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A Proof of Devaney–Nitecki Region for the Hénon Mapping using the Anti-Integrable Limit

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Abstract

We present in this note an alternative yet simple approach to the Devaney–Nitecki horseshoe region for the Hénon maps. Our approach is based on the anti-integrable limit and the implicit function theorem. We also highlight an application to the logistic maps.

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1 Introduction

For the celebrated Hénon map [7]

$$\mathcal{H}_{a,b} : (x, y) \mapsto (-a + y + x^2, -bx) \tag{1.1}$$

of $\mathbb{R}^2$, with a, b real parameters, Devaney and Nitecki [6] proved the following explicit parameter region

$$b \neq 0 \quad \text{and} \quad a > \frac{5 + 2\sqrt{5}}{4}(1 + |b|)^2 \tag{1.2}$$

for which the set consisting of all nonwandering points forms a hyperbolic horseshoe. This means that the restriction of Hénon map to its nonwandering set is topologically conjugate to the two-sided Bernoulli shift with two symbols. Their proof is based on a technique that is now referred as the "Conley–Moser conditions" (see for example [12]).

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In the enlightening paper [1], the anti-integrable limit for the Hénon map as $a \to \infty$ was established. It manifests a vivid picture on how the map is conjugate to the shift dynamics when it is restricted to the set of all bounded orbits and a is large. By utilizing the concept of anti-integrable limit of Aubry [2], Sterling and Meiss [16] also obtained the same parameter region as described in (1.2). In contrast to the geometrical argument involved in [6], the method used in [16] is more analytical.

The primary objective of this paper is intended to present a new yet simple approach to obtaining the Devaney–Nitecki parameter region. More precisely, we show that in the framework of anti-integrable limit, instead of the contraction mapping theorem argument used by [16], the Devaney–Nitecki region can also be obtained by using the implicit function theorem argument.

A noteworthy fact is that the Hénon map reduces to a one-dimensional quadratic map when $b = 0$. An advantage of our approach is that it can also offer a new and simple proof of a well-known fact that the restriction of the logistic map

$$x_{i+1} = f_\mu(x_i) = \mu x_i(1 - x_i), \qquad \mu \geq 0,$$

of $\mathbb{R}$ to its nonwandering set is topologically conjugate to the one-sided Bernoulli shift on two symbols when $\mu > 2 + \sqrt{5}$.

In the next section, we recall briefly the concept of anti-integrability. In Section 3, we prove the Devaney–Nitecki locus. In Section 4, we apply our approach to the logistic map.

We close the Introduction section with two remarks regarding obtaining better estimates of the horseshoe loci for the Hénon and logistic maps by taking advantage of the complex analysis.

Remark 1.1. When $a > 2(1 + |b|)^2$ and $b \neq 0$, Devaney and Nitecki [6] proved that the nonwandering set Ω is contained in a topological horseshoe $\Lambda = \bigcap_{n\in\mathbb{Z}} H^n_{a,b}(S)$, where S is the domain (3.3) defined later in Section 3. They also proved that the Hénon map restricted to its nonwandering set is topologically semi-conjugate to the two-sided shift with two symbols. By means of complex analysis techniques, it has been shown that the semi-conjugacy is in fact a conjugacy and $\Omega = \Lambda$ (see [8, 11, 13]). In particular, Mummert's proof [13] is based on the idea of Sterling and Meiss [16] but in the complex variable setting.

Remark 1.2. It is well-known that the restriction of logistic map to the invariant set $\bigcap_{n=0}^{\infty} f_\mu^{-n}([0,1])$ is topologically conjugate to the Bernoulli shift with two symbols not only when $\mu > 2 + \sqrt{5}$ but also $\mu > 4$. For approach by complex analysis, we refer the reader to [15], where the Poincaré metric and the Schwarz lemma are employed. (For approach by making use of repelling hyperbolicity of the invariant set, see comments in [5] and the references therein.)

2 Anti-Integrability

A dynamical systems is, in Aubry's sense [2], at the anti-integrable limit if it becomes nondeterministic and reduces to a subshift of finite type. The following definition originates from [1] and was re-written in [4] to fit the current situation.

A family of C^1-diffeomorphisms f_ϵ of $\mathbb{R}^n$, parametrized by ϵ,

$$z_{i+1} = f_\epsilon(z_i), \quad i \in \mathbb{Z}, \tag{2.1}$$

is called *anti-integrable* when $\epsilon \to 0$ if

- there exists a family of functions $L_\epsilon : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, parametrized by ϵ, such that the recurrence relation defined by $L_\epsilon(z_i, z_{i+1}) = 0$ is equivalent to (2.1) for nonzero ϵ;
- the limit
$$\lim_{\epsilon \to 0} L_\epsilon(z_i, z_{i+1}) = L_0(z_i, z_{i+1})$$
exists and the solution of $L_0(z_i, z_{i+1}) = 0$ can be determined by z_i alone;
- the set Σ of solutions $\{z_i\}_{i\in\mathbb{Z}}$ of $L_0(z_i, z_{i+1}) = 0$ for all i can be characterized bijectively by a subset of $\mathfrak{S}^{\mathbb{Z}}$ of infinite sequences with $\mathfrak{S}$ a certain finite set.

The limit $\epsilon \to 0$ is called the *anti-integrable limit* of f_ϵ. We call a sequence $\{z_i\}_{i\in\mathbb{Z}}$ comprising the solutions of $L_0(z_i, z_{i+1}) = 0$ for all i an *anti-integrable orbit* or *anti-integrable solution* of the map f_ϵ when $\epsilon \to 0$.

A remarkable significance of the anti-integrable limit is as follows. Endow the set $\mathfrak{S}$ with the discrete topology and the set $\mathfrak{S}^{\mathbb{Z}}$ with the product topology. Then, at the anti-integrable limit, the system is virtually a subshift with $\#(\mathfrak{S})$ symbols, where $\#(\mathfrak{S})$ is the cardinality of the set $\mathfrak{S}$.

For maps satisfying some nondegeneracy condition, the theory of anti-integrable limit says that the embedded symbolic dynamics at the limit persists to perturbations. Let $l_\infty := \{\mathbf{z} |\ \mathbf{z} = \{z_i\}_{i\in\mathbb{Z}},\ z_i \in \mathbb{R}^n,\ \text{bounded}\}$ endowed with the sup norm be the Banach space of bounded sequences in $\mathbb{R}^n$. Define a map $F : l_\infty \times \mathbb{R} \to l_\infty$ by

$$F(\mathbf{z}, \epsilon) = \{F_i(\mathbf{z}, \epsilon)\}_{i\in\mathbb{Z}}$$

with

$$F_i(\mathbf{z}, \epsilon) = L_\epsilon(z_i, z_{i+1}), \tag{2.2}$$

then the theory can be formulated rigorously by several steps (see for example [2,4,10], an application to high-dimensional Hénon-like maps [14], and a recent survey [3] for applying to discrete Lagrangian systems).

- A bounded anti-integrable orbit $\mathbf{z}^\dagger$ is precisely such that $F(\mathbf{z}^\dagger, 0) = 0$. Let $\Sigma \subset (\mathbb{R}^n)^{\mathbb{Z}}$ be the set consisting of all bounded anti-integrable orbits.

- Assume $F(\mathbf{z},\epsilon)$ is C^1 in a neighbourhood of $(\mathbf{z}^\dagger,0)$. If the linear map

$$D_{\mathbf{z}}F(\mathbf{z}^\dagger,0): l_\infty \to l_\infty, \tag{2.3}$$

which is the partial derivative of F at $(\mathbf{z}^\dagger,0)$ with respect to $\mathbf{z}$, is invertible, then the implicit function theorem implies there exists ϵ_0 and a unique C^1-function

$$\begin{aligned} \mathbf{z}^*(\cdot;\mathbf{z}^\dagger):\mathbb{R} &\to l_\infty, \\ \epsilon &\mapsto \mathbf{z}^*(\epsilon;\mathbf{z}^\dagger)=\{z_i^*(\epsilon;\mathbf{z}^\dagger)\}_{i\in\mathbb{Z}} \end{aligned} \tag{2.4}$$

such that $F(\mathbf{z}^*(\epsilon;\mathbf{z}^\dagger),\epsilon)=\mathbf{0}$ and $\mathbf{z}^*(0;\mathbf{z}^\dagger)=\mathbf{z}^\dagger$ for $0\le|\epsilon|<\epsilon_0$.

- Suppose the above assumptions are fulfilled for every $\mathbf{z}^\dagger\in\Sigma$ and ϵ_0 is independent of $\mathbf{z}^\dagger$. Let the projection $\mathbf{z}=(\cdots,z_{-1},z_0,z_1,\cdots)\mapsto z_0\in\mathbb{R}^n$ be denoted by π. The composition of mappings

$$\mathbf{z}^\dagger \overset{\Phi_\epsilon}{\longmapsto} \mathbf{z}^*(\epsilon;\mathbf{z}^\dagger) \overset{\pi}{\longmapsto} z_0^*(\epsilon;\mathbf{z}^\dagger)$$

is a continuous bijection with the product topology. (The proof of the continuity can be found, for example, in [4, 14].)

- Let the set $\mathcal{A}_\epsilon$ be defined by

$$\begin{aligned} \mathcal{A}_\epsilon &:= \bigcup_{\mathbf{z}^\dagger\in\Sigma}\pi(\mathbf{z}^*(\epsilon;\mathbf{z}^\dagger)) \\ &= \bigcup_{\mathbf{z}^\dagger\in\Sigma} z_0^*(\epsilon;\mathbf{z}^\dagger). \end{aligned}$$

Let σ be the left shift in $(\mathbb{R}^n)^{\mathbb{Z}}$, $\sigma(\mathbf{z})=\mathbf{z}'=(\cdots,z'_{-1},z'_0,z'_1,\cdots)$, with $z'_i=z_{i+1}$. Under the assumption $\sigma(\Sigma)=\Sigma$, the following diagram commutes when $0<|\epsilon|<\epsilon_0$.

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ {\scriptstyle\pi\circ\Phi_\epsilon}\downarrow & & \downarrow{\scriptstyle\pi\circ\Phi_\epsilon} \\ \mathcal{A}_\epsilon & \xrightarrow{f_\epsilon} & \mathcal{A}_\epsilon \end{array}$$

Remark 2.1. An anti-integrable orbit $\mathbf{z}^\dagger$ is called *nondegenerate* if the differential map $D_{\mathbf{z}}F(\mathbf{z}^\dagger,0)$ in (2.3) is invertible. Likewise, the orbit $\mathbf{z}^*(\epsilon;\mathbf{z}^\dagger)$ continued from an anti-integrable orbit $\mathbf{z}^\dagger$ is called nondegenerate if $D_{\mathbf{z}}F(\mathbf{z}^*(\epsilon;\mathbf{z}^\dagger),\epsilon)$ is invertible. If $|\epsilon|<\epsilon_0$, then $\mathbf{z}^*(\epsilon;\mathbf{z}^\dagger)$ is a nondegenerate orbit.

The following proposition provides a useful method to estimate a lower bound of ϵ_0. Its proof is easy (see for example [9]), thus we omit it.

Proposition 2.2. *Assume $\mathbf{z}^\dagger$ is a nondegenerate anti-integrable solution of $F(\mathbf{z},0)=0$, and ϵ_0 the maximal value such that the unique continuation $\mathbf{z}^*(\epsilon;\mathbf{z}^\dagger)$ is valid when $0\le\epsilon<\epsilon_0$. If ϵ' is a positive real number and satisfies*

$$\|D_{\mathbf{z}}F(\mathbf{z}^*(\epsilon';\mathbf{z}^\dagger),\epsilon')-D_{\mathbf{z}}F(\mathbf{z}^\dagger,0)\|<\frac{1}{\|D_{\mathbf{z}}F(\mathbf{z}^\dagger,0)^{-1}\|},$$

then one can conclude that $0<\epsilon'<\epsilon_0$.

3 Proof of the Devaney–Nitecki Region for the Hénon Family

To start with, we need a bounded domain with which the bounded orbits of the Hénon map are confined. The following result is first proved in [6].

Proposition 3.1. *Suppose $b\neq 0$ and let $\{(x_i,y_i)\}_{i\in\mathbb{Z}}$ be a bounded orbit of the Hénon map* (1.1)*, then*

- $\sup\limits_{i\in\mathbb{Z}}|x_i|\le R$ *if $a>0$,*
- $R_*<\inf\limits_{i\in\mathbb{Z}}|x_i|$ *if $a>2(1+|b|)^2$,*

where

$$R=\frac{1+|b|+\sqrt{(1+|b|)^2+4a}}{2}$$

and R_* satisfies

$$R_*^2=a-(1+|b|)R. \tag{3.1}$$

Proof. Our proof for the upper bound is adapted from [13]. Let $M=\sup\limits_{i\in\mathbb{Z}}|x_i|$, then for any $\delta>0$ there exists $t\in\mathbb{Z}$ such that $|x_t|>M-\delta$ and so $M\ge|x_{t+1}|\ge -a-|b|M+(M-\delta)^2$. Consequently, $M^2-(1+|b|)M-a\le 0$, which implies $\sup\limits_{i\in\mathbb{Z}}|x_i|\le R$.

For the lower bound, because (x_i,y_i) must belong to the intersection $\mathcal{H}_{a,b}^{-1}([-R,R]\times[-|b|R,|b|R])\cap\mathcal{H}_{a,b}([-R,R]\times[-|b|R,|b|R])$ for every $i\in\mathbb{Z}$, we infer that $|x_i|>R_*$, where R_* satisfies $\mathcal{H}_{a,b}(-R_*,|b|R)=(-R,bR_*)$. (See also Figure 3.1(a) of this paper and Figure 4 of [6].) And, the last equality gives rise to (3.1). □

Remark 3.2. Note that $R_*>0$ if $a>2(1+|b|)^2$ and $R_*=0$ if $a=2(1+|b|)^2$.

It is convenient to consider the Hénon map in the following form

$$H_{a,b}(x,y)=(\sqrt{a}(1-x^2)+by,-x). \tag{3.2}$$

It is easy to see that the two maps (1.1) and (3.2) are equivalent by the transformation $(x, y) \mapsto (-\sqrt{a}x, -\sqrt{a}by)$. We emphasize that they are equivalent only if both a and b are nonzero and of finite value. Figure 3.1(a) depicts the image and pre-image of domain

$$S = \{(x, y)| -r \le x \le r, \ -r \le y \le r\} \tag{3.3}$$

for an area-preserving Hénon map of the form (3.2) for $a = 10$ and depicts the position of the point r_*, where

$$\begin{aligned} r_* &= \frac{R_*}{\sqrt{a}} \\ &= \sqrt{1 - \frac{(1+|b|)}{2a}\sqrt{(1+|b|)^2 + 4a} - \frac{(1+|b|)^2}{2a}} \end{aligned}$$

and

$$\begin{aligned} r &= \frac{R}{\sqrt{a}} \\ &= \sqrt{1 + \frac{(1+|b|)}{2a}\sqrt{(1+|b|)^2 + 4a} + \frac{(1+|b|)^2}{2a}} \\ &= \frac{1}{2\sqrt{a}}\left\{(1+|b|) + \sqrt{(1+|b|)^2 + 4a}\right\}. \end{aligned}$$

In the figure, the image of the horizontal line segment (red colour) connecting the two points $(-r, -r)$ and $(r, -r)$ is the red parabola, while the image of the line segment (blue colour) connecting $(-r, r)$ and (r, r) is the blue parabola. The pre-image of the vertical line segment (green colour) connecting $(-r, -r)$ and $(-r, r)$ is the green parabola, while the pre-image of the line segment (black colour) connecting $(r, -r)$ and (r, r) is the black parabola.

Rescale the parameter by letting

$$\epsilon = 1/\sqrt{a},$$

then $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ is an orbit of $H_{a,b}$ if and only if $\{x_i\}_{i\in\mathbb{Z}}$ satisfies $x_i = -y_{i+1}$ and the following recurrence relation

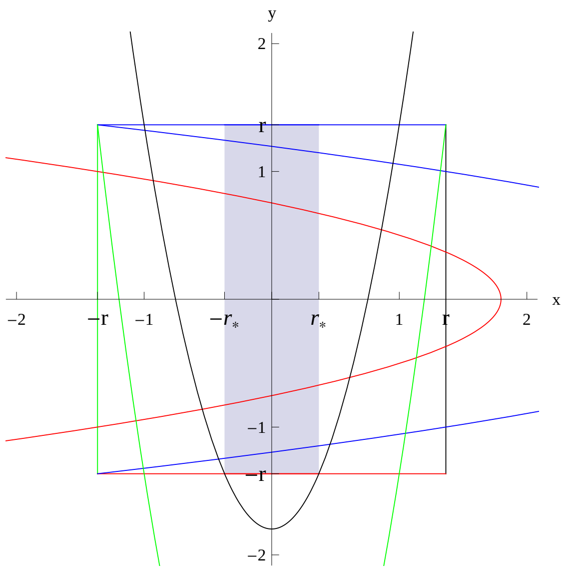
$$\epsilon(x_{i+1} + bx_{i-1}) + x_i^2 - 1 = 0$$

for each integer i. Let $\mathbf{x} = \{x_i\}_{i\in\mathbb{Z}}$ be an element of the Banach space l_∞ of bounded sequences in $\mathbb{R}$. Define $\mathcal{F}(\mathbf{x}, \epsilon) = \{\mathcal{F}_i(\mathbf{x}, \epsilon)\}_{i\in\mathbb{Z}}$ by

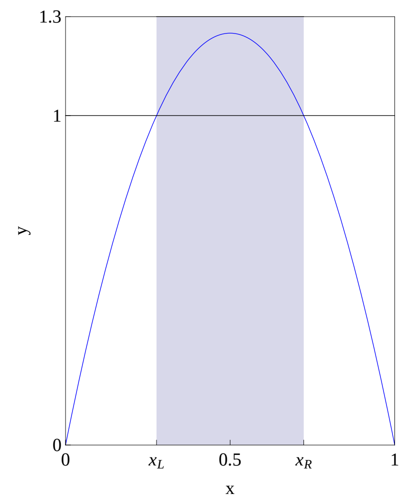
$$\mathcal{F}_i(\mathbf{x}, \epsilon) = \epsilon(x_{i+1} + bx_{i-1}) + x_i^2 - 1, \tag{3.4}$$

then $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ is a bounded orbit of $H_{a,b}$ if and only if $\mathcal{F}(\mathbf{x}, \epsilon) = 0$.

The following result provides an alternative proof of the Devaney–Nitecki locus. (Notice that the inequality (3.5) below is equivalent to inequality (1.2).)



(a)



(b)

Figure 3.1: (a) The image and pre-image of the domain S for the orientation-preserving Hénon map $H_{a,1}$ with $a = 10$. Notice that the intersection of the image and pre-image consists of four disjoint sets. (b) The graph of $5x(1-x)$ and corresponding x_L and x_R.

Theorem 3.3. *Let* $\mathcal{F} : l_\infty \times \mathbb{R} \to l_\infty$ *be defined as* (3.4). *Providing*

$$b \neq 0 \quad and \quad \epsilon < \frac{2}{\sqrt{5+2\sqrt{5}}(1+|b|)}, \tag{3.5}$$

there corresponds a unique C^1*-family of points* $\mathbf{x}^*(\epsilon;\mathbf{x}^\dagger) = \{x_i^*(\epsilon;\mathbf{x}^\dagger)\}_{i\in\mathbb{Z}}$ *parametrized by* ϵ *in* l_∞ *for any anti-integrable orbit* $\mathbf{x}^\dagger$ *such that* $\mathbf{x}^*(0;\mathbf{x}^\dagger) = \mathbf{x}^\dagger$ *and* $\mathcal{F}(\mathbf{x}^*(\epsilon;\mathbf{x}^\dagger),\epsilon) = \mathbf{0}$.

Proof. Certainly $\mathcal{F}$ is a C^1-map. Its partial derivative at $(\mathbf{x},\epsilon)$ with respect to $\mathbf{x}$ is a linear map which in matrix form is

$$D_{\mathbf{x}}\mathcal{F}(\mathbf{x},\epsilon) = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & \epsilon & 2x_{-1} & \epsilon b & & & \\ & & \epsilon & 2x_0 & \epsilon b & & \\ & & & \epsilon & 2x_1 & \epsilon b & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

It is easy to see that $\mathcal{F}(\mathbf{x},0) = 0$ if and only if $\mathbf{x} \in \{\pm 1\}^{\mathbb{Z}}$. Consequently, $D_{\mathbf{x}}\mathcal{F}(\mathbf{x}^\dagger,0)$ is invertible because it is a diagonal matrix with entries ± 2. We then have

$$\|D_{\mathbf{x}}\mathcal{F}(\mathbf{x}^\dagger,0)^{-1}\| = \frac{1}{2}.$$

We also have

$$D_{\mathbf{x}}\mathcal{F}(\mathbf{x}^*(\epsilon;\mathbf{x}^\dagger),\epsilon) - D_{\mathbf{x}}\mathcal{F}(\mathbf{x}^\dagger,0) =$$
$$\begin{pmatrix} \ddots & \ddots & & \ddots & & & & \\ & \epsilon & 2x^*_{-1}-2x^\dagger_{-1} & & \epsilon b & & & \\ & & \epsilon & & 2x^*_0-2x^\dagger_0 & & \epsilon b & \\ & & & & \epsilon & 2x^*_1-2x^\dagger_1 & \epsilon b & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

a tri-diagonal matrix. (In the above equation, we have used $x_i^* = x_i^*(\epsilon;\mathbf{x}^\dagger)$ for all $i \in \mathbb{Z}$ for simplicity sake.) Thus,

$$\|D_{\mathbf{x}}\mathcal{F}(\mathbf{x}^*(\epsilon;\mathbf{x}^\dagger),\epsilon) - D_{\mathbf{x}}\mathcal{F}(\mathbf{x}^\dagger,0)\| = \epsilon + 2\sup_{i\in\mathbb{Z}} |x_i^*(\epsilon;\mathbf{x}^\dagger) - x_i^\dagger| + \epsilon|b|.$$

According to Proposition 3.1, the fact that $\mathbf{x}^*(\epsilon;\mathbf{x}^\dagger)$ is a bounded orbit implies that

$$x_i^*(\epsilon;\mathbf{x}^\dagger) \in [-r,-r_*) \cup (r_*, r]$$

for all $i \in \mathbb{Z}$. Because $x_i^*(\epsilon;\mathbf{x}^\dagger)$ is a continuation of $x_i^\dagger \in \{\pm 1\}$, we have

$$|x_i^*(\epsilon;\mathbf{x}^\dagger) - x_i^\dagger| \le 1 - r_* \quad \forall i \in \mathbb{Z}.$$

(It is not difficult to verify that $r-1 < 1-r_*$.) Then, the inequality

$$r_* > \frac{\epsilon}{2}(1+|b|) \tag{3.6}$$

guarantees the following condition

$$\|D_{\mathbf{x}}\mathcal{F}(\mathbf{x}^*(\epsilon;\mathbf{x}^\dagger),\epsilon) - D_{\mathbf{x}}\mathcal{F}(\mathbf{x}^\dagger,0)\| < \frac{1}{\|D_{\mathbf{x}}\mathcal{F}(\mathbf{x}^\dagger,0)^{-1}\|}.$$

As a consequence, we conclude from (3.6) and Proposition 2.2 that $\epsilon < \epsilon_0$ if $\epsilon < 2\left(\sqrt{5+2\sqrt{5}}(1+|b|)\right)^{-1}$. □

Remark 3.4. $\mathcal{F}(\mathbf{x},\epsilon)$ defined in (3.4) is a function from the Banach space of bounded sequences in $\mathbb{R}$ to $\mathbb{R}$ by $\mathcal{F}_i(\mathbf{x},\epsilon) = \epsilon(x_{i+1}+bx_{i-1})+x_i^2-1$. If we set $z_i = (x_i, -x_{i-1}) = (x_i, y_i)$, then can define a function $F_i(\mathbf{z},\epsilon)$ from the Banach space of bounded sequences $\mathbf{z} = \{z_i\}_{i\in\mathbb{Z}}$ in $\mathbb{R}^2$ to $\mathbb{R}^2$ by $F_i(\mathbf{z},\epsilon) = \big(\epsilon(x_{i+1}-by_i)+x_i^2-1,\ y_{i+1}+x_i\big)$. It is easy to see that for all integer i we have $\mathcal{F}_i(\mathbf{x},\epsilon) = 0$ if and only if $F_i(\mathbf{z},\epsilon) = 0$. Let $L_\epsilon : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ be defined by $L_\epsilon(z_i,z_{i+1}) = \big(\epsilon(x_{i+1}-by_i)+x_i^2-1,\ y_{i+1}+x_i\big)$. In this situation, F_i is in the form of (2.2). The limit $\lim_{\epsilon\to 0} L_\epsilon(z_i,z_{i+1}) = \big(x_i^2-1,\ y_{i+1}+x_i\big) = L_0(z_i,z_{i+1})$. Apparently, the solution of $L_0(z_i,z_{i+1}) = 0$ is determined by z_i alone. The reason that we employ the function $\mathcal{F}_i(\mathbf{x},\epsilon)$ rather than $F_i(\mathbf{z},\epsilon)$ in this section is that the calculation of taking derivative $D_{\mathbf{x}}\mathcal{F}_i(\mathbf{x},\epsilon)$ is less complicated than that of $D_{\mathbf{z}}F_i(\mathbf{z},\epsilon)$.

4 Estimating Shift Locus for the Logistic Maps

We proceed to investigate the family of logistic maps $x \mapsto \mu x(1-x)$. The logistic map is anti-integrable at the limit $\mu \to \infty$ [5]. To see this, let

$$\epsilon = 1/\mu,$$

and rewrite the logistic map as another map $F(\cdot, \epsilon)$ in the space $l_\infty := \{\mathbf{x}|\ \mathbf{x} = \{x_0, x_1, x_2, \ldots\},\ x_i \in \mathbb{R},\ \text{bounded}\}$ of bounded sequences with the sup norm:

$$\begin{aligned} F : l_\infty \times \mathbb{R} &\to l_\infty, \\ (\mathbf{x}, \epsilon) &\mapsto F(\mathbf{x}, \epsilon) = \{F_0(\mathbf{x}, \epsilon), F_1(\mathbf{x}, \epsilon), \ldots\} \end{aligned}$$

with $F_i(\mathbf{x}, \epsilon) = -\epsilon x_{i+1} + x_i(1 - x_i)$. It is readily to see that $\mathbf{x}$ is a bounded orbit of the logistic map if and only if it solves $F(\mathbf{x}, \epsilon) = \mathbf{0}$. Let Σ denote the space of sequences of 0's and 1's, $\Sigma = \{\mathbf{a}|\ \mathbf{a} = \{a_i\}_{i=0}^{\infty},\ a_i = 0 \text{ or } 1\}$. As a consequence,

$$F(\mathbf{x}^\dagger, 0) = 0 \quad \Longleftrightarrow \quad \mathbf{x}^\dagger \in \Sigma.$$

Theorem 4.1. *Providing $\epsilon < 1/(2+\sqrt{5})$, there corresponds a unique C^1-family of points $\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger) = \{x_i^*(\epsilon; \mathbf{x}^\dagger)\}_{i\in\mathbb{N}}$ in l_∞ parametrized by ϵ for any anti-integrable orbit $\mathbf{x}^\dagger$ such that $\mathbf{x}^*(0; \mathbf{x}^\dagger) = \mathbf{x}^\dagger$ and $F(\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger), \epsilon) = 0$.*

Proof. The proof follows the same line as that of Theorem 3.3. Obviously, F is a C^1-map with its partial derivative with respect to $\mathbf{x}$ a linear map, which can be realized in matrix form as

$$D_{\mathbf{x}}F(\mathbf{x}, \epsilon) = \begin{pmatrix} 1-2x_0 & -\epsilon & 0 & \cdots \\ 0 & 1-2x_1 & -\epsilon & \cdots \\ 0 & 0 & 1-2x_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Accordingly, $D_{\mathbf{x}}F(\mathbf{x}^\dagger, 0)$ is invertible because it is a diagonal matrix with entries ± 1. We have

$$\|D_{\mathbf{x}}F(\mathbf{x}^\dagger, 0)^{-1}\| = 1.$$

Then, as claimed in (2.4), there is ϵ_0 and a unique C^1-function $\mathbf{x}^*(\cdot; \mathbf{x}^\dagger) : \mathbb{R} \to l_\infty$ such that $F(\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger), \epsilon) = 0$ provided $0 \le \epsilon < \epsilon_0$. We have

$$D_{\mathbf{x}}F(\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger), \epsilon) - D_{\mathbf{x}}F(\mathbf{x}^\dagger, 0) = \begin{pmatrix} -2x_0^* + 2x_0^\dagger & -\epsilon & 0 & \cdots \\ 0 & -2x_1^* + 2x_1^\dagger & -\epsilon & \cdots \\ 0 & 0 & -2x_2^* + 2x_2^\dagger & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(In the above equation, we have used $x_i^* = x_i^*(\epsilon; \mathbf{x}^\dagger)$ for all $i \in \mathbb{N}$ for simplicity sake.) Evidently,

$$x_i^*(\epsilon; \mathbf{x}^\dagger) \in [0, x_L] \cup [x_R, 1]$$

for all $i \in \mathbb{N}$, where (see Figure 3.1(b))

$$x_L = \frac{1 - \sqrt{1 - 4\epsilon}}{2} \qquad \text{and} \qquad x_R = \frac{1 + \sqrt{1 - 4\epsilon}}{2}.$$

And, because $x_i^*(\epsilon; \mathbf{x}^\dagger)$ is a continuation of $x_i^\dagger \in \{0, 1\}$, we obtain

$$\begin{aligned} \|D_{\mathbf{x}}F(\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger), \epsilon) - D_{\mathbf{x}}F(\mathbf{x}^\dagger, 0)\| &= 2 \sup_{i \in \mathbb{N}} |x_i^* - x_i^\dagger| + \epsilon \\ &= 2x_L + \epsilon \\ &= 1 - \sqrt{1 - 4\epsilon} + \epsilon. \end{aligned}$$

In the light of Proposition 2.2, we infer that $\epsilon_0 < -2 + \sqrt{5}$ (or equivalently $\mu > 2 + \sqrt{5}$). $\square$

Remark 4.2. Theorem 4.1 was already presented in [5]. There, the proof of the invertibility of $D_{\mathbf{x}}F(\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger), \epsilon)$ relies on the expanding property that the absolute value of the derivative of the logistic map f_μ on $f_\mu^{-1}([0, 1])$ is strictly greater than one when $\mu > 2 + \sqrt{5}$. By taking advantage of Proposition 2.2, the present proof here does not require the knowledge of the expanding property.

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