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Weak Solutions for Implicit Differential Equations with Hilfer–Hadamard Fractional Derivative

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Abstract

In this paper, by applying Mönch's fixed point theorem associated with the technique of measure of weak noncompactness, we present some existence of weak solutions for a class of functional implicit fractional differential equations of Hilfer–Hadamard type.

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1 Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, bio-engineering, and other applied sciences [17, 29]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas et al. [1, 2], Kilbas et al. [23], Samko et al. [28], and Zhou [33].

The measure of weak noncompactness was introduced by De Blasi [11]. The strong measure of noncompactness was developed first by Banaś and Goebel [5] and subsequently

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developed and used in many papers; see for example, Akhmerov et al. [4], Alvárez [10], Benchohra et al. [8], Guo et al. [15], and the references therein. In [8,26], the authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers obtained other results by application of the technique of measure of weak noncompactness; see [2,6,7] and the references therein.

Implicit functional differential equations have been considered by many authors [3, 12, 31]. Our intention is to extend the results to implicit differential equations of fractional order. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative [13, 14, 17–19, 30, 32] and other problems with Hilfer–Hadamard fractional derivative [20, 21]. In this paper, we discuss the existence of weak solutions for the problem of implicit Hilfer–Hadamard fractional differential equations of the form

$$\begin{cases} ({}^H D_1^{\alpha,\beta} u)(t) = f(t, u(t), ({}^H D_1^{\alpha,\beta} u)(t));\ t \in I := [1, T], \\ \\ ({}^H I_1^{1-\gamma} u)(t)\Big|_{t=1} = \phi, \end{cases} \tag{1.1}$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $T > 1$, $\phi \in E$, $f : I \times E \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual E^*, such that E is the dual of a weakly compactly generated Banach space X, ${}^H I_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$, and ${}^H D_1^{\alpha,\beta}$ is the Hilfer–Hadamard fractional derivative of order α and type β.

Our goal in this work is to give some existence results for implicit Hilfer–Hadamard fractional differential equations.

2 Preliminaries

Let C be the Banach space of all continuous functions v from I into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t \in I} \|v(t)\|_E.$$

As usual, $\mathrm{AC}(I)$ denotes the space of absolutely continuous functions from I into E. We denote by $\mathrm{AC}^1(I)$ the space defined by

$$\mathrm{AC}^1(I) := \{w : I \to E : w' \in \mathrm{AC}(I)\},$$

where $w'(t) = \frac{\mathrm{d}}{\mathrm{d}t} w(t)$ $t \in I$. Let

$$\delta = t\frac{\mathrm{d}}{\mathrm{d}t},\ q > 0,\ n = [q] + 1,$$

where $[q]$ is the integer part of q. Define the space

$$\mathrm{AC}_\delta^n := \{u : I \to E :\ \delta^{n-1}(u) \in \mathrm{AC}(I)\}.$$

Let $\gamma \in (0,1]$. By $\mathrm{C}_{\gamma,\ln}(I)$, $\mathrm{C}_{\gamma}(I)$, and $\mathrm{C}^1_{\gamma}(I)$, we denote the weighted spaces of continuous functions defined by

$$\mathrm{C}_{\gamma,\ln}(I) = \{w : I \to E : \ \widetilde{w} \in \mathrm{C}\},$$

where $\widetilde{w}(t) = (\ln t)^{1-\gamma} w(t)$, $t \in I$, with the norm

$$\|w\|_{\mathrm{C}_{\gamma,\ln}} := \sup_{t\in I} \|\widetilde{w}(t)\|_E,$$

$$\mathrm{C}_{\gamma}(I) = \{w : (1,T] \to E : \ \bar{w} \in \mathrm{C}\},$$

where $\bar{w}(t) = t^{1-\gamma} w(t)$, $t \in (1,T]$, with the norm

$$\|w\|_{\mathrm{C}_{\gamma}} := \sup_{t\in I} \|\bar{w}(t)\|_E,$$

and

$$\mathrm{C}^1_{\gamma}(I) = \{w \in \mathrm{C} : \ w' \in \mathrm{C}_{\gamma}\},$$

with the norm

$$\|w\|_{\mathrm{C}^1_{\gamma}} := \|w\|_{\infty} + \|w'\|_{\mathrm{C}_{\gamma}}.$$

In the following, we denote $\|w\|_{\mathrm{C}_{\gamma,\ln}}$ by $\|w\|_{\mathrm{C}}$. Let $(E,w) = (E, \sigma(E,E^*))$ be the Banach space E with its weak topology.

Definition 2.1. A Banach space X is called weakly compactly generated (WCG, for short) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.2. A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any (u_n) in E with $u_n \to u$ in (E,w) then $h(u_n) \to h(u)$ in (E,w)).

Definition 2.3 (See [27]). The function $u : I \to E$ is said to be Pettis integrable on I provided there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))\mathrm{d}s$ for all $\phi \in E^*$, where the integral on the right-hand side is assumed to exist in the sense of Lebesgue; by definition, $u_J = \int_J u(s)\mathrm{d}s$.

Let $\mathrm{P}(I,E)$ be the space of all E-valued Pettis integrable functions on I and $\mathrm{L}^1(I,E)$ be the Banach space of Bochner integrable functions $u : I \to E$. Define the class $\mathrm{P}_1(I,E)$ by

$$\mathrm{P}_1(I,E) = \{u \in \mathrm{P}(I,E) : \ \varphi(u) \in \mathrm{L}^1(I,E) \quad \text{for every} \quad \varphi \in E^*\}.$$

The space $\mathrm{P}_1(I,E)$ is normed by

$$\|u\|_{\mathrm{P}_1} = \sup_{\varphi\in E^*,\ \|\varphi\|\le 1} \int_1^T |\varphi(u(x))|\mathrm{d}\lambda x,$$

where λ stands for the Lebesgue measure on I.

The following result is due to Pettis (see [27, Theorem 3.4 and Corollary 3.41]).

Proposition 2.4 (See [27]). *If $u \in \mathrm{P}_1(I,E)$ and h is a measurable and essentially bounded E-valued function, then $uh \in \mathrm{P}_1(I,E)$.*

For all that follows, the symbol "$\int$" denotes the Pettis integral.

Now, we give some results and properties of fractional calculus.

Definition 2.5 (Riemann–Liouville fractional integral [1,23,28]). The left-sided mixed Riemann–Liouville integral of order $r > 0$ of a function $w \in \mathrm{L}^1(I)$ is defined by

$$(I_1^r w)(t) = \frac{1}{\Gamma(r)} \int_1^t (t-s)^{r-1} w(s)\mathrm{d}s \quad \text{for a.e.} \quad t \in I,$$

where $\Gamma(\cdot)$ is the (Euler) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t}\mathrm{d}t;\ \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in \mathrm{C}$, we have $I_0^r w \in \mathrm{C}$ and

$$(I_1^{r_1} I_1^{r_2} w)(t) = (I_1^{r_1+r_2} w)(t) \quad \text{for a.e.} \quad t \in I.$$

Definition 2.6 (Riemann–Liouville fractional derivative [1,23,28]). The Riemann–Liouville fractional derivative of order $r > 0$ of a function $w \in \mathrm{L}^1(I)$ is defined by

$$\begin{aligned}(D_1^r w)(t) &= \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} I_1^{n-r} w\right)(t) \\ &= \frac{1}{\Gamma(n-r)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_1^t (t-s)^{n-r-1} w(s)\mathrm{d}s \quad \text{for a.e.} \quad t \in I,\end{aligned}$$

where $n = [r]+1$ and $[r]$ is the integer part of r.

In particular, if $r \in (0,1]$, then

$$\begin{aligned}(D_1^r w)(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t} I_1^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{\mathrm{d}}{\mathrm{d}t} \int_1^t (t-s)^{-r} w(s)\mathrm{d}s \quad \text{for a.e.} \quad t \in I.\end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$, and $w \in \mathrm{C}_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows:

$$(D_1^r I_1^r w)(t) = w(t) \quad \text{for all} \quad t \in (1,T].$$

Moreover, if $I_1^{1-r} w \in \mathrm{C}_{1-\gamma}^1(I)$, then the following composition is proved in [28]:

$$(I_1^r D_1^r w)(t) = w(t) - \frac{(I_1^{1-r} w)(1^+)}{\Gamma(r)} t^{r-1} \quad \text{for all} \quad t \in (1,T].$$

Definition 2.7 (Caputo fractional derivative [1, 23, 28]). The Caputo fractional derivative of order $r > 0$ of a function $w \in \mathrm{L}^1(I)$ is defined by

$$\begin{aligned}({}^cD_1^r w)(t) &= \left(I_1^{n-r}\frac{\mathrm{d}^n}{\mathrm{d}t^n}w\right)(t)\\ &= \frac{1}{\Gamma(n-r)}\int_1^t (t-s)^{n-r-1}\frac{\mathrm{d}^n}{\mathrm{d}s^n}w(s)\mathrm{d}s \quad \text{for a.e.} \quad t\in I.\end{aligned}$$

In particular, if $r \in (0, 1]$, then

$$\begin{aligned}({}^cD_1^r w)(t) &= \left(I_1^{1-r}\frac{\mathrm{d}}{\mathrm{d}t}w\right)(t)\\ &= \frac{1}{\Gamma(1-r)}\int_1^t (t-s)^{-r}\frac{\mathrm{d}}{\mathrm{d}s}w(s)\mathrm{d}s \quad \text{for a.e.} \quad t\in I.\end{aligned}$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [16, 23] for a more detailed analysis.

Definition 2.8 (Hadamard fractional integral [16, 23]). The Hadamard fractional integral of order $q > 0$ of a function $g \in \mathrm{L}^1(I, E)$ is defined as

$$({}^HI_1^q g)(x) = \frac{1}{\Gamma(q)}\int_1^x \left(\ln\frac{x}{s}\right)^{q-1}\frac{g(s)}{s}\mathrm{d}s,$$

provided the integral exists.

Example 2.9. Let $0 < q < 1$. Then

$${}^HI_1^q \ln t = \frac{1}{\Gamma(2+q)}(\ln t)^{1+q} \quad \text{for a.e.} \quad t \in [0, e].$$

Remark 2.10. Let $g \in \mathrm{P}_1(I, E)$. For every $\varphi \in E^*$, we have

$$\varphi({}^HI_1^q g)(t) = ({}^HI_1^q \varphi g)(t) \quad \text{for a.e.} \quad t \in I.$$

Analogous to the Riemann–Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way.

Definition 2.11 (Hadamard fractional derivative [16, 23]). The Hadamard fractional derivative of order $q > 0$ applied to the function $w \in \mathrm{AC}_\delta^n$ is defined by

$$({}^HD_1^q w)(x) = \delta^n({}^HI_1^{n-q} w)(x).$$

In particular, if $q \in (0, 1]$, then

$$({}^HD_1^q w)(x) = \delta({}^HI_1^{1-q} w)(x).$$

Example 2.12. Let $0 < q < 1$. Then

$$ {}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)} (\ln t)^{1-q} \quad \text{for a.e.} \quad t \in [0, e]. $$

It has been shown (see, e.g., Kilbas [22, Theorem 4.8]) that in the space $\mathrm{L}^1(I, E)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.,

$$ ({}^H D_1^q)({}^H I_1^q w)(x) = w(x). $$

From [23, Theorem 2.3], we have

$$ ({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)} (\ln x)^{q-1}. $$

Analogous to the Hadamard fractional calculus, the Caputo–Hadamard fractional derivative is defined as follows.

Definition 2.13 (Caputo–Hadamard fractional derivative)**.** The Caputo–Hadamard fractional derivative of order $q > 0$ applied to the function $w \in \mathrm{AC}_\delta^n$ is defined by

$$ ({}^{Hc} D_1^q w)(x) = ({}^H I_1^{n-q} \delta^n w)(x). $$

In particular, if $q \in (0, 1]$, then

$$ ({}^{Hc} D_1^q w)(x) = ({}^H I_1^{1-q} \delta w)(x). $$

In [17], R. Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases (see also [18, 19, 30]).

Definition 2.14 (Hilfer fractional derivative)**.** Let $\alpha \in (0, 1)$, $\beta \in [0, 1]$, $w \in \mathrm{L}^1(I)$, $I_1^{(1-\alpha)(1-\beta)} w \in \mathrm{AC}^1(I)$. The Hilfer fractional derivative of order α and type β of w is defined by

$$ (D_1^{\alpha,\beta} w)(t) = \left(I_1^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} I_1^{(1-\alpha)(1-\beta)} w \right)(t) \quad \text{for a.e.} \quad t \in I. \tag{2.1} $$

Now we list some properties of the Hilfer fractional derivative. Let $\alpha \in (0, 1)$, $\beta \in [0, 1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in \mathrm{L}^1(I)$.

1. The operator $(D_1^{\alpha,\beta} w)(t)$ can be written as

$$ (D_1^{\alpha,\beta} w)(t) = \left(I_1^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} I_1^{1-\gamma} w \right)(t) = \left(I_1^{\beta(1-\alpha)} D_1^\gamma w \right)(t) \quad \text{for a.e.} \quad t \in I. $$

Moreover, the parameter γ satisfies

$$ \gamma \in (0, 1], \ \gamma \geq \alpha, \ \gamma > \beta, \ 1 - \gamma < 1 - \beta(1 - \alpha). $$

2. The generalization (2.1) for $\beta = 0$ coincides with the Riemann–Liouville derivative and for $\beta = 1$ coincides with the Caputo derivative:
$$D_1^{\alpha,0} = D_1^{\alpha} \quad \text{and} \quad D_1^{\alpha,1} = {}^cD_1^{\alpha}.$$
3. If $D_1^{\beta(1-\alpha)}w$ exists and is in $\mathrm{L}^1(I)$, then
$$(D_1^{\alpha,\beta} I_1^{\alpha} w)(t) = (I_1^{\beta(1-\alpha)} D_1^{\beta(1-\alpha)} w)(t) \quad \text{for a.e.} \quad t \in I.$$
Furthermore, if $w \in \mathrm{C}_\gamma(I)$ and $I_1^{1-\beta(1-\alpha)}w \in \mathrm{C}_\gamma^1(I)$, then
$$(D_1^{\alpha,\beta} I_1^{\alpha} w)(t) = w(t) \quad \text{for a.e.} \quad t \in I.$$
4. If $D_1^{\gamma}w$ exists and is in $\mathrm{L}^1(I)$, then
$$(I_1^{\alpha} D_1^{\alpha,\beta} w)(t) = (I_1^{\gamma} D_1^{\gamma} w)(t) = w(t) - \frac{I_1^{1-\gamma}(1^+)}{\Gamma(\gamma)} t^{\gamma-1} \quad \text{for a.e.} \quad t \in I.$$

From the Hadamard fractional integral, the Hilfer–Hadamard fractional derivative (introduced for the first time in [20]) is defined as follows.

Definition 2.15 (Hilfer–Hadamard fractional derivative)**.** Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $w \in \mathrm{L}^1(I)$, and ${}^HI_1^{(1-\alpha)(1-\beta)}w \in \mathrm{AC}^1(I)$. The Hilfer–Hadamard fractional derivative of order α and type β applied to the function w is defined by
$$\begin{aligned}({}^HD_1^{\alpha,\beta}w)(t) &= \left({}^HI_1^{\beta(1-\alpha)}({}^HD_1^{\gamma}w)\right)(t) \\ &= \left({}^HI_1^{\beta(1-\alpha)}\delta({}^HI_1^{1-\gamma}w)\right)(t) \quad \text{for a.e.} \quad t \in I.\end{aligned} \tag{2.2}$$

This new fractional derivative (2.2) may be viewed as interpolating the Hadamard fractional derivative and the Caputo–Hadamard fractional derivative. Indeed, for $\beta = 0$, this derivative reduces to the Hadamard fractional derivative, and when $\beta = 1$, we recover the Caputo–Hadamard fractional derivative:
$${}^HD_1^{\alpha,0} = {}^HD_1^{\alpha} \quad \text{and} \quad {}^HD_1^{\alpha,1} = {}^{Hc}D_1^{\alpha}.$$

From [21, Theorem 21], we concluded the following lemma.

Lemma 2.16. *Let $f : I \times E \times E \to E$ be such that $f(\cdot,u,v) \in \mathrm{C}_{\gamma,\ln}(I)$ for any $u,v \in \mathrm{C}_{\gamma,\ln}(I)$. Then* (1.1) *is equivalent to the problem of obtaining the solution of the equation*
$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^HI_1^{\alpha}g)(t), g(t)\right),$$
and if $g \in \mathrm{C}_{\gamma,\ln}$ is the solution of this equation, then
$$u(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^HI_1^{\alpha}g)(t).$$

Definition 2.17 (See [11])**.** Let E be a Banach space, Ω_E the set of bounded subsets of E and B_1 the unit ball of E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by

$$\beta(X) = \inf\{\varepsilon > 0 : \text{ there exists a weakly compact } \Omega \subset E \text{ such that } X \subset \varepsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B$ implies $\beta(A) \le \beta(B)$,

(b) $\beta(A) = 0$ if and only if A is weakly relatively compact,

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$,

(d) $\beta(\overline{A}^{\omega}) = \beta(A)$, where $\overline{A}^{\omega}$ denotes the weak closure of A,

(e) $\beta(A + B) \le \beta(A) + \beta(B)$,

(f) $\beta(\lambda A) = |\lambda|\beta(A)$,

(g) $\beta(\mathrm{conv}(A)) = \beta(A)$,

(h) $\beta(\cup_{|\lambda| \le h} \lambda A) = h\beta(A)$.

The next result follows directly from the Hahn–Banach theorem.

Proposition 2.18. *If E is a normed space and $x_0 \in E \setminus \{0\}$, then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$, let us denote

$$V(t) = \{v(t) : \ v \in V\}, \ t \in I$$

and

$$V(I) = \{v(t) : \ v \in V, \ t \in I\}.$$

Lemma 2.19 (See [15])**.** *Let $H \subset C$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on I,*

$$\beta_C(H) = \max_{t \in I} \beta(H(t)),$$

and

$$\beta\left(\int_I u(s)\mathrm{d}s\right) \le \int_I \beta(H(s))\mathrm{d}s,$$

where $H(s) = \{u(s) : u \in H, \ s \in I\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

For our purpose, we will need the following fixed point theorem.

Theorem 2.20 (See [25]). *Let Q be a nonempty, closed, convex, and equicontinuous subset of a metrizable locally convex vector space $\mathrm{C}(I,E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{\mathrm{conv}}(\{0\} \cup T(V)) \quad \textit{implies} \quad V \ \textit{is relatively weakly compact} \tag{2.3}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence of Weak Solutions

Let us start by defining what we mean by a weak solution of (1.1).

Definition 3.1. By a weak solution of (1.1), we mean a measurable function $u \in \mathrm{C}_{\gamma,\ln}$ that satisfies the condition $({}^H I_1^{1-\gamma} u)(1^+) = \phi$ and the equation

$$({}^H D_1^{\alpha,\beta} u)(t) = f(t, u(t), ({}^H D_1^{\alpha,\beta} u)(t)) \quad \text{on} \quad I.$$

The following hypotheses will be used in the sequel.

(H_1) The functions $v \to f(t,v,w)$ and $w \to f(t,v,w)$ are weakly sequentially continuous for a.e. $t \in I$,

(H_2) For each $v, w \in E$, the function $t \to f(t,v,w)$ is Pettis integrable a.e. on I,

(H_3) There exists $p \in \mathrm{C}(I,[0,\infty))$ such that for all $\varphi \in E^*$, we have

$$|\varphi(f(t,u,v))| \leq \frac{p(t)\|u\|_E}{1 + \|\varphi\| + \|u\|_E + \|v\|_E} \text{ for a.e. } t \in I \text{ and each } u, v \in E,$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta(f(t, B, {}^H D_1^{\alpha,\beta} B) \leq (\ln t)^{1-\gamma} p(t) \beta(B),$$

where ${}^H D_1^{\alpha,\beta} B = \{{}^H D_1^{\alpha,\beta} w : w \in B\}$.

Now we present our main existence theorem. Set

$$p^* = \sup_{t \in I} p(t).$$

Theorem 3.2. *Assume that the hypotheses (H_1)–(H_4) hold. If*

$$L := \frac{p^* (\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} < 1, \tag{3.1}$$

then (1.1) *has at least one weak solution defined on I.*

Proof. Consider the operator $N : \mathrm{C}_{\gamma,\ln} \to \mathrm{C}_{\gamma,\ln}$ defined by

$$(Nu)(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g)(t), \tag{3.2}$$

where $g \in \mathrm{C}_{\gamma,\ln}$ is such that

$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g)(t), g(t)\right).$$

First, notice that the hypotheses imply that for each $u, g \in C_{\gamma,\ln}$, the function

$$t \mapsto \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s)$$

is Pettis integrable over I, and

$$t \mapsto f\left(t, \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g)(t), g(t)\right) \quad \text{for a.e.} \quad t \in I$$

is Pettis integrable. Thus, the operator N is well defined. Let $R > 0$ be such that $R > L$ and consider the set

$$\begin{aligned} Q = \Big\{ & u \in \mathrm{C}_{\gamma,\ln} : \|u\|_{\mathrm{C}} \le R \quad \text{and} \quad \|(\ln t_2)^{1-\gamma}u(t_2) - (\ln t_1)^{1-\gamma}u(t_1)\|_E \\ & \le L\left(\ln \frac{t_2}{t_1}\right)^{\alpha} \\ & + \frac{p^*}{\Gamma(\alpha)} \int_1^{t_1} \left| (\ln t_2)^{1-\gamma}\left(\ln \frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln \frac{t_1}{s}\right)^{\alpha-1} \right| \mathrm{d}s \Big\}. \end{aligned}$$

Clearly, the subset Q is closed, convex, and equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 2.20. The proof will be given in three steps.

Step 1. N maps Q into itself

Let $u \in Q$, $t \in I$ and assume that $(Nu)(t) \neq 0$. Then there exists $\varphi \in E^*$ such that

$$\|(\ln t)^{1-\gamma}(Nu)(t)\|_E = |\varphi((\ln t)^{1-\gamma}(Nu)(t))|.$$

Thus

$$\|(\ln t)^{1-\gamma}(Nu)(t)\|_E = \varphi\left(\frac{\phi}{\Gamma(\gamma)} + \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} g(s)\frac{\mathrm{d}s}{s}\right),$$

where $g \in \mathrm{C}_{\gamma,\ln}$ is such that

$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g)(t), g(t)\right).$$

Then

$$\begin{aligned}\|(\ln t)^{1-\gamma}(Nu)(t)\|_E &\le \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}|\varphi(g(s))|\frac{\mathrm{d}s}{s}\\ &\le \frac{p^*(\ln T)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{\mathrm{d}s}{s}\\ &\le \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\\ &= L<R.\end{aligned}$$

Next, let $t_1, t_2 \in I$ be such that $t_1 < t_2$, and let $u \in Q$ with

$$(\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1)\neq 0.$$

Then there exists $\varphi \in E^*$ such that

$$\begin{aligned}&\|(\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1)\|_E\\ &\qquad\qquad = |\varphi((\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1))|\end{aligned}$$

and $\|\varphi\| = 1$. Hence,

$$\begin{aligned}&\|(\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1)\|_E\\ &= |\varphi((\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1))|\\ &\le \varphi\left((\ln t_2)^{1-\gamma}\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{g(s)}{s\Gamma(\alpha)}\mathrm{d}s-(\ln t_1)^{1-\gamma}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\frac{g(s)}{s\Gamma(\alpha)}\mathrm{d}s\right),\end{aligned}$$

where $g \in \mathrm{C}_{\gamma,\ln}$ is such that

$$g(t)=f\left(t,\frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1}+({}^HI_1^\alpha g)(t),g(t)\right).$$

Then

$$\begin{aligned}&\|(\ln t_2)^{1-\gamma}(Nu)(t_2)-(\ln t_1)^{1-\gamma}(Nu)(t_1)\|_E\\ &\le (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{|\varphi(g(s))|}{s\Gamma(\alpha)}\mathrm{d}s\\ &\quad+\int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}-(\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\frac{|\varphi(g(s))|}{s\Gamma(\alpha)}\mathrm{d}s\\ &\le (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{p(s)}{s\Gamma(\alpha)}\mathrm{d}s\\ &\quad+\int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}-(\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\frac{p(s)}{s\Gamma(\alpha)}\mathrm{d}s.\end{aligned}$$

Thus, we get

$$\begin{aligned}\|(\ln t_2)^{1-\gamma}(Nu)(t_2) - (\ln t_1)^{1-\gamma}(Nu)(t_1)\|_E &\le \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\left(\ln\frac{t_2}{t_1}\right)^{\alpha} \\ &+ L\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right| \mathrm{d}s.\end{aligned}$$

Hence $N(Q) \subset Q$.

Step 2. N **is weakly-sequentially continuous**

Let $\{u_n\}$ be a sequence in Q and let $u_n(t) \to u(t)$ in (E,ω) for each $t \in I$. Fix $t \in I$. Since f satisfies the assumption (H_1), we have that $f(t, u_n(t), ({}^H D_1^{\alpha,\beta} u_n)(t))$ converges weakly uniformly to $f(t, u(t), (D_0^{\alpha,\beta} u)(t))$. Hence the Lebesgue dominated convergence theorem for Pettis integrals implies that $(Nu_n)(t)$ converges weakly uniformly to $(Nu)(t)$ in (E,ω), for each $t \in I$. Thus, $N(u_n) \to N(u)$. Hence, $N : Q \to Q$ is weakly-sequentially continuous.

Step 3. The implication (2.3) **holds**

Let V be a subset of Q such that $\overline{V} = \overline{\mathrm{conv}}(N(V) \cup \{0\})$. Obviously

$$V(t) \subset \overline{\mathrm{conv}}(NV)(t) \cup \{0\}),\ t \in I.$$

Further, as V is bounded and equicontinuous, by [9, Lemma 3], the function $t \to v(t) = \beta(V(t))$ is continuous on I. From (H_3), (H_4), Lemma 2.19, and the properties of the measure β, for any $t \in I$, we have

$$\begin{aligned}(\ln t)^{1-\gamma}v(t) &\le \beta((\ln t)^{1-\gamma}(NV)(t) \cup \{0\}) \\ &\le \beta((\ln t)^{1-\gamma}(NV)(t)) \\ &\le \frac{(\ln T)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} p(s)\beta(V(s))\frac{\mathrm{d}s}{s} \\ &\le \frac{(\ln T)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} (\ln s)^{1-\gamma}p(s)v(s)\frac{\mathrm{d}s}{s} \\ &\le L\|v\|_{\mathrm{C}}.\end{aligned}$$

Thus

$$\|v\|_{\mathrm{C}} \le L\|v\|_{\mathrm{C}}.$$

From (3.1), we get $\|v\|_{\mathrm{C}} = 0$, that is, $v(t) = \beta(V(t)) = 0$, for each $t \in I$. Then, by [24, Theorem 2], V is weakly relatively compact in C. Applying now Theorem 2.20, we conclude that N has a fixed point which is a weak solution of (1.1). □

4 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

As an application of our results, we consider the problem for Hilfer–Hadamard fractional differential equations of the form

$$\begin{cases} ({}^H D_1^{\frac{1}{2},\frac{1}{2}} u_n)(t) = f_n(t, u_n(t), ({}^H D_1^{\frac{1}{2},\frac{1}{2}} u_n)(t)); \ t \in [1, e], \\ ({}^H I_1^{\frac{1}{4}} u_n)(t)|_{t=1} = (0, 0, \ldots, 0, \ldots), \end{cases} \tag{4.1}$$

where

$$f_n(t, u(t), v(t)) = \frac{ct^2}{1 + \|u_n(t)\|_E + \|v_n(t)\|_E} \frac{u_n(t)}{e^{t+4}}, \quad t \in [1, e]$$

with

$$u = (u_1, u_2, \ldots, u_n, \ldots) \quad \text{and} \quad c := \frac{e^3}{8}\sqrt{\pi}.$$

Set

$$f = (f_1, f_2, \ldots, f_n, \ldots).$$

Clearly, the function f is continuous. For each $u, v \in E$ and $t \in [1, e]$, we have

$$\|f(t, u(t), v(t))\|_E \leq ct^2 \frac{1}{e^{t+4}}.$$

Hence, the hypothesis (H_3) is satisfied with $p^* = ce^{-3}$. We shall show that condition (3.1) holds with $T = e$. Indeed,

$$\frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} = \frac{2ce^{-3}}{\sqrt{\pi}} = \frac{1}{4} < 1.$$

Simple computations show that all conditions of Theorem 3.2 are satisfied. It follows that the problem (4.1) has at least one weak solution defined on $[1, e]$.

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