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Necessary and Sufficient Condition for Asymptotic Behaviour of Solutions of a Class of First-Order Impulsive Systems

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Abstract

In this work, the necessary and sufficient condition for asymptotic behaviour of a class of first-order neutral impulsive systems is established. Here, our impulse satisfies a discrete neutral nonlinear equation of similar type.

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1 Introduction

Consider a class of nonlinear neutral first-order differential equations of the form

$$(y(t) + p(t)y(t - \tau))' + q(t)G(y(t - \sigma)) = 0, \tag{1.1}$$

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where $\tau > 0$, $\sigma \geq 0$ are real constants, $G \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing such that $xG(x) > 0$ for $x \neq 0$, $q, r \in C(\mathbb{R}_+, \mathbb{R}_+)$. Let $\tau_k,\ k \in \mathbb{N}$ be the fixed moments of impulsive effect for (1.1) with the properties $0 < \tau_1 < \tau_2 < ...$, $\lim_{k\to\infty} \tau_k = \infty$ and

$$\Delta(y(\tau_k) + p(\tau_k)y(\tau_k - \tau)) + r(\tau_k)G(y(\tau_k - \sigma)) = 0,\ \ k \in \mathbb{N},$$

where $p \in PC(\mathbb{R}_+, \mathbb{R})$, $p(\tau_k), r(\tau_k)$ are constants for $k \in \mathbb{N}$, Δ is the difference operator defined by

$$\Delta(y(\tau_k) + p(\tau_k)y(\tau_k - \tau)) = y(\tau_k + 0) + p(\tau_k)y(\tau_k - \tau + 0) - y(\tau_k - 0)$$
$$-p(\tau_k)y(\tau_k - \tau - 0),\ \ y(\tau_k - 0) = y(\tau_k)\ \ and\ \ y(\tau_k - \tau - 0) = y(\tau_k - \tau),\ k \in \mathbb{N}.$$

In this work, our objective is to establish the necessary and sufficient condition governing the impulse operators acting on (1.1) which we denote as the impulsive system

$$(E) \qquad \begin{cases} (y(t) + p(t)y(t - \tau))' + q(t)G(y(t - \sigma)) = 0,\ t \neq \tau_k,\ k \in \mathbb{N} \\ \Delta(y(\tau_k) + p(\tau_k)y(\tau_k - \tau)) + r(\tau_k)G(y(\tau_k - \sigma)) = 0,\ \ k \in \mathbb{N} \end{cases}$$

so that its solutions are asymptotically converging to zero for various ranges of $p(t)$.

The motivation of the present work has come from the work of [8]. In [8], Tripathy has studied (E) and established the sufficient conditions for oscillation and nonoscillation of solutions of (E) for any $|p(t)| < \infty$, when G is linear, sublinear and superlinear. In this direction, we refer some related works ([2]– [7], [9], [10]) to the readers and the references cited therein.

Definition 1.1. A function $y : [-\rho, +\infty) \to \mathbb{R}$ is said to be a solution of (E) with initial function $\o \in C([-\rho, 0], \mathbb{R})$, if $y(t) = \o(t)$ for $t \in [-\rho, 0]$, $y \in PC(\mathbb{R}_+, \mathbb{R})$, $z(t) = y(t) + p(t)y(t - \tau)$ is continuously differentiable for $t \in \mathbb{R}_+$, and $y(t)$ satisfies (E) for all sufficiently large $t \geq 0$, where $\rho = \max\{\tau, \sigma\}$ and $PC(\mathbb{R}_+, \mathbb{R})$ is the set of all functions $U : \mathbb{R}_+ \to \mathbb{R}$ which are continuous for $t \in \mathbb{R}_+, t \neq \tau_k, k \in \mathbb{N}$, continuous from the left-side for $t \in \mathbb{R}_+$, and have discontinuity of the first kind at the points $\tau_k \in \mathbb{R}_+, k \in \mathbb{N}$.

Definition 1.2. A nontrivial solution $y(t)$ of (E) is said to be nonoscillatory, if there exists a point $t_0 \geq 0$ such that $y(t)$ has a constant sign for $t \geq t_0$. Otherwise, the solution $y(t)$ is said to be oscillatory.

Definition 1.3. A solution $y(t)$ of (E) is said to be regular, if it is defined on some interval $[T_y, +\infty) \subset [t_0, +\infty)$ and

$$\sup\{|y(t)| : t \geq T_y\} > 0$$

for every $T_y \geq T$. A regular solution $y(t)$ of (E) is said to be eventually positive (eventually negative), if there exists $t_1 > 0$ such that $y(t) > 0$ $(y(t) < 0)$ for $t \geq t_1$.

2 Main Results

In this section, we establish the necessary and sufficient condition for asymptotic behavior of solutions of the impulsive system (E). We need the following lemma for our use in the sequel.

Lemma 2.1 (See [1]). *Let* $p, y, z \in C([0,\infty),\mathbb{R})$ *be such that* $z(t) = y(t)+p(t)y(t-\tau)$, $t \geq \tau > 0$, $y(t) > 0$ *for* $t \geq t_1 > \tau$, $\liminf_{t\to\infty} y(t) = 0$ *and* $\lim_{t\to\infty} z(t) = L$ *exists. Let* p *satisfy any one of following conditions:*

i) $0 \leq a_1 \leq p(t) \leq a_2 < 1$,

ii) $1 < a_3 \leq p(t) \leq a_4 < \infty$,

iii) $-\infty < -a_5 \leq p(t) \leq 0$,

where $a_i > 0$, $1 \leq i \leq 5$. *Then* $L = 0$.

Theorem 2.2. *Let* $0 \leq p(t) \leq a_2 < 1$, $t \in \mathbb{R}_+$. *Let* G *be Lipschitzian on the intervals of the form* $[a, b]$, $0 < a < b < \infty$. *Then every regular solution of* (E) *converges to zero as* $t \to \infty$ *if and only if*

(A_1) $\displaystyle\int_0^\infty q(t)dt + \sum_{k=1}^{\infty} r(\tau_k) = \infty.$

Proof. Suppose that (A_1) holds. Let $y(t)$ be a regular solution of (E) on $[t_y,\infty]$, $t_y \geq 0$. Let the regular solution $y(t) > 0$ for $t \geq t_y$. Setting

$$z(t) = y(t) + p(t)y(t-\tau), t \geq t_0 = t_y + \rho \tag{2.1}$$

in (E), it follows that

$$\begin{aligned} z'(t) &= -q(t)G(y(t-\sigma)) < 0, \\ \triangle z(\tau_k) &= -r(\tau_k)G(y(\tau_k-\sigma)) < 0,\ k \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Hence $z(t)$ is a decreasing function for $t \geq t_1 > t_0$. Since $z(t) > 0$ for $t \geq t_0$, then $\lim_{t\to\infty} z(t)$ exists and thus $\lim_{k\to\infty} z(\tau_k)$ exists. Consequently, $z(t) \geq y(t)$ implies that $y(t)$ is bounded. We claim that $\liminf_{t\to\infty} y(t) = 0$. If not, there exist $t_3 > t_2$ and $\beta > 0$ such that $y(t-\sigma) \geq \beta > 0$ for $t \geq t_3$. Ultimately,

$$\int_{t_3}^{t} q(s)G(y(s-\sigma))ds + \sum_{t_3\leq\tau_k<t} r(\tau_k)G(y(\tau_k-\sigma)) \geq G(\beta)\left[\int_{t_3}^{t} q(s)ds + \sum_{t_3\leq\tau_k<t} r(\tau_k)\right],$$

and this tends to ∞ as $t \to \infty$, due to (A_1). On the other hand, we integrate the system (2.2) from t_3 to $t(> t_3)$ to obtain

$$[z(s)]_{t_3}^{t} + \int_{t_3}^{t} q(s)G(y(s-\sigma))ds - \sum_{t_3 \le \tau_k < t} \Delta z(\tau_k) = 0,$$

that is,

$$\int_{t_3}^{t} q(s)G(y(s-\sigma))ds + \sum_{t_3 \le \tau_k < t} r(\tau_k)G(y(\tau_k - \sigma)) = -[z(s)]_{t_3}^{t}$$
$$< \infty, \quad as \ \ t \to \infty,$$

a contradiction. So, our claim holds. Consequently, $\lim_{t\to\infty} z(t) = 0$ due to Lemma 2.1. As a result,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}(y(t) + p(t)y(t-\tau)) \\ &\ge \limsup_{t\to\infty} y(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} y(t) = 0$, that is, $\lim_{t\to\infty} y(t) = 0$ for $t \neq \tau_k$, $k \in \mathbb{N}$. We may note that $\{y(\tau_k - 0)\}$ and $\{y(\tau_k + 0)\}$ are real valued sequences and the continuity of y will imply that $\lim_{k\to\infty} y(\tau_k - 0) = 0$ and $\lim_{k\to\infty} y(\tau_k + 0) = 0$ due to $\liminf_{t\to\infty} y(t) = 0$ and $\limsup_{t\to\infty} y(t) = 0$ respectively. Hence for all t and τ_k, $\lim_{t\to\infty} y(t) = 0$. The case $y(t) < 0$ for $t \ge t_y$ is similar.

Next, we suppose that (A_1) does not hold. We need to show that the impulsive system admits a nonoscillatory regular solution which does not tend to zero as $t \to \infty$ when the limit exists. If possible, let there exist $t_1 > 0$ such that

$$\int_{t_1}^{\infty} q(s)ds + \sum_{k=1}^{\infty} r(\tau_k) < \frac{1-a_2}{10L},$$

where $L = \max\{L_1, G(1)\}$, L_1 is the Lipschitz constant of G on $\left[\frac{2(1-a_2)}{5}, 1\right]$. For $t_2 > t_1$, we set $X = BC([t_2, \infty), \mathbb{R})$, the space of real valued bounded continuous functions on $[t_2, \infty)$. Clearly, X is a Banach space with respect to sup norm defined by

$$||x|| = \sup\{|x(t)| : t \ge t_2\}.$$

Let us define

$$S = \left\{ u \in X : \ \frac{2(1-a_2)}{5} \le u(t) \le 1, \ t \ge t_2 \right\}.$$

It is easy to see that S is a closed and convex subspace of X. Let $T : S \to S$ be defined by

$$Ty(t) = \begin{cases} Ty(t_2+\rho), \quad t \in [t_2, t_2+\rho] \\ -p(t)y(t-\tau) + \dfrac{2+3a_2}{5} + \displaystyle\int_t^\infty q(s)G(y(s-\sigma))ds \\ + \displaystyle\sum_{t_2 \le \tau_k < t} r(\tau_k)G(y(\tau_k - \sigma)), t \ge t_2 + \rho. \end{cases}$$

For every $y \in S$,

$$\begin{aligned} Ty(t) &\le \frac{2+3a_2}{5} + G(1)\left[\int_t^\infty q(s)ds + \sum_{t_2 \le \tau_k < t} r(\tau_k)\right] \\ &< \frac{2+3a_2}{5} + \frac{1-a_2}{10} = \frac{1+a_2}{2} < 1 \end{aligned}$$

and

$$\begin{aligned} Ty(t) &\ge -p(t)y(t-\tau) + \frac{2+3a_2}{5} \\ &\ge -a_2 + \frac{2+3a_2}{5} = \frac{2(1-a_2)}{5} \end{aligned}$$

implies that $Ty \in S$. Now, for $x_1, x_2 \in S$

$$\begin{aligned} |Tx_1(t) - Tx_2(t)| \le\ & |p(t)||x_1(t-\tau) - x_2(t-\tau)| \\ &+ L_1 \int_t^\infty q(s)|x_1(s-\sigma) - x_2(s-\sigma)|ds \\ &+ L_1 \sum_{t_2 \le \tau_k < t} r(\tau_k)|x_1(\tau_k - \sigma) - x_2(\tau_k - \sigma)|, \end{aligned}$$

that is,

$$\begin{aligned} |Tx_1(t) - Tx_2(t)| &\le a_2||x_1 - x_2|| + L_1||x_1 - x_2||\left[\int_t^\infty q(s)ds + \sum_{t_2 \le \tau_k < t} r(\tau_k)\right] \\ &< \left(a_2 + \frac{1-a_2}{10}\right)||x_1 - x_2|| \end{aligned}$$

implies that

$$||Tx_1 - Tx_2|| \le \mu||x_1 - x_2||,$$

that is, T is a contraction with $\mu = a_2 + \dfrac{1-a_2}{10} = \dfrac{1+9a_2}{10} < 1$. Since S is complete, then by Banach's fixed point theorem T has a unique fixed point on $\left[\dfrac{2(1-a_2)}{5}, 1\right]$.

Hence $Ty = y$ and

$$y(t) = \begin{cases} y(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -p(t)y(t-\tau) + \dfrac{2+3a_2}{5} + \displaystyle\int_t^\infty q(s)G(y(s-\sigma))ds \\ + \displaystyle\sum_{t_2 \le \tau_k < t} r(\tau_k)G(y(\tau_k - \sigma)), t \ge t_2+\rho \end{cases}$$

is a regular nonoscillatory solution of the system (E) on $\left[\dfrac{2(1-a_2)}{5}, 1\right]$ such that $\lim_{t\to\infty} y(t) \neq 0$. Therefore, (A_1) is necessary. This completes the proof. □

Theorem 2.3. *Let $1 < a_3 \le p(t) \le a_4 < \infty$, $t \in \mathbb{R}_+$ and $a_3^2 > a_4$. Suppose that G is Lipschitzian on the intervals of the form $[a,b]$, $0 < a < b < \infty$. Then every solution of the system (E) converges to zero as $t \to \infty$ if and only if (A_1) holds.*

Proof. Sufficient part is same as in the proof of Theorem 2.2. For the necessary part, we suppose that (A_1) does not hold. It is possible to find a $t_1 > 0$ such that

$$\int_{t_1}^\infty q(t)dt + \sum_{k=1}^\infty r(\tau_k) < \frac{a_3-1}{2L},$$

where $L = \max\{L_1, L_2\}$, L_1 is the Lipschitz constant of G on $[a,b]$ and $L_2 = G(b)$ such that

$$a = \frac{2c({a_3}^2 - a_4) - a_4(a_3-1)}{2{a_3}^2 a_4}$$

$$b = \frac{a_3 - 1 + 2c}{2a_3}, \qquad c > \frac{a_4(a_3-1)}{2({a_3}^2 - a_4)} > 0.$$

Let $X = BC([t_2,\infty),\mathbb{R})$ be the space of real valued bounded continuous functions on $[t_2,\infty)$. Clearly, X is a Banach space with respect to sup norm defined by

$$||x|| = \sup\{|x(t)| : t \ge t_2\}.$$

Define

$$S = \{u \in X : \ a \le u(t) \le b, \ t \ge t_2\}.$$

It is easy to verify that S is a closed convex subspace of X. Let $T : S \to S$ be such that

$$Ty(t) = \begin{cases} Ty(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -\dfrac{y(t+\tau)}{p(t+\tau)} + \dfrac{c}{p(t+\tau)} + \dfrac{1}{p(t+\tau)}[\displaystyle\int_{t+\tau}^\infty q(s)G(y(s-\sigma))ds \\ + \displaystyle\sum_{t_2 \le \tau_k < t+\tau} r(\tau_k)G(y(\tau_k-\sigma))], & t \ge t_2+\rho. \end{cases}$$

For every $y \in S$,

$$Ty(t) \leq \frac{G(b)}{p(t+\tau)}\left[\int_{t+\tau}^{\infty} q(s)ds + \sum_{t_2 \leq \tau_k < t+\tau} r(\tau_k)\right] + \frac{c}{p(t+\tau)}$$
$$\leq \frac{1}{a_3}\left[\frac{a_3-1}{2} + c\right] = b$$

and

$$Ty(t) \geq -\frac{y(t+\tau)}{p(t+\tau)} + \frac{c}{p(t+\tau)} > -\frac{b}{a_3} + \frac{c}{a_4}$$
$$= -\frac{a_3 - 1 + 2c}{2a_3^2} + \frac{c}{a_4} = \frac{2c({a_3}^2 - a_4) - a_4(a_3-1)}{2{a_3}^2 a_4} = a$$

implies that $Ty \in S$. For $x_1, x_2 \in S$

$$|Tx_1(t) - Tx_2(t)| \leq \frac{1}{|p(t+\tau)|}|x_1(t+\tau) - x_2(t+\tau)|$$
$$+ \frac{L_1}{|p(t+\tau)|}[\int_{t+\tau}^{\infty} q(s)|x_1(s-\sigma) - x_2(s-\sigma)|ds$$
$$+ \sum_{t_2 \leq \tau_k < t+\tau} r(\tau_k)|x_1(\tau_k - \sigma) - x_2(\tau_k - \sigma)|],$$

that is,

$$|Tx_1(t) - Tx_2(t)| \leq \frac{1}{a_3}||x_1 - x_2|| + \frac{L_1}{a_3}||x_1 - x_2||\left[\int_{t+\tau}^{\infty} q(s)ds + \sum_{t_2 \leq \tau_k < t+\tau} r(\tau_k)\right]$$
$$< \left(\frac{1}{a_3} + \frac{a_3 - 1}{2a_3}\right)||x_1 - x_2||$$

implies that

$$||Tx_1 - Tx_2|| \leq \mu||x_1 - x_2||,$$

that is, T is a contraction with $\mu = \left(\frac{1}{a_3} + \frac{a_3-1}{2a_3}\right) < 1$. Hence by Banach's fixed point theorem T has a unique fixed point which is a regular nonoscillatory solution of the system (E) on $[a, b]$. Thus the proof is complete. □

Theorem 2.4. *Let* $-1 < -a_5 \leq p(t) \leq 0$, $t \in \mathbb{R}_+$, $a_5 > 0$. *Then every solution of the system* (E) *converges to zero as* $t \to \infty$ *if and only if* (A_1) *holds.*

Proof. Proceeding as in the proof of Theorem 2.2, we have (2.2). Hence, $z(t)$ is monotonic on $[t_2, \infty)$, $t_2 > t_1$. Let $z(t) > 0$ for $t \geq t_2$. Then $\lim_{t\to\infty} z(t)$ exists and thus

$\lim_{k\to\infty} z(\tau_k)$ exists. If $z(t) < 0$ for $t \geq t_2$, then we claim that $y(t)$ is bounded. Otherwise, there exists $\{\eta_n\}$ such that $\eta_n \to \infty, y(\eta_n) \to \infty$ as $n \to \infty$ and

$$y(\eta_n) = \max\{y(s) : t_2 \leq s \leq \eta_n\}.$$

Therefore,

$$\begin{aligned} z(\eta_n) &= y(\eta_n) + p(\eta_n)y(\eta_n - \tau) \\ &\geq (1 - a_5)y(\eta_n) \to +\infty, \quad as \ \ n \to \infty, \end{aligned}$$

a contradiction to the fact that $z(t) > 0$. So, our claim holds. Consequently, $\lim_{t\to\infty} z(t)$ exists and $\lim_{k\to\infty} z(\tau_k)$ exists. Hence for any $z(t)$, $y(t)$ is bounded. Using the same type of argument as in the proof of Theorem 2.2, it is easy to show that $\liminf_{t\to\infty} y(t) = 0$ and by Lemma 2.1, $\lim_{t\to\infty} z(t) = 0$. Indeed,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}(y(t) + p(t)y(t - \tau)) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(-a_5 y(t - \tau)) \\ &= (1 - a_5) \limsup_{t\to\infty} y(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} y(t) = 0$. The rest of the proof follows from Theorem 2.2.

Next, we suppose that (A_1) does not hold. Let there exists $t_1 > 0$ such that

$$\int_{t_1}^{\infty} q(t)dt + \sum_{k=1}^{\infty} r(\tau_k) < \frac{1 - a_5}{5G(1)}.$$

For $t_2 > t_1$, let $X = BC([t_2, \infty), \mathbb{R})$ be the space of all real valued bounded continuous functions defined on $[t_2, \infty)$. Clearly, X is a Banach space with respect to sup norm defined by

$$||x|| = \sup\{|x(t)| : t \geq t_2\}.$$

Let $K = \{x \in X : x(t) \geq 0, \ t \geq t_2\}$. Thus X is a partially ordered Banach space (see for e.g [1], p. 30). For $u, v \in X$, we define $u \leq v$ if and only if $u - v \in K$. Let

$$S = \left\{ y \in X : \ \frac{1 - a_5}{5} \leq y(t) \leq 1, \ t \geq t_2 \right\}.$$

If $y_0(t) = \dfrac{1 - a_5}{5}$, then $y_0 \in S$ and $y_0 =$ g.l.b S. Further, if $\phi \subset S^* \subset S$, then

$$S^* = \left\{ y \in X : \ l_1 \leq y(t) \leq l_2, \ \frac{1 - a_5}{5} \leq l_1, \ l_2 \leq 1 \right\}.$$

Let $v_0(t) = l_2'$ for $t \geq t_3 > t_2$ and $l_2' = \sup\{l_2 : \frac{1-a_5}{5} \leq l_2 \leq 1\}$. Then $v_0 \in S$ and $v_0 =$ l.u.b S^*. Define $T : S \to S$ by

$$Ty(t) = \begin{cases} Ty(t_3), \quad t \in [t_2, t_3] \\ -p(t)y(t-\tau) + \frac{1-a_5}{5} + \int_t^\infty q(s)G(y(s-\sigma))ds \\ + \sum_{t_3 \leq \tau_k < t} r(\tau_k)G(y(\tau_k - \sigma)), t \geq t_3. \end{cases}$$

For every $y \in S, Ty(t) \geq \frac{1-a_5}{5}$ and

$$\begin{aligned} Ty(t) &\leq a_5 + \frac{1-a_5}{5} + G(1)\left[\int_t^\infty q(s)ds + \sum_{t_3 \leq \tau_k < t} r(\tau_k)\right] \\ &< \frac{2+3a_5}{5} < 1 \end{aligned}$$

implies that $Ty \in S$. For $y_1, y_2 \in S$, it is easy to verify that $y_1 \leq y_2$ implies that $Ty_1 \leq Ty_2$. Hence by Knaster–Tarski fixed point theorem (see [1, Theorem 1.7.3]), T has a unique fixed point such that $\lim_{t\to\infty} y(t) \neq 0$. This completes the proof. □

Theorem 2.5. *Let* $-\infty < -a_6 \leq p(t) \leq -a_7 < -1$, $t \in \mathbb{R}_+$, $a_6, a_7 > 0$. *Let* G *be Lipschitzian on the intervals of the form* $[a, b]$, $0 < a < b < \infty$. *Then every bounded solutions of* (E) *converges to zero as* $t \to \infty$ *if and only if* (A_1) *holds.*

Proof. The proof of the theorem follows from the proof of the Theorem 2.4. For the necessary part, we mention the following:

$$\int_{t_1}^\infty q(t)dt + \sum_{k=1}^\infty r(\tau_k) < \frac{a_7 - 1}{2L},$$

where $L = \max\{L_1, L_2\}$, L_1 is the Lipschitz constant of G on $[a, b]$, $L_2 = G(b)$ such that

$$a = \frac{2ca_7 - a_6(a_7-1)}{2a_6a_7}, \quad b = \frac{c}{a_7 - 1}, \quad c > \frac{a_6(a_7-1)}{2a_7} > 0,$$

and

$$Ty(t) = \begin{cases} Ty(t_2+\rho), \quad t \in [t_2, t_2+\rho] \\ -\frac{y(t+\tau)}{p(t+\tau)} - \frac{c}{p(t+\tau)} + \frac{1}{p(t+\tau)}[\int_{t+\tau}^\infty q(s)G(y(s-\sigma))ds \\ + \sum_{t_2 \leq \tau_k < t+\tau} r(\tau_k)G(y(\tau_k - \sigma))], \quad t \geq t_2 + \rho. \end{cases}$$

The rest of the proof follows from Theorem 2.3. This completes the proof. □

Remark 2.6. In the above theorems, G could be linear, sublinear or superlinear.

Remark 2.7. Lemma 2.1 does not include $p(t) \equiv 1$, for all t (see for e.g [1]). The present analysis does not allow the case $p(t) \equiv -1$, for all t. Hence in our discussion, a necessary and sufficient condition is established except $p(t) = \pm 1$. It seems that a different approach is necessary to study the case $p(t) = \pm 1$.

We conclude this section with the following example to show feasibility and effectiveness of our main results.

Example 2.8. Consider the impulsive system

$$(E_1) \qquad \begin{cases} (y(t) + e^{t-1}y(t-1))' + e^{2t-6}y^3(t-2) = 0, \;\; t \neq \tau_k, \; t > 2, \; k \in \mathbb{N} \\ \Delta(y(\tau_k) + e^{k-1}y(\tau_k - 1)) + r(\tau_k)y^3(\tau_k - 2)) = 0, \; k \in \mathbb{N}, \end{cases}$$

where $r(\tau_k) = \dfrac{e^h(e^{-k} - e^{-(k+1)} + 1 - e^{-1})}{e^{-3(k-2)}}$, $\Delta y(\tau_k) = y(\tau_k + h) - y(\tau_k - h)$, $\tau_k = k \in \mathbb{N}$ and $G(u) = u^3$. Clearly, all conditions of Theorem 2.3 hold true for (E_1). Therefore, every solution of (E_1) converges to zero as $t \to \infty$.

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