Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 12, Number 2, pp. 85–105 (2017)
http://campus.mst.edu/adsa

Oscillation and Asymptotic Behavior of Solutions of Odd-Order Difference Equations of Mixed Type

Martin Bohner
Missouri S&T, Department of Mathematics and Statistics
Rolla, MO 65409-0020, USA
bohner@mst.edu

S. A. Rupadevi and Srinivasan Selvarangam
Presidency College, Department of Mathematics
Chennai 600 005, India
selvarangam.9962@gmail.com

Ethiraju Thandapani
University of Madras
Ramanujan Institute for Advanced Study in Mathematics
Chennai 600 005, India
ethandapani@yahoo.co.in

Abstract

In this paper, we present some new sufficient conditions for the oscillation of certain odd-order nonlinear neutral difference equations, using the arithmetic-geometric mean inequality. Examples are provided in order to illustrate the main results.

AMS Subject Classifications: 39A10.
Keywords: Odd order, neutral difference equation, oscillation, asymptotic behavior, mixed type.

1 Introduction

In this paper, we are concerned with the odd-order mixed-type neutral difference equation of the form

$$\Delta^m(x_n + ax_{n-\tau_1} + bx_{n+\tau_2}) + p_n x^{\alpha}_{n-\sigma_1} + q_n x^{\beta}_{n+\sigma_2} = 0 \tag{1.1}$$

Received February 2, 2017; Accepted July 26, 2017
Communicated by Mehmet Ünal

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \ldots\}$, and $n_0 \in \mathbb{N}_0$, subject to the following conditions:

(i) $\{p_n\}$ and $\{q_n\}$ are positive real sequences for all $n \in \mathbb{N}(n_0)$;

(ii) a and b are nonnegative real numbers, $\tau_1, \tau_2, \sigma_1, \sigma_2 \in \mathbb{N}_0$;

(iii) α and β are ratios of odd positive integers, and $m \geq 3$ is an odd integer.

Let $\theta = \max\{\tau_1, \sigma_1\}$. By a solution of (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ and satisfying (1.1) for all $n \geq n_0$. A nontrivial solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise. Equation (1.1) is said to be almost oscillatory if its solution are either oscillatory or converge to zero asymptotically.

Since difference equations have important applications in population dynamics, biology, probability theory, computer science, and many other fields, there is a permanent interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of various types of even-order/odd-order difference equations, see the references in this paper and the references therein.

For the oscillation of odd-order difference equations, see [1–3,6–8,10–13,15,16,18, 19,21–23]. Regarding higher-order mixed-type neutral difference equations, Agarwal and Grace [4], Grace [9], Thandapani and Kavitha [17], considered several third-order mixed-type neutral difference equations and established sufficient conditions for the oscillation of all solutions.

In [5], Agarwal, Grace and Bohner considered the mth-order mixed-type neutral difference equation (1.1) with $\alpha = \beta = 1$, $p_n \equiv p$, and $q_n \equiv q$, and established some sufficient conditions for the oscillation of (1.1). Motivated by the observation, in this paper we investigate the oscillatory behavior of solutions of (1.1), and hence the results obtained in this paper complement and generalize those in [3–13,15–19,21–23].

After some preliminaries given in Section 2, Section 3 offers sufficient conditions which ensure that every solution $\{x_n\}$ of (1.1) is either oscillatory or tends to zero as $n \to \infty$. Examples are provided in Section 4 in order to illustrate the main results.

2 Preliminary Lemmas

In this section, we present some lemmas, which are useful in proving the main results. We write

$$z_n = x_n + a x_{n-\tau_1} + b x_{n+\tau_2}.$$

Lemma 2.1 (See [17]). *Let a, b, c are positive quantities not all equal. Then*

$$a^\alpha + b^\alpha + c^\alpha \geq \frac{1}{3^{\alpha-1}}(a+b+c)^\alpha \quad \textit{if} \quad \alpha \geq 1$$

and

$$a^\alpha + b^\alpha + c^\alpha \geq (a+b+c)^\alpha \quad \textit{if} \quad 0 < \alpha \leq 1.$$

Lemma 2.2 (See [14]). *Let $\alpha > 1$ be a ratio of odd positive integers. Assume that $\{d_n\}$ is a positive real sequence defined for all $n \geq n_0$ and there exists $\lambda > \frac{1}{\sigma} ln\alpha$ such that*

$$\lim_{n\to\infty} \inf \left[d_n \exp(-e^{\lambda n})\right] > 0. \tag{2.1}$$

Then all solutions of the difference equation

$$\Delta y_n + d_n y_{n-\sigma}^{\alpha} = 0 \tag{2.2}$$

are oscillatory.

Lemma 2.3 (See [14]). *Let $0 < \alpha < 1$ be a ratio of odd positive integers. Assume that $\{d_n\}$ is a positive real sequence defined for all $n \geq n_0$. If*

$$\sum_{n=n_0}^{\infty} d_n = \infty, \tag{2.3}$$

then every solution of the first-order difference equation (2.2) *is oscillatory.*

Lemma 2.4 (See [1]). *Assume that $\{d_n\}$ is a nonnegative real sequence defined for all $n \geq n_0$ and*

$$\lim_{n\to\infty} \inf \sum_{j=n-\sigma}^{n-1} d_j > \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1}, \tag{2.4}$$

where $\sigma \in \mathbb{N}$. Then the first-order difference inequality

$$\Delta w_n + d_n w_{n-\sigma} \leq 0 \tag{2.5}$$

has no eventually positive solution.

Lemma 2.5 (See [20]). *Assume that $\alpha > 0$ is a ratio of odd positive integers, $k \in \mathbb{N}$, and $\{d_n\}$ is a nonnegative real sequence not identically zero for infinitely many values of n. If the difference inequality*

$$\Delta w_n + d_n w_{n-k}^{\alpha} \leq 0 \tag{2.6}$$

has eventually positive solution, then so does the difference equation

$$\Delta w_n + d_n w_{n-k}^{\alpha} = 0.$$

Lemma 2.6 (See [1]). *Let $\{u_n\}$ be a sequence of positive real numbers with $\{\Delta^m u_n\}$ be of constant sign eventually and not identically zero eventually. Then there exists an integer $l \in \{0, 1, 2, \ldots, m\}$ with $(m + l)$ odd for $\Delta^m u_n \leq 0$, and $(m + l)$ even for $\Delta^m u_n \geq 0$ and for $N > 0$ such that*

$$\Delta^j u_n > 0 \quad \textit{for} \quad j = 0, 1, 2, 3, \ldots, (l - 1)$$

and

$$(-1)^{j+l} \Delta^j u_n > 0 \quad \textit{for} \quad j = l, l + 1, \ldots, m - 1$$

for all $n \geq N$.

Lemma 2.7 (See [1]). *Let $\{u_n\}$ be a sequence of positive real numbers with $\Delta^m u_n \le 0$ and not identically zero eventually. Then there exists a large integer N such that*

$$u_n \ge \frac{(n-N)^{(m-1)}}{(m-1)!}\Delta^{m-1}u_{2^{m-l-1}n} \quad \text{for} \quad n \ge N,\ l \in \{0,1,2,\ldots,m\},$$

where $u^{(j)} = u(u-1)(u-2)\cdots(u-j+1)$. Further if $\{u_n\}$ is increasing, then

$$u_n \ge \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}u_n \quad \text{for all} \quad n \ge 2^{m-1}. \tag{2.7}$$

3 Oscillation Results

In this section, we obtain some sufficient conditions for all solutions of (1.1) to be almost oscillatory. For our convenience, we introduce the notations

$$P_n = \min\{p_{n-\tau_1}, p_n, p_{n+\tau_2}\},$$
$$Q_n = \min\{q_{n-\tau_1}, q_n, q_{n+\tau_2}\},$$

and

$$R_n = K_1P_n + K_2Q_n,$$

where K_1 and K_2 are some positive constants.

Theorem 3.1. *Assume that the conditions*

(C_1) $\displaystyle\sum_{n=n_0}^{\infty} n^{m-1}R_n = \infty,$

(C_2) $\alpha < 1 < \beta$

hold. If the first-order difference inequality

$$\Delta w_n + \frac{A_n}{(1+d_1+d_2)}\frac{\lambda}{(m-1)!}(n-\sigma_1)^{(m-1)}w_{n+\tau_1-\sigma_1} \le 0, \tag{3.1}$$

where

$$A_n = \eta_1^{-\eta_1}\eta_2^{-\eta_2}P_n^{\eta_1}\left(\frac{Q_n}{3^{\beta-1}}\right)^{\eta_2}, \quad \eta_1 = \frac{\beta-1}{\beta-\alpha}, \quad \eta_2 = \frac{1-\alpha}{\beta-\alpha},$$
$$d_1 = \begin{cases} a^\alpha \text{ if } a \le 1 \\ a^\beta \text{ if } a \ge 1, \end{cases} \qquad d_2 = \begin{cases} b^\alpha \text{ if } b \le 1 \\ b^\beta \text{ if } b \ge 1, \end{cases}$$

has no positive solution for some $\lambda \in (0,1)$, and for all $n \ge n_0$, then every solution of (1.1) *is almost oscillatory.*

Proof. Assume the contrary. Let $\{x_n\}$ be a nonoscillatory solution of (1.1), which does not converge to zero. Without loss of generality, we may assume that $\{x_n\}$ is a positive solution of (1.1) (since the proof for the negative case is similar). Then there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\sigma_1} > 0$, and $x_{n-\tau_1} > 0$ for all $n \geq n_1$. By definition of z_n, we have $z_n > 0$ for all $n \geq n_1$. Now from (1.1), we obtain

$$\Delta^m z_n = -p_n x^{\alpha}_{n-\sigma_1} - q_n x^{\beta}_{n+\sigma_2} \leq 0 \tag{3.2}$$

for all $n \geq n_1$. First, we shall prove that $\Delta z_n > 0$ for all $n \geq n_1$. If not, then $\Delta z_n \leq 0$ for all $n \geq n_1$. That is, $\{z_n\}$ is a positive decreasing sequence, and hence $\lim_{n\to\infty} z_n = M > 0$ and

$$\lim_{n\to\infty} \Delta^k z_n = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots, (m-1). \tag{3.3}$$

Now we discuss the different cases for a and b.

Case (i)

Suppose $a \leq 1$ and $b \leq 1$. Then, from (1.1), we have

$$a^{\alpha}\Delta^m z_{n-\tau_1} + a^{\alpha} p_{n-\tau_1} x^{\alpha}_{n-\sigma_1-\tau_1} + a^{\alpha} q_{n-\tau_1} x^{\beta}_{n+\sigma_2-\tau_1} = 0, \quad n \geq n_1 \tag{3.4}$$

and

$$b^{\alpha}\Delta^m z_{n+\tau_2} + b^{\alpha} p_{n+\tau_2} x^{\alpha}_{n-\sigma_1+\tau_2} + b^{\alpha} q_{n+\tau_2} x^{\beta}_{n+\sigma_2+\tau_2} = 0, \quad n \geq n_1. \tag{3.5}$$

Now combining (1.1), (3.4), and (3.5), we obtain

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^{\alpha}\Delta^{m-1} z_{n-\tau_1} + b^{\alpha}\Delta^{m-1} z_{n+\tau_2}) + P_n(x^{\alpha}_{n-\sigma_1} + a^{\alpha} x^{\alpha}_{n-\sigma_1-\tau_1} + b^{\alpha} x^{\alpha}_{n-\sigma_1+\tau_2})\\ + Q_n(x^{\beta}_{n+\sigma_2} + a^{\alpha} x^{\beta}_{n+\sigma_2-\tau_1} + b^{\alpha} x^{\beta}_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Since $a \leq 1$, $b \leq 1$, and $\beta > \alpha$, the last inequality becomes

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^{\alpha}\Delta^{m-1} z_{n-\tau_1} + b^{\alpha}\Delta^{m-1} z_{n+\tau_2}) + P_n(x^{\alpha}_{n-\sigma_1} + a^{\alpha} x^{\alpha}_{n-\sigma_1-\tau_1} + b^{\alpha} x^{\alpha}_{n-\sigma_1+\tau_2})\\ + Q_n(x^{\beta}_{n+\sigma_2} + a^{\beta} x^{\beta}_{n+\sigma_2-\tau_1} + b^{\beta} x^{\beta}_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1} z_n + a^{\alpha}\Delta^{m-1} z_{n-\tau_1} + b^{\alpha}\Delta^{m-1} z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n+\sigma_2} \leq 0, \quad n \geq n_1. \tag{3.6}$$

Case (ii)

Suppose $a \geq 1$ and $b \geq 1$. Then, from (1.1), we have

$$a^\beta \Delta^m z_{n-\tau_1} + a^\beta p_{n-\tau_1} x^\alpha_{n-\sigma_1-\tau_1} + a^\beta q_{n-\tau_1} x^\beta_{n+\sigma_2-\tau_1} = 0, \quad n \geq n_1 \tag{3.7}$$

and

$$b^\beta \Delta^m z_{n+\tau_2} + b^\beta p_{n+\tau_2} x^\alpha_{n-\sigma_1+\tau_2} + b^\beta q_{n+\tau_2} x^\beta_{n+\sigma_2+\tau_2} = 0, \quad n \geq n_1. \tag{3.8}$$

Now combining (1.1), (3.7), and (3.8), we obtain

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^\beta \Delta^{m-1} z_{n-\tau_1} + b^\beta \Delta^{m-1} z_{n+\tau_2}) + P_n(x^\alpha_{n-\sigma_1} + a^\beta x^\alpha_{n-\sigma_1-\tau_1} + b^\beta x^\alpha_{n-\sigma_1+\tau_2}) \\ + Q_n(x^\beta_{n+\sigma_2} + a^\beta x^\beta_{n+\sigma_2-\tau_1} + b^\beta x^\beta_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Since $a \geq 1$, $b \geq 1$, and $\alpha < \beta$, the last inequality becomes

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^\beta \Delta^{m-1} z_{n-\tau_1} + b^\beta \Delta^{m-1} z_{n+\tau_2}) + P_n(x^\alpha_{n-\sigma_1} + a^\alpha x^\alpha_{n-\sigma_1-\tau_1} + b^\alpha x^\alpha_{n-\sigma_1+\tau_2}) \\ + Q_n(x^\beta_{n+\sigma_2} + a^\beta x^\beta_{n+\sigma_2-\tau_1} + b^\beta x^\beta_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1} z_n + a^\beta \Delta^{m-1} z_{n-\tau_1} + b^\beta \Delta^{m-1} z_{n+\tau_2}) + P_n z^\alpha_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^\beta_{n+\sigma_2} \leq 0, \quad n \geq n_1. \tag{3.9}$$

Case (iii)

Now suppose $a \leq 1$ and $b \geq 1$. Then, from (1.1), we have

$$a^\alpha \Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1} x^\alpha_{n-\sigma_1-\tau_1} + a^\alpha q_{n-\tau_1} x^\beta_{n+\sigma_2-\tau_1} = 0, \quad n \geq n_1 \tag{3.10}$$

and

$$b^\beta \Delta^m z_{n+\tau_2} + b^\beta p_{n+\tau_2} x^\alpha_{n-\sigma_1+\tau_2} + b^\beta q_{n+\tau_2} x^\beta_{n+\sigma_2+\tau_2} = 0, \quad n \geq n_1. \tag{3.11}$$

Combining (1.1), (3.10), and (3.11), we obtain

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\beta \Delta^{m-1} z_{n+\tau_2}) + P_n(x^\alpha_{n-\sigma_1} + a^\alpha x^\alpha_{n-\sigma_1-\tau_1} + b^\beta x^\alpha_{n-\sigma_1+\tau_2}) \\ + Q_n(x^\beta_{n+\sigma_2} + a^\alpha x^\beta_{n+\sigma_2-\tau_1} + b^\beta x^\beta_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Now using $a \leq 1$, $b \geq 1$, and $\alpha < \beta$, the last inequality becomes

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\beta \Delta^{m-1} z_{n+\tau_2}) + P_n(x^\alpha_{n-\sigma_1} + a^\alpha x^\alpha_{n-\sigma_1-\tau_1} + b^\alpha x^\alpha_{n-\sigma_1+\tau_2}) \\ + Q_n(x^\beta_{n+\sigma_2} + a^\beta x^\beta_{n+\sigma_2-\tau_1} + b^\beta x^\beta_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\beta \Delta^{m-1} z_{n+\tau_2}) + P_n z^\alpha_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^\beta_{n+\sigma_2} \leq 0, \quad n \geq n_1. \tag{3.12}$$

Case (iv)

Suppose $a \geq 1$ and $b \leq 1$. Then, from (1.1), we have

$$a^{\beta}\Delta^m z_{n-\tau_1} + a^{\beta} p_{n-\tau_1} x^{\alpha}_{n-\sigma_1-\tau_1} + a^{\beta} q_{n-\tau_1} x^{\beta}_{n+\sigma_2-\tau_1} = 0, \quad n \geq n_1 \tag{3.13}$$

and

$$b^{\alpha}\Delta^m z_{n+\tau_2} + b^{\alpha} p_{n+\tau_2} x^{\alpha}_{n-\sigma_1+\tau_2} + b^{\alpha} q_{n+\tau_2} x^{\beta}_{n+\sigma_2+\tau_2} = 0, \quad n \geq n_1. \tag{3.14}$$

Now combining (1.1), (3.13), (3.14), and $\alpha < \beta$, we obtain

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^{\beta}\Delta^{m-1} z_{n-\tau_1} + b^{\alpha}\Delta^{m-1} z_{n+\tau_2}) + P_n(x^{\alpha}_{n-\sigma_1} + a^{\beta} x^{\alpha}_{n-\sigma_1-\tau_1} + b^{\alpha} x^{\alpha}_{n-\sigma_1+\tau_2}) \\ + Q_n(x^{\beta}_{n+\sigma_2} + a^{\beta} x^{\beta}_{n+\sigma_2-\tau_1} + b^{\alpha} x^{\beta}_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

In view of $a \geq 1$, $b \leq 1$, and $\alpha < \beta$, the last inequality becomes

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^{\beta}\Delta^{m-1} z_{n-\tau_1} + b^{\alpha}\Delta^{m-1} z_{n+\tau_2}) + P_n(x^{\alpha}_{n-\sigma_1} + a^{\alpha} x^{\alpha}_{n-\sigma_1-\tau_1} + b^{\alpha} x^{\alpha}_{n-\sigma_1+\tau_2}) \\ + Q_n(x^{\beta}_{n+\sigma_2} + a^{\beta} x^{\beta}_{n+\sigma_2-\tau_1} + b^{\beta} x^{\beta}_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1} z_n + a^{\beta}\Delta^{m-1} z_{n-\tau_1} + b^{\alpha}\Delta^{m-1} z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n+\sigma_2} \leq 0, \quad n \geq n_1. \tag{3.15}$$

Now (3.6), (3.9), (3.12), and (3.15) can be written as

$$\Delta(\Delta^{m-1} z_n + d_1\Delta^{m-1} z_{n-\tau_1} + d_2\Delta^{m-1} z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n+\sigma_2} \leq 0, \quad n \geq n_1. \tag{3.16}$$

Summing (3.16) from n to ∞ $(m-1)$ times, and then summing the resulting inequality from n_1 (where n_1 is large enough) to ∞, we obtain

$$\sum_{n=n_1}^{\infty} \frac{(n-n_1)^{m-1}}{(m-1)!} \left[P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n+\sigma_2}\right] < \infty. \tag{3.17}$$

Since $\{z_n\}$ is bounded, the last inequality implies that

$$\sum_{n=N}^{\infty} n^{m-1} R_n < \infty$$

for all $n \geq N \geq n_1$, which is a contradiction to (C_1). Thus, $\Delta z_n > 0$ eventually. Now using monotonicity of z_n, (3.16) becomes

$$\Delta(\Delta^{m-1} z_n + d_1\Delta^{m-1} z_{n-\tau_1} + d_1\Delta^{m-1} z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n-\sigma_1} \leq 0, \quad n \geq n_1. \tag{3.18}$$

Let $u_1\eta_1 = P_n z^{\alpha}_{n-\sigma_1}$ and $u_2\eta_2 = \dfrac{Q_n}{3^{\beta-1}} z^{\beta}_{n-\sigma_1}$. Using the arithmetic-geometric mean inequality

$$\frac{u_1\eta_1 + u_2\eta_2}{\eta_1+\eta_2} \geq (u_1^{\eta_1} u_2^{\eta_2})^{\frac{1}{\eta_1+\eta_2}}$$

and the fact $\eta_1 + \eta_2 = 1$, we get

$$P_n z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n-\sigma_1} \geq \eta_1^{-\eta_1}\eta_2^{-\eta_2} P_n^{\eta_1} \left(\frac{Q_n}{3^{\beta-1}}\right)^{\eta_2} z_{n-\sigma_1} = A_n z_{n-\sigma_1}, \quad n \geq n_1. \tag{3.19}$$

Now using (3.19) in (3.18), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + A_n z_{n-\sigma_1} \leq 0 \tag{3.20}$$

for all $n \geq n_1$. Using (2.7) in (3.20), we obtain

$$\begin{aligned}\Delta(\Delta^{m-1}z_n &+ d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) \\ &+ A_n\frac{\lambda}{(m-1)!}(n-\sigma_1)^{(m-1)}\Delta^{m-1}z_{n-\sigma_1} \leq 0, \quad n \geq n_1.\end{aligned} \tag{3.21}$$

By setting $\Delta^{m-1}z_n = y_n$, we see that $y_n > 0$ and $\Delta y_n \leq 0$. That is, $\{y_n\}$ is a positive decreasing solution of the inequality

$$\Delta(y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2}) + A_n\frac{\lambda}{(m-1)!}(n-\sigma_1)^{(m-1)}y_{n-\sigma_1} \leq 0 \tag{3.22}$$

for all $n \geq n_1$. Now by denoting $y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2} = w_n$ and using the monotonicity of y_n, we get

$$w_n \leq (1 + d_1 + d_2)y_{n-\tau_1} \quad \text{for all} \quad n \geq n_1.$$

Using the last inequality in the inequality (3.22), we see that $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n + \frac{A_n}{(1+d_1+d_2)}\frac{\lambda}{(m-1)!}(n-\sigma_1)^{(m-1)}w_{n+\tau_1-\sigma_1} \leq 0, \quad n \geq n_1,$$

which is a contradiction to (3.1). This completes the proof. □

Theorem 3.2. *Assume that* (C_1) *and*

(C_3) $\beta < 1 < \alpha$

hold. If the first-order difference inequality

$$\Delta w_n + \frac{B_n}{(1+d_3+d_4)}\frac{\lambda}{(m-1)!}(n-\sigma_1)^{(m-1)}w_{n+\tau_1-\sigma_1} \leq 0, \tag{3.23}$$

where

$$B_n = \eta_1^{-\eta_1}\eta_2^{-\eta_2}\left(\frac{P_n}{3^{\alpha-1}}\right)^{\eta_1} Q_n^{\eta_2}, \quad \eta_1 = \frac{\alpha-1}{\alpha-\beta}, \quad \eta_2 = \frac{1-\beta}{\alpha-\beta}$$

$$d_3 = \begin{cases} a^\alpha \text{ if } a \geq 1 \\ a^\beta, \text{ if } a \leq 1, \end{cases} \qquad d_4 = \begin{cases} b^\alpha \text{ if } b \geq 1 \\ b^\beta \text{ if } b \leq 1, \end{cases}$$

has no positive solution for some $\lambda \in (0,1)$ *and for all* $n \geq n_0$*, then every solution of* (1.1) *is almost oscillatory.*

Proof. The proof is similar to that of Theorem 3.1, and hence the details are omitted. □

Theorem 3.3. *Assume that* (C_1) *and*

(C_4) $1 < \alpha < \beta$

hold. If the first-order difference inequality

$$\Delta w_n + \frac{C_n}{(1+d_1+d_2)}\frac{\lambda}{(m-1)!}(n-\sigma_1)^{(m-1)} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0, \tag{3.24}$$

where

$$C_n = \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{3^{\beta-1}} P_n^{\eta_1} Q_n^{\eta_2}, \quad \eta_1 = \frac{\alpha-1}{\beta-1}, \quad \eta_2 = \frac{\beta-\alpha}{\beta-1}$$

and d_1, d_2 *are as in Theorem 3.1, has no positive solution for some* $\lambda \in (0,1)$ *and for all* $n \geq n_0$*, then every solution of* (1.1) *is almost oscillatory.*

Proof. Assume the contrary. Let $\{x_n\}$ be a nonoscillatory solution of (1.1), which does not converge to zero. Without loss of generality, we may assume that $\{x_n\}$ is a positive solution of (1.1) (since the proof for the negative case is similar). Then there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\sigma_1} > 0$, and $x_{n-\tau_1} > 0$ for all $n \geq n_1$. By definition of z_n, we have $z_n > 0$ for all $n \geq n_1$. Now from (1.1), we obtain

$$\Delta^m z_n = -p_n x_{n-\sigma_1}^\alpha - q_n x_{n+\sigma_2}^\beta \leq 0 \quad \text{for all} \quad n \geq n_1. \tag{3.25}$$

First, we shall prove that $\Delta z_n > 0$ for all $n \geq n_1$. If not, then $\Delta z_n \leq 0$ for all $n \geq n_1$. That is, $\{z_n\}$ is a positive decreasing sequence, and hence $\lim_{n\to\infty} z_n = M > 0$ and

$$\lim_{n\to\infty} \Delta^k z_n = 0 \quad \text{for} \quad k = 1,2,3,\ldots,(m-1). \tag{3.26}$$

Now we discuss the different cases for a and b.

Case (i)

Suppose $a \leq 1$ and $b \leq 1$. Then, from (1.1), we get

$$a^{\alpha}\Delta^{m} z_{n-\tau_1} + a^{\alpha} p_{n-\tau_1} x^{\alpha}_{n-\sigma_1-\tau_1} + a^{\alpha} q_{n-\tau_1} x^{\beta}_{n+\sigma_2-\tau_1} = 0, \quad n \geq n_1 \tag{3.27}$$

and

$$b^{\alpha}\Delta^{m} z_{n+\tau_2} + b^{\alpha} p_{n+\tau_2} x^{\alpha}_{n-\sigma_1+\tau_2} + b^{\alpha} q_{n+\tau_2} x^{\beta}_{n+\sigma_2+\tau_2} = 0, \quad n \geq n_1. \tag{3.28}$$

Now combining (1.1), (3.27), and (3.28), we obtain

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^{\alpha}\Delta^{m-1} z_{n-\tau_1} + b^{\alpha}\Delta^{m-1} z_{n+\tau_2}) + P_n(x^{\alpha}_{n-\sigma_1} + a^{\alpha} x^{\alpha}_{n-\sigma_1-\tau_1} + b^{\alpha} x^{\alpha}_{n-\sigma_1+\tau_2}) \\ + Q_n(x^{\beta}_{n+\sigma_2} + a^{\alpha} x^{\beta}_{n+\sigma_2-\tau_1} + b^{\alpha} x^{\beta}_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Since $a \leq 1$, $b \leq 1$, and $\beta > \alpha$, the last inequality yields

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^{\alpha}\Delta^{m-1} z_{n-\tau_1} + b^{\alpha}\Delta^{m-1} z_{n+\tau_2}) + P_n(x^{\alpha}_{n-\sigma_1} + a^{\alpha} x^{\alpha}_{n-\sigma_1-\tau_1} + b^{\alpha} x^{\alpha}_{n-\sigma_1+\tau_2}) \\ + Q_n(x^{\beta}_{n+\sigma_2} + a^{\beta} x^{\beta}_{n+\sigma_2-\tau_1} + b^{\beta} x^{\beta}_{n+\sigma_2+\tau_2}) \leq 0, \quad n \geq n_1.\end{aligned}$$

Now using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1} z_n + a^{\alpha}\Delta^{m-1} z_{n-\tau_1} + b^{\alpha}\Delta^{m-1} z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}} z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n+\sigma_2} \leq 0, \quad n \geq n_1.$$

The proofs for the other cases of a and b are similar to that of Theorem 3.1. Therefore, for all cases of a and b, we have the inequality

$$\Delta(\Delta^{m-1} z_n + d_1\Delta^{m-1} z_{n-\tau_1} + d_2\Delta^{m-1} z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}} z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n+\sigma_2} \leq 0, \quad n \geq n_1. \tag{3.29}$$

Now summing (3.29) from n to ∞ up to $(m-1)$ times, and then summing the resulting inequality from n_1 (where n_1 is large enough) to ∞, we obtain

$$\sum_{n=n_1}^{\infty} \frac{(n-n_1)^{m-1}}{(m-1)!3^{\beta-1}} \left[P_n z^{\alpha}_{n-\sigma_1} + Q_n z^{\beta}_{n+\sigma_2}\right] < \infty. \tag{3.30}$$

Since $\{z_n\}$ is bounded, (3.30) implies that

$$\sum_{n=N}^{\infty} n^{m-1} R_n < \infty$$

for all $n \geq N \geq n_1$, which is a contradiction to (C_1). Thus, $\Delta z_n > 0$ eventually. Now, using the monotonicity of z_n, (3.30) becomes

$$\Delta(\Delta^{m-1} z_n + d_1\Delta^{m-1} z_{n-\tau_1} + d_2\Delta^{m-1} z_{n+\tau_2}) + \frac{P_n}{3^{\beta-1}} z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n-\sigma_1} \leq 0, \quad n \geq n_1. \tag{3.31}$$

Let
$$u_1\eta_1 = \frac{P_n}{3^{\beta-1}} z^{\alpha}_{n-\sigma_1} \quad \text{and} \quad u_2\eta_2 = \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n-\sigma_1}.$$
Using the arithmetic-geometric mean inequality and the fact $\eta_1 + \eta_2 = 1$, we get
$$\frac{P_n}{3^{\beta-1}} z^{\alpha}_{n-\sigma_1} + \frac{Q_n}{3^{\beta-1}} z^{\beta}_{n-\sigma_1} \geq \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{3^{\beta-1}} P_n^{\eta_1} Q_n^{\eta_2} z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} = C_n z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1}. \tag{3.32}$$
Now using (3.32) in (3.31), we obtain
$$\Delta(\Delta^{m-1} z_n + d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) + C_n z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} \leq 0 \tag{3.33}$$
for all $n \geq n_1$. From (2.7) and (3.33), we obtain
$$\begin{aligned}\Delta(\Delta^{m-1} z_n &+ d_1 \Delta^{m-1} z_{n-\tau_1} + d_2 \Delta^{m-1} z_{n+\tau_2}) \\ &+ C_n \frac{\lambda}{(m-1)!} (n-\sigma_1)^{(m-1)} \Delta^{m-1} z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} \leq 0 \end{aligned} \tag{3.34}$$
for all $n \geq n_1$. By setting $\Delta^{m-1} z_n = y_n$, we see that $y_n > 0$ and $\Delta y_n \leq 0$. That is, $\{y_n\}$ is a positive decreasing solution of the equation
$$\Delta(y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2}) + C_n \frac{\lambda}{(m-1)!} (n-\sigma_1)^{(m-1)} y^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} \leq 0 \tag{3.35}$$
for all $n \geq n_1$. Now, by denoting $y_n + d_1 y_{n-\tau_1} + d_2 y_{n+\tau_2} = w_n$ and using the monotonicity of y_n, we get
$$w_n \leq (1 + d_1 + d_2) y_{n-\tau_1} \quad \text{for all} \quad n \geq n_1.$$
From the last inequality and (3.35), we see that $\{w_n\}$ is a positive solution of the inequality
$$\Delta w_n + \frac{C_n}{(1+d_1+d_2)} \frac{\lambda}{(m-1)!} (n-\sigma_1)^{(m-1)} w^{\alpha\eta_1+\beta\eta_2}_{n+\tau_1-\sigma_1} \leq 0, \quad n \geq n_1,$$
which is a contradiction to (3.24). This completes the proof. □

Theorem 3.4. *Assume that* (C_1) *and*

(C_5) $1 < \beta < \alpha$

hold. If the first-order difference inequality
$$\Delta w_n + \frac{D_n}{(1+d_3+d_4)(m-1)!} (n-\sigma_1)^{(m-1)} w^{\alpha\eta_1+\beta\eta_2}_{n+\tau_1-\sigma_1} \leq 0, \tag{3.36}$$
where
$$D_n = \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{3^{\alpha-1}} P_n^{\eta_1} Q_n^{\eta_2}, \quad \eta_1 = \frac{\beta-1}{\alpha-1}, \quad \eta_2 = \frac{\alpha-\beta}{\alpha-1}$$
and d_3, d_4 *are as in Theorem 3.2, has no positive solution for some* $\lambda \in (0,1)$ *and for all* $n \geq n_0$*, then every solution of* (1.1) *is almost oscillatory.*

Proof. The proof is similar to that of Theorem 3.3, and hence the details are omitted. □

Theorem 3.5. *Assume that* (C_1) *and*

(C_6) $\alpha < \beta < 1$

hold. If the first-order difference inequality

$$\Delta w_n + \frac{E_n}{(1+d_1+d_2)} \frac{\lambda}{(m-1)!}(n-\sigma_1)^{(m-1)} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \le 0, \tag{3.37}$$

where

$$E_n = \eta_1^{-\eta_1}\eta_2^{-\eta_2} P_n^{\eta_1} Q_n^{\eta_2}, \quad \eta_1 = \frac{\beta-\alpha}{1-\alpha}, \quad \eta_2 = \frac{1-\beta}{1-\alpha}$$

and d_1, d_2 *are as in Theorem 3.1, has no positive solution for some* $\lambda \in (0,1)$ *and for all* $n \ge n_0$*, then every solution of* (1.1) *is almost oscillatory.*

Proof. Assume the contrary. Let $\{x_n\}$ be a nonoscillatory solution of (1.1), which does not converge to zero. Then, proceeding as in Theorem 3.1, we have the following four cases for a and b.

Case (i)

From (1.1), we get

$$a^\alpha \Delta^m z_{n-\tau_1} + a^\alpha p_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\alpha + a^\alpha q_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\beta = 0, \quad n \ge n_1 \tag{3.38}$$

and

$$b^\alpha \Delta^m z_{n+\tau_2} + b^\alpha p_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\alpha + b^\alpha q_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\beta = 0, \quad n \ge n_1. \tag{3.39}$$

Now combining (1.1), (3.38), and (3.39), we obtain

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ + Q_n(x_{n+\sigma_2}^\beta + a^\alpha x_{n+\sigma_2-\tau_1}^\beta + b^\alpha x_{n+\sigma_2+\tau_2}^\beta) \le 0, \quad n \ge n_1.\end{aligned}$$

Since $a \le 1$, $b \le 1$, and $\alpha < \beta < 1$, the last inequality becomes

$$\begin{aligned}\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) + P_n(x_{n-\sigma_1}^\alpha + a^\alpha x_{n-\sigma_1-\tau_1}^\alpha + b^\alpha x_{n-\sigma_1+\tau_2}^\alpha) \\ + Q_n(x_{n+\sigma_2}^\beta + a^\beta x_{n+\sigma_2-\tau_1}^\beta + b^\beta x_{n+\sigma_2+\tau_2}^\beta) \le 0, \quad n \ge n_1.\end{aligned}$$

Now, using Lemma 2.1, we obtain

$$\Delta(\Delta^{m-1} z_n + a^\alpha \Delta^{m-1} z_{n-\tau_1} + b^\alpha \Delta^{m-1} z_{n+\tau_2}) + P_n z_{n-\sigma_1}^\alpha + Q_n z_{n+\sigma_2}^\beta \le 0, \quad n \ge n_1.$$

The proofs for the other cases of a and b are similar to that of Theorem 3.1. Therefore, for all cases of a and b, we have the inequality

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + Q_n z^{\beta}_{n+\sigma_2} \leq 0, \quad n \geq n_1. \tag{3.40}$$

Now summing (3.40) from n to ∞ up to $(m-1)$ times, and then summing the resulting inequality from n_1 to ∞, we obtain

$$\sum_{n=n_1}^{\infty} \frac{(n-n_1)^{m-1}}{(m-1)!}\left[P_n z^{\alpha}_{n-\sigma_1} + Q_n z^{\beta}_{n-\sigma_1}\right] < 0. \tag{3.41}$$

Using boundedness of z_n in the last inequality, we get

$$\sum_{n=n_1}^{\infty} n^{m-1}R_n < \infty,$$

which is a contradiction to (C_1). Hence, $\Delta z_n > 0$ eventually. From the monotonicity of z_n, (3.40) becomes

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + P_n z^{\alpha}_{n-\sigma_1} + Q_n z^{\beta}_{n-\sigma_1} \leq 0, \quad n \geq n_1. \tag{3.42}$$

Now set

$$u_1\eta_1 = P_n z^{\alpha}_{n-\sigma_1}, \quad u_2\eta_2 = Q_n z^{\beta}_{n-\sigma_1}, \quad \eta_1 = \frac{\beta-\alpha}{1-\alpha}, \quad \eta_2 = \frac{1-\beta}{1-\alpha}.$$

Then, by the arithmetic-geometric mean inequality

$$\frac{u_1\eta_1 + u_2\eta_2}{\eta_1+\eta_2} \geq (u_1^{\eta_1}u_2^{\eta_2})^{\frac{1}{\eta_1+\eta_2}},$$

we obtain

$$P_n z^{\alpha}_{n-\sigma_1} + Q_n z^{\beta}_{n-\sigma_1} \geq \eta_1^{-\eta_1}\eta_2^{-\eta_2}P_n^{\eta_1}Q_n^{\eta_2}z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} = E_n z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1}, \quad n \geq n_1. \tag{3.43}$$

Combining (3.42) and (3.43), we obtain

$$\Delta(\Delta^{m-1}z_n + d_1\Delta^{m-1}z_{n-\tau_1} + d_2\Delta^{m-1}z_{n+\tau_2}) + E_n z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} \leq 0. \tag{3.44}$$

From the last inequality, by taking $\Delta^{m-1}z_n = y_n$, we see that $y_n > 0$, $\Delta y_n \leq 0$, and

$$\Delta(y_n + d_1y_{n-\tau_1} + d_2y_{n+\tau_2}) + E_n z^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1} \leq 0. \tag{3.45}$$

Now let $y_n + d_1y_{n-\tau_1} + d_2y_{n+\tau_2} = w_n$. Then $w_n > 0$, and using $\Delta y_n \leq 0$, we get,

$$w_n \leq (1+d_1+d_2)y_{n-\tau_1} \quad \text{for all} \quad n \geq n_1. \tag{3.46}$$

Combining (3.45) and (3.46), we see that $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n + E_n\frac{\lambda(n-\sigma_1)^{(m-1)}}{(m-1)!(1+d_1+d_2)}w^{\alpha\eta_1+\beta\eta_2}_{n-\sigma_1+\tau_1} \leq 0, \quad \geq n_1,$$

which is a contradiction to (3.37). This completes the proof. □

Theorem 3.6. *Assume that* (C_1) *and*

(C_7) $\beta < \alpha < 1$

hold. If the first-order difference inequality

$$\Delta w_n + \frac{E_n}{(1+d_3+d_4)} \frac{\lambda}{(m-1)!}(n-\sigma_1)^{(m-1)} w_{n+\tau_1-\sigma_1}^{\alpha\eta_1+\beta\eta_2} \leq 0, \tag{3.47}$$

where

$$\eta_1 = \frac{\alpha-\beta}{1-\beta}, \quad \eta_2 = \frac{1-\alpha}{1-\beta}$$

and d_3, d_4 *are as in Theorem 3.2 and* E_n *is as defined in Theorem 3.5, has no positive solution, then every solution of* (1.1) *is almost oscillatory.*

Proof. The proof is similar to that of Theorem 3.5, and hence it is omitted. □

Corollary 3.7. *Assume that* (C_1), (C_2), *and* $\sigma_1 > \tau_1$ *hold. If*

$$\liminf_{n\to\infty} \sum_{s=n-(\sigma_1-\tau_1)}^{n-1} A_s(s-\sigma_1)^{(m-1)} > \frac{(1+d_1+d_2)(m-1)!}{\lambda} \left(\frac{\sigma_1-\tau_1}{\sigma_1-\tau_1+1}\right)^{\sigma_1-\tau_1+1}, \tag{3.48}$$

then every solution of (1.1) *is almost oscillatory.*

Proof. By Lemma 2.4, (3.48) guarantees that (3.1) has no positive solution. Now the result follows from Theorem 3.1. □

Corollary 3.8. *Assume that* (C_1), (C_3), *and* $\sigma_1 > \tau_1$ *hold. If*

$$\liminf_{n\to\infty} \sum_{n-(\sigma_1-\tau_1)}^{n-1} B_s(s-\sigma_1)^{(m-1)} > \frac{(1+d_3+d_4)(m-1)!}{\lambda} \left(\frac{\sigma_1-\tau_1}{\sigma_1-\tau_1+1}\right)^{\sigma_1-\tau_1+1}, \tag{3.49}$$

then every solution of (1.1) *is almost oscillatory.*

Proof. By Lemma 2.4, (3.49) guarantees that (3.23) has no positive solution. Now the result follows from Theorem 3.2. □

Note that for $\beta > \alpha > 1$,

$$\eta_1 = \frac{\alpha-1}{\beta-1} \quad \text{and} \quad \eta_2 = \frac{\beta-\alpha}{\beta-1} \quad \text{imply} \quad \alpha\eta_1 + \beta\eta_2 > 1.$$

Now using Theorem 3.3, we have the following corollary.

Corollary 3.9. *Assume that* (C_1)*,* (C_4)*, and* $\sigma_1 > \tau_1$ *hold. If there exists* $\mu > 0$ *such that*

$$\mu > \frac{1}{\sigma_1 - \tau_1} \ln(\alpha\eta_1 + \beta\eta_2)$$

and

$$\liminf_{n\to\infty} C_n (n - \sigma_1)^{(m-1)} \exp(-e^{\mu n}) > 0, \tag{3.50}$$

then every solution of (1.1) *is almost oscillatory.*

Proof. By Lemma 2.2, (3.50) guarantees that (3.24) has no positive solution. Now the result follows from Theorem 3.3. □

Corollary 3.10. *Assume that* (C_1)*,* (C_5)*, and* $\sigma_1 > \tau_1$ *hold. If there exists* $\mu > 0$ *such that*

$$\mu > \frac{1}{\sigma_1 - \tau_1} \ln(\alpha\eta_1 + \beta\eta_2)$$

and

$$\liminf_{n\to\infty} D_n (n - \sigma_1)^{(m-1)} \exp(-e^{\mu n}) > 0, \tag{3.51}$$

then every solution of (1.1) *is almost oscillatory.*

Proof. By Lemma 2.2, (3.51) guarantees that (3.36) has no positive solution. Now the result follows from Theorem 3.4. □

Note that for $\alpha < \beta < 1$,

$$\eta_1 = \frac{\beta - \alpha}{1 - \alpha} \quad \text{and} \quad \eta_2 = \frac{1 - \beta}{1 - \alpha} \quad \text{imply} \quad \alpha\eta_1 + \beta\eta_2 < 1.$$

Now using Theorem 3.5, we have the following corollary.

Corollary 3.11. *Assume that* (C_1) *and* (C_6) *hold. If*

$$\liminf_{n\to\infty} \sum_{n=n_0}^{\infty} E_s (s - \sigma_1)^{(m-1)} = \infty, \tag{3.52}$$

then every solution of (1.1) *is almost oscillatory.*

Proof. By Lemma 2.3, (3.52) guarantees that (3.37) has no positive solution. Now the result follows from Theorem 3.5. □

Note that for $\beta < \alpha < 1$,

$$\eta_1 = \frac{\alpha - \beta}{1 - \beta} \quad \text{and} \quad \eta_2 = \frac{1 - \alpha}{1 - \beta} \quad \text{imply} \quad \alpha\eta_1 + \beta\eta_2 < 1.$$

Now using Theorem 3.6, we have the following corollary.

Corollary 3.12. *Assume that* (C_1) *and* (C_7) *hold. If*

$$\lim_{n\to\infty} \inf \sum_{n=n_0}^{\infty} E_s(s-\sigma_1)^{(m-1)} = \infty, \tag{3.53}$$

then every solution of (1.1) *is almost oscillatory.*

Proof. By Lemma 2.3, (3.53) guarantees that (3.47) has no positive solution. Now the result follows from Theorem 3.6. □

Theorem 3.13. *Assume that* (C_1)*,* (C_2)*, and* $\sigma_1 \le \tau_1$ *hold. Further assume that there exists a real-valued function* $H : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ *such that*

$$\begin{aligned} H_{n,n} &= 0 \quad \textit{for} \quad n \ge n_0 > 0, \\ H_{n,s} &> 0 \quad \textit{for} \quad n > s \ge n_0, \\ \Delta_2 H_{n,s} &\le 0 \quad \textit{for} \quad n > s \ge n_0, \end{aligned}$$

where

$$\Delta_2 H_{n,s} = H_{n,s+1} - H_{n,s}.$$

If

$$\lim_{n\to\infty} \sup \frac{1}{H_{n,N}} \sum_{s=N}^{n-1} A_s H_{n,s} = \infty, \quad n \ge N \ge n_0, \tag{3.54}$$

then every solution of (1.1) *is almost oscillatory.*

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of (1.1), which does not tend to zero asymptotically. Without loss of generality, we may assume that $\{x_n\}$ is a positive solution of (1.1), which does not tend to zero asymptotically (since the proof for the negative case is similar). Then, proceeding as in the proof of Theorem 3.1, we obtain that $\Delta z_n > 0$ for all $n \ge n_1 \ge n_0$. Now define a function

$$w_n = \frac{\Delta^{m-1} z_n}{z_{n-\tau_1}} \quad \text{for all} \quad n \ge n_1. \tag{3.55}$$

Then $w_n > 0$ and

$$\Delta w_n = \frac{\Delta^m z_n}{z_{n-\tau_1}} - \frac{\Delta^{m-1} z_{n+1}}{z_{n-\tau_1} z_{n+1-\tau_1}} \Delta z_{n-\tau_1} \le \frac{\Delta^m z_n}{z_{n-\tau_1}}, \quad n \ge n_1.$$

Similarly, by defining v_n and u_n by

$$v_n = \frac{\Delta^{m-1} z_{n-\tau_1}}{z_{n-\tau_1}}, \quad u_n = \frac{\Delta^{m-1} z_{n+\tau_2}}{z_{n-\tau_1}}, \quad n \ge n_1,$$

we obtain $v_n > 0$ and $u_n > 0$ for all $n \geq n_1$ and

$$\Delta v_n \leq \frac{\Delta^m z_{n-\tau_1}}{z_{n-\tau_1}}, \quad \Delta u_n \leq \rho_n \frac{\Delta^m z_{n+\tau_2}}{z_{n-\tau_1}}, \quad n \geq n_1.$$

Now combining these inequalities, we obtain

$$\Delta w_n + a^\beta \Delta v_n + b^\beta \Delta u_n \leq \frac{1}{z_{n-\tau_1}}[\Delta^m z_n + a^\beta \Delta^m z_{n-\tau_1} + b^\beta \Delta^m z_{n+\tau_2}]$$

for all $n \geq n_1$. Now using (3.20) and the monotonicity of z_n, the last inequality becomes

$$\Delta w_n + a^\beta \Delta v_n + b^\beta \Delta u_n \leq -A_n.$$

Replacing n by s and multiplying the last inequality $H_{n,s}$ and then summing the resulting inequality from $N \geq n_1$ to $n-1$, we have

$$\sum_{s=N}^{n-1} A_s H_{n,s} \leq -\sum_{s=N}^{n-1} [\Delta w_s + a^\beta \Delta v_s + b^\beta \Delta u_s] H_{n,s}.$$

Now using summation by parts, we get

$$\begin{aligned}
\sum_{s=N}^{n-1} A_s H_{n,s} &\leq H_{n,N}[w_N + a^\beta v_N + b^\beta u_N] \\
&\quad + \sum_{s=N}^{n-1} [w_{s+1} + a^\beta v_{s+1} + b^\beta u_{s+1}] \Delta_2 H_{n,s} \\
&\leq [w_N + a^\beta v_N + b^\beta u_N] H_{n,N},
\end{aligned}$$

i.e.,

$$\frac{1}{H(n,N)} \sum_{s=N}^{n-1} A_s H_{n,s} \leq [w_N + a^\beta v_N + b^\beta u_N] \tag{3.56}$$

Taking $\limsup$ as $n \to \infty$ in (3.56), we obtain

$$\lim_{n\to\infty} \sup \frac{1}{H_{n,N}} \sum_{s=N}^{n-1} A_s H_{n,s} < \infty,$$

which is a contradiction to (3.54). The proof is complete. □

4 Examples

In this section, we present some examples in order to illustrate our main results.

Example 4.1. Consider the fifth-order difference equation

$$\Delta^5\left(x_n+\frac{1}{2}x_{n-1}+4x_{n+2}\right)+\frac{4^{-\frac{n}{3}}}{64}x_{n-3}^{\frac{1}{3}}+2^{2n+2}x_{n+2}^3=0. \tag{4.1}$$

Here,

$$m=5,\quad a=\frac{1}{2},\quad b=4,\quad p_n=\frac{4^{-\frac{n}{3}}}{64},\quad q_n=2^{2n+2},$$
$$\tau_1=1,\quad \tau_2=2,\quad \sigma_1=3,\quad \sigma_2=2,\quad \alpha=\frac{1}{3},\quad \beta=3.$$

Then we find

$$P_n=\frac{4^{-\frac{n}{3}}}{64},\quad Q_n=2^{2n+2},\quad R_n=\frac{4^{-\frac{n}{3}}}{64}+2^{2n+2},$$

and it is easy to see that

$$\sum_{n=n_0}^{\infty}n^4R_n=\infty$$

and

$$\lim_{n\to\infty}\inf\sum_{s=n-2}^{n-1}A_n(s-3)^{(4)}=\lim_{n\to\infty}\inf\frac{(250)^{1/4}}{48}[(n-5)^{(4)}+(n-4)^{(4)}]>\left(\frac{2}{3}\right)^3.$$

Therefore, all conditions of Corollary 3.7 are satisfied. Hence, every solution of (4.1) is almost oscillatory. In fact,

$$\{x_n\}=\left\{\frac{1}{2^n}\right\}$$

is one such nonoscillatory solution of (4.1) tending to zero as $n\to\infty$.

Example 4.2. Consider the third-order difference equation

$$\Delta^3(x_n+2x_{n-1}+\frac{1}{2}x_{n+2})+e^{e^n}x_{n-3}^5+2x_{n+2}^3=0. \tag{4.2}$$

Here,

$$m=3,\quad a=2,\quad b=\frac{1}{2},\quad \tau_1=1,\quad \tau_2=2,$$
$$\sigma_1=3,\quad \sigma_2=2,\quad \alpha=5,\quad \beta=3,\quad p_n=e^{e^n},\quad q_n=2.$$

Then we find

$$P_n=e^{e^n},\quad Q_n=2,\quad R_n=2+\exp(e^n),$$

and it is easy to see that

$$\sum_{n=n_0}^{\infty}n^3(2+\exp(e^n))=\infty$$

and

$$\lim_{n\to\infty}\inf D_n(n-\sigma_1)^{(m-1)}\exp(-e^{\mu_n}) = \lim_{n\to\infty}\inf \frac{(1/2)^{-1}(\exp(e^n))^{1/2}2^{1/2}}{3^4}(n-3)^{(2)} > 0.$$

Therefore, all conditions of Corollary 3.10 are satisfied. Hence, every solution of (4.2) is almost oscillatory.

Example 4.3. Consider the fifth-order difference equation

$$\Delta^5(x_n + x_{n-2} + x_{n+2}) + 4x_{n-3}^{\frac{1}{3}} + 100x_{n+2}^{\frac{1}{5}} = 0. \tag{4.3}$$

Here,

$$a = 1, \quad b = 1, \quad p_n = 4, \quad q_n = 100, \quad \alpha = \frac{1}{3}, \quad \beta = \frac{1}{5},$$
$$\tau_1 = 2, \quad \tau_2 = 2, \quad \sigma_1 = 3, \quad \sigma_2 = 2.$$

Then we find

$$P_n = 4, \quad Q_n = 100, \quad R_n = 104,$$

and it is easy to see that

$$\sum_{n=n_0}^{\infty} n^4 100 = \infty$$

and

$$\lim_{n\to\infty}\inf \sum_{s=n-1}^{n-1} E_n(s-3)^{(4)} = \lim_{n\to\infty}\inf \sum_{s=n-1}^{n-1} 24(5)^{5/6}(n-4)^{(4)} = \infty.$$

Therefore, all conditions of Corollary 3.12 are satisfied. Hence, every solution of (4.3) is almost oscillatory. In fact,

$$\{x_n\} = \{(-1)^{15n}\}$$

is one such oscillatory solution of (4.3).

Remark 4.4.

1. The established results are presented in a form, which is essentially new and include some of the existing results as special cases.
2. The existing results [4–6, 8, 10, 11, 15, 18–23] cannot be applied to (4.1)–(4.3) since $\alpha \neq \beta$.
3. The results of this paper may be extended to equation of the form

$$\Delta\left(a_n(\Delta^{m-1}(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2}))\right) + q_n x_{n-\sigma_1}^{\alpha} + p_n x_{n+\sigma_2}^{\beta} = 0$$

when

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n} = \infty \quad \text{or} \quad \sum_{n=n_0}^{\infty}\frac{1}{a_n} < \infty,$$

and the details are left to the reader.

References

[1] Ravi P. Agarwal. *Difference equations and inequalities*, volume 228 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, second edition, 2000. Theory, methods, and applications.

[2] Ravi P. Agarwal, Martin Bohner, Said R. Grace, and Donal O'Regan. *Discrete oscillation theory*. Hindawi Publishing Corporation, New York, 2005.

[3] Ravi P. Agarwal and Said R. Grace. Oscillation of higher-order nonlinear difference equations of neutral type. *Appl. Math. Lett.*, 12(8):77–83, 1999.

[4] Ravi P. Agarwal and Said R. Grace. Oscillation criteria for certain higher order difference equations. *Math. Sci. Res. J.*, 6(1):60–64, 2002.

[5] Ravi P. Agarwal, Said R. Grace, and Elvan Akın-Bohner. On the oscillation of higher order neutral difference equations of mixed type. *Dynam. Systems Appl.*, 11(4):459–469, 2002.

[6] Ravi P. Agarwal, Said R. Grace, and Donal O'Regan. On the oscillation of higher order difference equations. *Soochow J. Math.*, 31(2):245–259, 2005.

[7] Ravi P. Agarwal, Ethiraju Thandapani, and Patricia J. Y. Wong. Oscillations of higher-order neutral difference equations. *Appl. Math. Lett.*, 10(1):71–78, 1997.

[8] Y. Bolat and Ö. Akin. Oscillatory behaviour of a higher-order nonlinear neutral type functional difference equation with oscillating coefficients. *Appl. Math. Lett.*, 17(9):1073–1078, 2004.

[9] S. R. Grace. Oscillation of certain neutral difference equations of mixed type. *J. Math. Anal. Appl.*, 224(2):241–254, 1998.

[10] Emrah Karaman and Mustafa Kemal Yıldız. Oscillatory behavior of a higher-order nonlinear neutral type functional difference equation with oscillating coefficients. *Appl. Math. E-Notes*, 14:242–249, 2014.

[11] Wan-Tong Li. Oscillation of higher-order neutral nonlinear difference equations. *Appl. Math. Lett.*, 11(4):1–8, 1998.

[12] N. Parhi and Arun K. Tripathy. Oscillation of a class of nonlinear neutral difference equations of higher order. *J. Math. Anal. Appl.*, 284(2):756–774, 2003.

[13] Devarajalu Seghar, Srinivasan Selvarangam, and Ethiraju Thandapani. Oscillation criteria for even order nonlinear neutral difference equations. *Differ. Equ. Appl.*, 6(3):441–453, 2014.

[14] Xianhua Tang and Yuji Liu. Oscillation for nonlinear delay difference equations. *Tamkang J. Math.*, 32(4):275–280, 2001.

[15] Ethiraju Thandapani. Oscillation theorems for higher order nonlinear difference equations. *Indian J. Pure Appl. Math.*, 25(5):519–524, 1994.

[16] Ethiraju Thandapani, John R. Graef, and Paul W. Spikes. On existence of positive solutions and oscillations of neutral difference equations of odd order. *J. Differ. Equations Appl.*, 2(2):175–183, 1996. Difference equations: theory and applications (San Francisco, CA, 1995).

[17] Ethiraju Thandapani and Nagabhushanam Kavitha. Oscillatory behavior of solutions of certain third order mixed neutral difference equations. *Acta Math. Sci. Ser. B Engl. Ed.*, 33(1):218–226, 2013.

[18] Xiaoping Wang and Lihong Huang. Oscillation for an odd order delay difference equations with several delays. *Int. J. Qual. Theory Diff. Equ. Appl.*, 2(1):15–23, 2016.

[19] Mustafa K. Yildiz and Özkan Öcalan. Oscillation results for higher order nonlinear neutral delay difference equations. *Appl. Math. Lett.*, 20(3):243–247, 2007.

[20] Guang Zhang and Ying Gao. Positive solutions of higher-order nonlinear difference equations. *J. Systems Sci. Math. Sci.*, 19(2):157–161, 1999.

[21] Xiaoliang Zhou and Jurang Yan. Oscillatory and asymptotic properties of higher order nonlinear difference equations. *Nonlinear Anal.*, 31(3-4):493–502, 1998.

[22] Yong Zhou. Oscillations of higher-order linear difference equations. *Comput. Math. Appl.*, 42(3-5):323–331, 2001. Advances in difference equations, III.

[23] Yong Zhou. Existence of nonoscillatory solutions of higher-order neutral difference equations with general coefficients. *Appl. Math. Lett.*, 15(7):785–791, 2002.