$G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} G_1(u,v) \\ G_2(u,v) \end{pmatrix}.$$

and

$$\begin{aligned}
G_1(u,v) &= \frac{u\,(B_0\bar{x}\,(\bar{x}+1)\,\bar{y} - v\,(\bar{x}+1) - (2\bar{x}+1)\,(\bar{y}+1) + r)}{\bar{x}+u+1} \\
&\quad + \frac{u^2\,(-\,(-B_0\bar{x}\bar{y} + \bar{y} + v + 1)) - \bar{x}\,(\bar{x}\bar{y} + \bar{x} + \bar{y} - r + 1)}{\bar{x}+u+1} \\
G_2(u,v) &= B_0uv.
\end{aligned}$$

Hence, (3.1) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = J_{F_{B_0}}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + G\begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

Let us define the basis of $\mathbb{R}^2$ by $\Phi = (\mathbf{q}, \overline{\mathbf{q}})$ where $\mathbf{q} = \mathbf{q}(B_0)$. Then, we can represent (u, v) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}z + \overline{\mathbf{q}z}) = \begin{pmatrix} \dfrac{-B_0\bar{x}\bar{y}\left(\bar{z}+z\right)+i\left(z-\bar{z}\right)\Delta}{2B_0\bar{y}} \\ z+\bar{z} \end{pmatrix},$$

from which, we have

$$G\left(\Phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \begin{pmatrix} \dfrac{B_0\bar{y}\left(B_0\Gamma_2\bar{x}\bar{y}+\Gamma_1\right)+\left(z-\bar{z}\right)\Gamma}{B_0\bar{y}\left(\bar{x}\left(\bar{z}+z-2\right)-2\right)-i\Delta\left(z-\bar{z}\right)} \\ -\dfrac{\left(\bar{z}+z\right)\left(B_0\bar{x}\bar{y}\left(\bar{z}+z\right)-i\Delta\left(z-\bar{z}\right)\right)}{2\bar{y}} \end{pmatrix},$$

where

$$\begin{aligned}
\Gamma &= -i\Delta\bar{x}\left(\bar{y}\left(\bar{z}+z-2\right)+\left(\bar{z}+z\right)^2-2\right) \\
&\quad -2\bar{x}\left(z-\bar{z}\right)\left(\bar{y}+\bar{z}+z+1\right)+i\Delta\left(\bar{y}+\bar{z}+z+1\right) \\
\Gamma_1 &= \bar{x}\left(\bar{x}\left(\bar{y}\left(\left(\bar{z}-2\right)\bar{z}+(z-2)z+2\right)+\bar{z}^3+(z+2)\bar{z}^2\right.\right. \\
&\quad +(z-6)z\bar{z}-2\bar{z}+z(z(z+2)-2)+2)-\bar{y}\left(\bar{z}+z-2\right) \\
&\quad \left.-\left(\bar{z}+z-1\right)\left(\bar{z}+z+2\right)\right)+i\Delta\left(z-\bar{z}\right)\left(\bar{x}\left(\bar{x}\left(\bar{z}+z-2\right)-3\right)-1\right), \\
\Gamma_2 &= -\bar{x}\left(\bar{x}\left(\left(\bar{z}-2\right)\bar{z}+(z-2)z+2\right)-3\bar{z}-3z+4\right)+\bar{z}+z-2.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
g_{20} &= \left.\frac{\partial^2}{\partial z^2}G\left(\Phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\right|_{z=0} = \begin{pmatrix} \dfrac{\Gamma_3}{2B_0^2\left(\bar{x}+1\right)\bar{y}^2} \\ -B_0\bar{x}+\dfrac{i\Delta}{\bar{y}} \end{pmatrix} \\
g_{11} &= \left.\frac{\partial^2}{\partial z\partial\bar{z}}G\left(\Phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\right|_{z=0} = \begin{pmatrix} \dfrac{\Gamma_4}{2B_0^2\left(\bar{x}+1\right)\bar{y}^2} \\ -B_0\bar{x} \end{pmatrix} \\
g_{02} &= \left.\frac{\partial^2}{\partial\bar{z}\partial\bar{z}}G\left(\Phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\right|_{z=0} = \overline{g_{02}},
\end{aligned}$$

and

$$\begin{aligned}
K_{20} &= \begin{pmatrix} \dfrac{\Gamma_5}{B_0\left(\bar{x}+1\right)\bar{y}^2\left(B_0\bar{x}\bar{y}-3\right)} \\ \dfrac{\Gamma_6}{B_0\bar{x}\left(\bar{x}+1\right)\bar{y}\left(B_0\bar{x}\bar{y}-3\right)\left(B_0\bar{x}\bar{y}\left(B_0\bar{x}\bar{y}-i\Delta-4\right)+2i\Delta+2\right)} \end{pmatrix} \\
K_{11} &= \begin{pmatrix} \dfrac{\bar{x}}{\bar{y}} \\ -\dfrac{2}{\bar{x}+1}-B_0\bar{x}+1 \end{pmatrix}
\end{aligned}$$

$$K_{02} = \overline{K_{20}},$$

where

$$\begin{aligned}
\Gamma_3 &= \Delta^2 (\bar{y} (B_0 \bar{x} + B_0 - 1) - 1) + 2iB_0 \Delta \bar{y} (\bar{x} (B_0 \bar{y} - 1) - 1) \\
&\quad + B_0 \bar{x} \bar{y} \left(\bar{y} \left(B_0 \left(B_0 \bar{x}^2 \bar{y} - \bar{x} (B_0 \bar{y} + \bar{y} + 3) + 2\right) + 4\right) + 4\right) \\
\Gamma_4 &= \Delta^2 (1 - \bar{y} (B_0 \bar{x} + B_0 - 1)) \\
&\quad + B_0 \bar{x} \bar{y} (\bar{y} (B_0 (\bar{x} (-\bar{y}) (B_0 \bar{x} + B_0 - 1) + 7\bar{x} + 2) - 4) - 4) \\
\Gamma_5 &= (\bar{x} + 1) \bar{y} (B_0 \bar{x} (\bar{y} (B_0 \bar{x} - 1) - 3) + 4) \\
&\quad + i\Delta (\bar{x} (\bar{y} (B_0 \bar{x} + B_0 - 1) - 1) + \bar{y} - 1) \\
\Gamma_6 &= 2B_0 \bar{x} (\bar{x} (\bar{y} (1 - B_0 (\bar{x} (B_0 (\bar{x} + 1) \bar{y} - 4) + \bar{y} - 4)) - 4) + 3\bar{y} - 4) \\
&\quad + 2i\Delta (B_0 \bar{x} \bar{y} (B_0 \bar{x} (\bar{x} + 1) + 1) + (-\bar{x} - 1) (2B_0 \bar{x} + 1)).
\end{aligned}$$

Also,

$$\begin{aligned}
g_{21} &= \frac{\partial^3}{\partial z^2 \partial \bar{z}} G \left(\Phi \begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2} K_{20} z^2 + K_{11} z \bar{z} + \frac{1}{2} K_{02} \bar{z}^2 \right) \Bigg|_{z=0} = \\
&= \begin{pmatrix} \dfrac{B_0 \bar{x} \bar{y} (B_0 \bar{x} \bar{y} (B_0 \bar{x} \bar{y} \Delta_1 \bar{x} + \Delta_2) + \Delta_3) + \bar{x} \Delta_7}{2B_0^2 (B_0 \bar{x} \bar{y} - 3) \bar{x} (\bar{x} + 1)^2 \bar{y}^2 (-i (B_0 \bar{x} \bar{y} - 2) \Delta + B_0 \bar{x} \bar{y}^2 - 4B_0 \bar{x} \bar{y} + 2)} \\ \dfrac{B_0 \bar{x} \bar{y} (B_0 \bar{x} \bar{y} (2B_0 B_0 \bar{x} \bar{y} \Delta_4 \bar{x} + \Delta_5) + \Delta_6) + 2B_0 \bar{x} \Delta_8}{B_0 (B_0 \bar{x} \bar{y} - 3) \bar{x} (\bar{x} + 1) \bar{y}^2 (-i (B_0 \bar{x} \bar{y} - 2) \Delta + B_0 \bar{x} \bar{y}^2 - 4B_0 \bar{x} \bar{y} + 2)} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1 &= -72 + B_0 (-4i\Delta (\bar{x} ((\bar{x} - 3) \bar{y} + 5\bar{x} - 2) - \bar{y}) + \bar{x} (-4\bar{x} (8\bar{y} + 13) + 91\bar{y} + 56) \\
&\quad + 4 (7\bar{y} + 7i\Delta + 27)) \\
\Delta_2 &= i\Delta \bar{x} (4B_0 ((\bar{x} (6\bar{x} - 19) - 5) \bar{y} + 5 (\bar{x} - 3) (\bar{x} + 1)) + 9) \\
&\quad + 4B_0 \bar{x} (\bar{x} (19\bar{x} - 48) - 15) \bar{y} - 2\bar{x} (60B_0 (\bar{x} + 1) + \bar{x} - 70) - 2 \\
\Delta_3 &= -i\Delta \bar{x} (B_0 (\bar{x} (36\bar{x} - 127) - 28) \bar{y} - 8B_0 (\bar{x} + 1) (2\bar{x} + 5) + 36) \\
&\quad + 2B_0 \bar{x} (23 - 19 (\bar{x} - 2) \bar{x}) \bar{y} + 4 (\bar{x} + 1) (\bar{x} (B_0 (7\bar{x} + 11) + 2) + 2)
\end{aligned}$$

$$\begin{aligned}
\Delta_4 &= i\Delta (\bar{x} (\bar{y} + 5) - \bar{y} + 5) + 8\bar{x}\bar{y} + 13\bar{x} - 9\bar{y} + 13 \\
\Delta_5 &= 1 + \bar{x} (-2iB_0 \Delta (6\bar{x}\bar{y} + 5\bar{x} - 7\bar{y} + 5) + 2B_0 (27 - 19\bar{x}) \bar{y} + 1) \\
\Delta_6 &= -4 + \bar{x} (-4 + B_0 (2i\Delta (9\bar{x}\bar{y} - 4\bar{x} - 15\bar{y} - 4) + 19\bar{x}\bar{y} - 14\bar{x} - 55\bar{y} - 14)) \\
\Delta_7 &= 4iB_0 \Delta (\bar{x} + 1) \left(B_0 \bar{x} \bar{y}^4 (\bar{x} - 1) - \bar{x} - 3\bar{y} - 1\right) \\
&\quad + 4B_0 \bar{y} \left(B_0 \bar{x} \bar{y}^4 ((\bar{x} - 3) \bar{x} - 1) - 4 (\bar{x} + 1)^2\right) \\
&\quad + B_0 \bar{x} \bar{y}^4 (9 - 4B_0 (\bar{x} + 1) (B_0 \bar{x} \bar{y} (\bar{x} - 1) - 7\bar{x} + 9)) \\
\Delta_8 &= -B_0 \bar{x} \bar{y}^4 (i\Delta (\bar{x} + 1) + \bar{x}\bar{y} + 7\bar{x} - \bar{y} + 7) + B_0 \bar{x} \bar{y}^5 (\bar{x} + 1) \\
&\quad + i\Delta (\bar{x} + 7\bar{y} + 1) + 4 (\bar{x} + 1) \bar{y}.
\end{aligned}$$

We can easily find the vector

$$\mathbf{p} = \left(-\frac{iB_0\bar{y}}{\sqrt{B_0\bar{x}\bar{y}\,(4 - B_0\bar{x}\bar{y})}}, -\frac{2i}{iB_0\bar{x}\bar{y} + \sqrt{B_0\bar{x}\bar{y}\,(4 - B_0\bar{x}\bar{y})} - 4i} \right)$$

so $\mathbf{p}A = \lambda\mathbf{p}$ and $\mathbf{pq} = 1$. This gives

$$\begin{aligned} a(B_0) &= \frac{1}{2}\mathrm{Re}(\mathbf{p}g_{21}\bar{\lambda}) = \\ &= \frac{\bar{x}\,(B_0\,(-\bar{x}\bar{y}\,(B_0\bar{x} + B_0 - 1)\,(\bar{x}\,(B_0\bar{x} + B_0 - 1) + 1) + \bar{x}^2 - 1) - \bar{x} + 2)}{2\,(\bar{x} + 1)^2}. \end{aligned}$$

By using (1.2), (1.4) and (3.6) we obtain

$$a(B_0) = -\frac{\bar{x}\,(\bar{x}\bar{y} + \bar{x}^2 - \bar{y}^2 + \bar{y})}{2\,(\bar{x} + 1)^3\,\bar{y}^2} < 0,$$

since $\bar{x}\bar{y} + \bar{x}^2 - \bar{y}^2 + \bar{y} > (r - 1)^2 > 0$. Now, we have

$$\begin{aligned} \rho_0 &= \sqrt{-\frac{d}{a}(B - B_0)} = \\ &= \sqrt{\frac{(B - B_0)\,(\bar{x} + 1)^2\,\bar{y}}{B_0\bar{x}\bar{y}\,(B_0\bar{x} + B_0 - 1)\,(\bar{x}\,(B_0\bar{x} + B_0 - 1) + 1) - B_0\bar{x}^2 + \bar{x} + B_0 - 2}}. \end{aligned}$$

The rest of the proof follows from Corollary 2.1. □

References

[1] Z. AlSharawi, S. K. Sasmal and J. Chattopathyay Persistance and bifurcation in two dinamical discrete system, *Advances in Dynamical Systems*, ISSN 0973-5321, Volume 12, Number 2, 63–83 (2017).

[2] J. K. Hale and H. Kocak, *Dynamics and bifurcations*. Texts in Applied Mathematics, 3. Springer-Verlag, New York, 1991.

[3] E. J. Janowski and M. R. S. Kulenović, Attractivity and global stability for linearizable difference equations, *Comput. Math. Appl.* 57 (2009), no. 9, 1592–1607.

[4] T. Khyat, M. R. S. Kulenović and E. Pilav, The Neimark–Sacker bifurcation and asymptotic approximation of the invariant curve of a certain difference equation, *J.Computational Analysis and Applications*, Vol.13. No. 8, 2017.

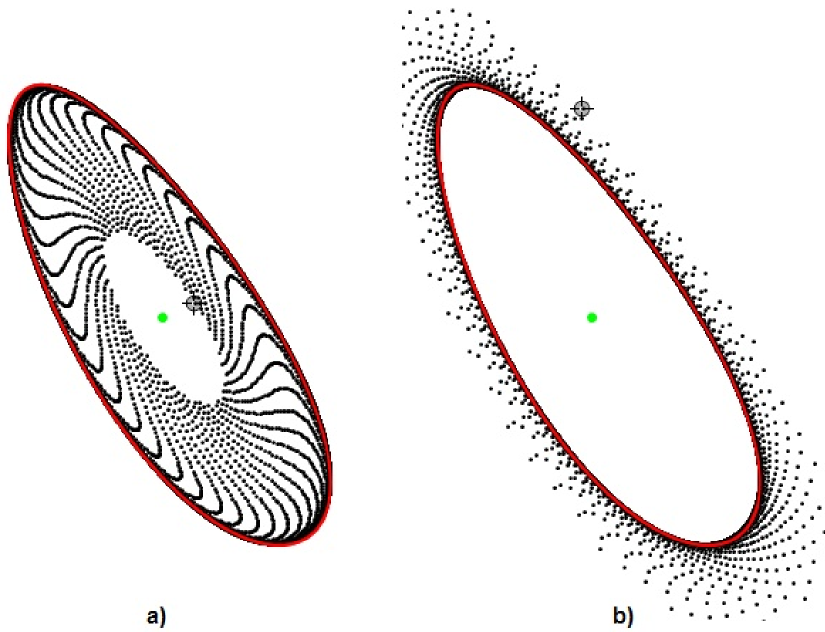

Figure 3.1: Trajectories and invariant curve for $r = 35$, $\mu = 0.6$, $B = 0.117647$, $B_0 = 0.117513$.

[5] T. Khyat, M. R. S. Kulenović and E. Pilav, The invariant curve caused by Neimark–Sacker bifurcation of a perturbaded Beverton–Holt difference equation, *International Journal of Difference Equations*, ISSN 0973-6069, Volume 12, Number 2, 267–280 (2017).

[6] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations, with Open Problems and Conjectures*, Chapman& Hall/CRC Press, 2001.

[7] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.

[8] M. R. S. Kulenović and O. Merino, A global attractivity result for maps with invariant boxes. *Discrete Contin. Dyn. Syst. Ser. B* 6(2006), 97–110.

[9] M. R. S. Kulenović, E. Pilav and E. Silić, Naimark–Sacker bifurcation of second order quadratic fractional difference equation, *J. Comp. Math. Sciences*, 4 (2014), 1025–1043.

[10] M. R. S. Kulenović, E. Pilav and E. Silić, Local Dynamics and Global Attractivity of a Certain Second Order Quadratic Fractional Difference Equation, *Adv. Difference Equ.* 2014, 32 pp.

[11] Y. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer, NewYork, 1998.

[12] Murakami, K., 2002, *The invariant curve caused by NeimarkSacker bifurcation*. Dynamics of Continuous, Discrete and Impulsive Systems, 9(1), 121–132.

[13] C. Robinson, *Stability, Symbolic Dynamics, and Chaos*, CRC Press, Boca Raton, 1995.

[14] Y. H. Wan, *Computation of the stability condition for the Hopf bifurcation of diffeomorphisms on* $\mathbb{R}^2$, SIAM J. Appl. Math. 34(1) (1978), 167–175.

[15] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*. Second edition. Texts in Applied Mathematics, 2. Springer-Verlag, New York, 2003.