Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 11, Number 2, pp. 81–92 (2016)
http://campus.mst.edu/adsa

On Algebraic Properties of the Discrete Raising and Lowering Operators, Associated with the N-dimensional Discrete Fourier Transform

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Abstract

By using the algebraic technique, developed by J. Schwinger, we employ probably the simplest approach to deriving the fundamental symmetry property of the eigenvectors of the N-dimensional discrete (finite) Fourier transform. This enables us to give a precise algebraic interpretation to the raising and lowering difference operators, which are building blocks for the self-adjoint discrete number operator that governs those eigenvectors. Thus from the algebraic point of view the lowering and raising operators, and, consequently, the discrete number operator, turn out to be elements of the *group algebra* over the group of unitary transformations of the N-dimensional complex vector space, spanned by the eigenvectors of the N-dimensional discrete Fourier transform.

AMS Subject Classifications: 39A10, 39A12, 42A38.
Keywords: Discrete Fourier Transform, Raising and Lowering Operators, Schwinger, Group Algebra.

Received November 11, 2016; Accepted November 27, 2016
Communicated by Sandra Pinelas

1 Introduction

Recently, we have proposed in [1] a strategy for finding explicit forms of eigenvectors of the discrete (finite) Fourier transform (DFT) operator $\mathbf{\Phi}^{(N)}$ by constructing a self-adjoint discrete number operator $\mathcal{N}^{(N)} = \mathbf{b}_N^{\dagger}\,\mathbf{b}_N$ in terms of the lowering and raising difference operators $\mathbf{b}_N$ and $\mathbf{b}_N^{\dagger}$, which are defined by the intertwinning relations

$$\mathbf{b}_N\,\mathbf{\Phi}^{(N)} = \mathrm{i}\,\mathbf{\Phi}^{(N)}\,\mathbf{b}_N\,, \qquad \mathbf{\Phi}^{(N)}\,\mathbf{b}_N^{\dagger} = \mathrm{i}\,\mathbf{b}_N^{\dagger}\,\mathbf{\Phi}^{(N)}\,. \tag{1.1}$$

Since thus introduced discrete number operator $\mathcal{N}^{(N)}$ commutes with the DFT operator $\mathbf{\Phi}^{(N)}$, the ability to solve a difference eigenvalue problem

$$\mathcal{N}^{(N)}\,\vec{f}_n^{(N)} = \lambda_n^{(N)}\,\vec{f}_n^{(N)}\,, \qquad 0 \le n \le N-1\,, \tag{1.2}$$

then enables one to define an analytical form of desired set of eigenvectors for the latter operator $\mathbf{\Phi}^{(N)}$.

This idea of making use an analogy with the continuous case of the classical Fourier transform integral operator $\mathcal{F}$ was then studied in detail in [2] for the case of the 5D DFT operator $\mathbf{\Phi}^{(5)}$. In particular, it was confirmed that the eigenvalues of the 5D discrete number operator $\mathcal{N}^{(5)}$ are represented by distinct nonnegative numbers $\lambda_n^{(5)}$, $0 \le n \le 4$, so that this operator $\mathcal{N}^{(5)}$ can be used to systematically classify eigenvectors of the 5D DFT operator $\mathbf{\Phi}^{(5)}$. Hence, despite the fact that the 5D lowering and raising operators $\mathbf{b}_N$ and $\mathbf{b}_N^{\dagger}$ turn out to be more complicated difference operators than the first-order differential ones $\mathbf{a} = \dfrac{1}{\sqrt{2}}\left(x + \dfrac{d}{dx}\right)$ and $\mathbf{a}^{\dagger} = \dfrac{1}{\sqrt{2}}\left(x - \dfrac{d}{dx}\right)$ in the continuous case [8], the 5D number operator $\mathcal{N}^{(5)}$ is the appropriate discrete version of the quantum number operator $\mathbf{N} = \mathbf{a}^{\dagger}\,\mathbf{a}$ and it does resolve the ambiguity caused by the well-known degeneracy of the eigenvalues of the DFT $\mathbf{\Phi}^{(5)}$.

It is the main goal of this short paper to establish how the lowering and raising difference operators $\mathbf{b}_N$ and $\mathbf{b}_N^{\dagger}$, defined by the intertwinning relations (1.1), are interconnected with those operators of unitary transformations of N-dimensional complex vector space $\mathbb{C}^N$, which is spanned by the ND DFT eigenvectors $\vec{f}_n^{(N)}$. Although the group-theoretic interpretation of the eigenvectors $\vec{f}_n^{(N)}$ has long been clarified (see, for example, [3, 4]), we find it more appropriate to employ a different, simpler approach to describing those well-known symmetry properties of $\vec{f}_n^{(N)}$. It turns out that one can effectively use J. Schwinger's technique [11–15] for finding an explicit form of all the unitary transformations of $\mathbb{C}^N$, built over the eigenvectors $\vec{f}_n^{(N)}$. The main reason for choosing J. Schwinger's approach is that it actually provides us with the desired interpretation of the lowering and raising operators $\mathbf{b}_N$ and $\mathbf{b}_N^{\dagger}$ as elements of the *group algebra* $\mathbb{C}\,[\mathcal{U}(N)]$ over the group $\mathcal{U}(N)$ of unitary transformations of the N-dimensional complex vector space $\mathbb{C}^N$, associated with the eigenvectors $\vec{f}_n^{(N)}$.

2 Schwinger's Unitary Operator Basis

Recall that the eigenvectors $\vec{f}_n^{(N)}$ of the N-dimensional discrete Fourier transform are defined on the finite set of points $x_n = n,\ 0 \le n \le N-1$, on the real line $\mathbb{R}$. It was Gauss' idea to associate this set of points $\{0, 1, ...N-1\}$ with the quotient ring $\mathbb{Z}/N\mathbb{Z}$ whose elements are equivalence classes one gets upon identifying any two integers from $\mathbb{Z}$ to be the same if they are congruent modulo N (see, for example, [16]). Thus the domain of definition of the eigenvectors $\vec{f}_n^{(N)}$ is commonly interpreted as the cyclic group $\mathbb{Z}/N\mathbb{Z} = \{q^n \,|\, 0 \le n \le N-1,\, q^N = 1\}$, where the element $q \in \mathbb{Z}/N\mathbb{Z}$ is of order N. As for the range of the ND DFT eigenvectors $\vec{f}_n^{(N)}$, they span N-dimensional complex vector space $\mathbb{C}^N$ with the standard real basis vectors $\vec{e}_n$, whose components are of the form $(\vec{e}_n)_k = \delta_{nk},\ 0 \le n, k \le N-1$. As usual, one defines Hermitian structure on this N-dimensional vector space over $\mathbb{C}$ with elements $\vec{z} = \{z_0, z_1, ...z_{N-1}\}$ by

$$\Big(\vec{z}_1, \vec{z}_2\Big) = \sum_{k=0}^{N-1} (\vec{z}_1)^*_k\, (\vec{z}_2)_k, \tag{2.1}$$

where $\vec{z}_1, \vec{z}_2 \in \mathbb{C}^N$ and the symbol $*$ denotes complex conjugate. Then those linear transformations $\mathbf{U}$ of $\mathbb{C}^N$, which preserve the Hermitian structure (2.1) on $\mathbb{C}^N$,

$$\Big(\mathbf{U}\vec{z}_1, \mathbf{U}\vec{z}_2\Big) = (\vec{z}_1, \vec{z}_2), \tag{2.2}$$

are called *unitary transformations* of $\mathbb{C}^N$. The set of all unitary transformations of $\mathbb{C}^N$ is known to form a group $\mathcal{U}(N)$, called the *group of unitary transformations* of $\mathbb{C}^N$. To construct an explicit form of this group it is the most convenient to employ J. Schwinger's technique for studying algebraic properties of unitary transformations by building appropriate *unitary operator bases*. The main idea of Schwinger's approach, developed in [11–15], is that it is sufficient to choose a complementary pair of unitary operators on $\mathbb{C}^N$ because they generate a complete orthonormal basis and therefore together supply the foundation for a full description of the complex vector space $\mathbb{C}^N$, associated in the case under study with the ND DFT eigenvectors $\vec{f}_n^{(N)}$.

In order to proceed to this task under consideration, let us start by emphasizing that similar to the continuous case, associated with the Heisenberg–Weyl group [10], it is convenient to single out first the *center* $\mathcal{U}_c(N)$ of the group $\mathcal{U}(N)$ with N elements of the form $u_k := q^k\mathbf{I}$, $k = 0, 1, ..., N-1$, which represents the *normal subgroup* of $\mathcal{U}(N)$, $\mathcal{U}_c(N) \subset \mathcal{U}(N)$. Then to find out an explicit form of remaining elements of the group $\mathcal{U}(N)$, that do not have such simple structures as elements of the center $\mathcal{U}_c(N)$, it will be sufficient to concentrate our efforts on the *quotient group* $\mathcal{U}(N)/\mathcal{U}_c(N)$ of $\mathcal{U}(N)$ *by the normal subgroup* $\mathcal{U}_c(N)$.[1]

[1]Recall also that the map taking an element $u \in \mathcal{U}(N)$ to the coset $u\mathcal{U}_c(N)$ is called the *canonical homomorphism* [7].

We are now in a position to illustrate how to use the J. Schwinger method for constructing explicitly all the unitary transformations of $\mathbb{C}^N$, built over the eigenvectors $\vec{f}_n^{(N)}$. Define a unitary operator $\mathbf{V}$ on $\mathbb{C}^N$ by *cyclic permutation*

$$\mathbf{V}\,\vec{e}_n \;=\; \vec{e}_{n+1}, \qquad \vec{e}_N = \vec{e}_0\,, \tag{2.3}$$

which represents a mapping of the orthonormal system $\{\vec{e}_n\}_{n=0}^{N-1}$ onto itself. The matrix form of (2.3) is $V_{k,l}\,(\vec{e}_n)_l \;=\; (\vec{e}_{n+1})_k$, from which it follows that the operator $\mathbf{V}$ in the $\vec{e}_n$-basis is associated with the $(N \times N)$-matrix

$$\Big(V_{k,l}\Big) = \Big(\delta_{k,l+1}\Big) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{2.4}$$

where $\delta_{k,N} \;=\; \delta_{k,0}$. The repetition of $\mathbf{V}$ defines linearly independent unitary operators, $\mathbf{V}^m\,\vec{e}_n \;=\; \vec{e}_{n+m}$, until we arrive at

$$\mathbf{V}^N\,\vec{e}_n \;=\; \vec{e}_{n+N} \;=\; \vec{e}_n\,; \tag{2.5}$$

thus the operator $\mathbf{V}$ is a cyclic operator of period N, $\mathbf{V}^N \;=\; I$. The eigenvalues of $\mathbf{V}$ obey the same equation and they are given by the N distinct complex numbers q^k, $q := \exp(2\pi\mathrm{i}/N)$, $0 \le k \le N-1$. Indeed, from the definition $\mathbf{V}\,\vec{v}_n = \lambda_n \vec{v}_n$ it follows at once that $\mathbf{V}^N\,\vec{v}_n \;=\; (\lambda_n)^N \vec{v}_n \;=\; \vec{v}_n$; thus $(\lambda_n)^N \;=\; 1$ and, consequently, $\lambda_n \;=\; q^n$. Now from the matrix form $V_{k,l}\,(\vec{v}_n)_l = q^n(\vec{v}_n)_k$ one concludes that components of the eigenvectors $\vec{v}_n$ of the operator $\mathbf{V}$ are interconnected by the relations $q^n(\vec{v}_n)_k = (\vec{v}_n)_{k-1}$, $0 \le k \le N-1$, $(\vec{v}_n)_{-1} = (\vec{v}_n)_{N-1}$. This means that these eigenvectors are explicitly given as

$$\vec{v}_n = (\vec{v}_n)_0 \left\{1, q^{(N-1)n}, q^{(N-2)n}, ..., q^n\right\}^T,$$

so that the orthonormalized with respect to the Hermitian structure (2.1) eigenvectors $\vec{v}_n$ of the unitary operator $\mathbf{V}$ have the following form

$$\vec{v}_n = \frac{1}{\sqrt{N}} \left\{1, q^{(N-1)n}, q^{(N-2)n}, ..., q^n\right\}^T. \tag{2.6}$$

Observe that the matrix elements of the ND DFT operator $\mathbf{\Phi}^{(N)}$ can be expressed in terms of the eigenvectors (2.6) of the unitary operator $\mathbf{V}$ as

$$\Phi^{(N)}_{m,n} := \frac{1}{\sqrt{N}}\,q^{mn} = \Big(\vec{v}^{\,*}_m\Big)_n. \tag{2.7}$$

At the next step one similarly defines the second unitary operator $\mathbf{U}$, but this time by cyclic permutation

$$\mathbf{U}\,\vec{v}_n \;=\; \vec{v}_{n-1}, \qquad \vec{v}_{-1} = \vec{v}_{N-1}\,, \tag{2.8}$$

of the eigenvectors $\vec{v}_n$ of the first unitary operator $\mathbf{V}$.

As in the first case, it is not hard to show that thus defined operator $\mathbf{U}$ is represented by a diagonal matrix of the form $\mathbf{U} = diag\,\{1, q, q^2, ..., q^{N-1}\}$, that is, it is a cyclic operator of period N, $\mathbf{U}^N = I$. Therefore from the definition $\mathbf{U}\,\vec{u}_n = \lambda_n\,\vec{u}_n$ it follows that $(\lambda_n)^N = 1$, or $\lambda_n = q^n$. Now from the matrix form $U_{k,l}\,(\vec{u}_n)_l = q^n(\vec{u}_n)_k$ one concludes that components of the eigenvectors of the unitary operator $\mathbf{U}$ are interconnected by the relations $q^k\delta_{kl}(\vec{u}_n)_l = q^n(\vec{u}_n)_k$. This means that $(\vec{u}_n)_k = \delta_{nk}$ and therefore $\vec{u}_n = \vec{e}_n$; thus the key point of Schwinger's method is that the eigenvectors of the operator $\mathbf{U}$ are the same vectors $\vec{e}_n$, from which the construction of $\mathbf{V}$ and $\mathbf{U}$ started,

$$\mathbf{U}\,\vec{e}_n \;=\; q^n\,\vec{e}_n\,. \tag{2.9}$$

Observe that from (2.3) and (2.9) it is clear now that the operators $\mathbf{V}$ and $\mathbf{U}$ satisfy Weyl commutation relations

$$\mathbf{U}\,\mathbf{V} \;=\; q\,\mathbf{V}\,\mathbf{U} \tag{2.10}$$

and as a consequence of the latter result, one also has

$$\mathbf{U}^k\,\mathbf{V}^l \;=\; q^{kl}\,\mathbf{V}^l\,\mathbf{U}^k\,. \tag{2.11}$$

Also, the first operator $\mathbf{V}$ is actually *unitarily similar* to the second operator $\mathbf{U}$,

$$\mathbf{U} \;=\; \mathbf{\Phi}^{(N)}\,\mathbf{V}\left(\mathbf{\Phi}^{(N)}\right)^{\dagger}, \tag{2.12}$$

with the transforming matrix, associated with the ND DFT operator $\mathbf{\Phi}^{(N)}$.[2]

J. Schwinger proved in [13] that the two unitary operators $\mathbf{V}$ and $\mathbf{U}$ are the generators of a complete orthonormal basis in the quotient group $\mathcal{U}(N)/\mathcal{U}_c(N)$ of $\mathcal{U}(N)$, such as the set of N^2 operators

$$\mathbf{X}(m,n) \;=\; \frac{1}{\sqrt{N}}\,\mathbf{V}^m\,\mathbf{U}^n\,, \qquad 0 \le m, n \le N-1\,, \tag{2.13}$$

and therefore together $\mathbf{V}$ and $\mathbf{U}$ supply a complementary pair of unitary operators. Moreover, J. Schwinger emphasizes that "operators having the algebraic properties of $\mathbf{V}$ and $\mathbf{U}$ have long been known from the work of H.Weyl [17], but what has been lacking is an appreciation of these operators as generators of a complete operator basis for any N, and of their incompatibility, as summarized in the attribute of complementarity. Nor has it been clearly recognized that an *a priori* classification of all possible types of physical degrees of freedom emerges from these considerations"(see third footnote on p.579 in [13]).

[2] It is interesting to notice that J. Schwinger never mentions the discrete Fourier transform in his papers [11–15], although some of formulas in them do explicitly contain the kernel of the DFT. Moreover, the same happens in the much later publications [5, 6], where the authors employ Schwinger's technique in order to analyze discrete quantum phase spaces and to extend Weyl–Wigner transformation to those particular degrees of freedom, which are described by a finite number of states.

3 Main Results

What does directly follow from J. Schwinger's approach (but was not mentioned in [13]) is that one can readily construct an explicit form of all elements of the group of unitary transformations of $\mathbb{C}^N$. The point is that the product of two elements from the set $\mathbf{X}(m,n)$ is equal to

$$\mathbf{X}(m,n)\,\mathbf{X}(m',n') = \frac{1}{N}\,\mathbf{V}^m\,\underbrace{\mathbf{U}^n\,\mathbf{V}^{m'}}\,\mathbf{U}^{n'}$$

$$= \frac{q^{nm'}}{N}\,\mathbf{V}^{m+m'}\mathbf{U}^{n+n'} = \frac{q^{nm'}}{\sqrt{N}}\,\mathbf{X}(m+m',n+n')\,, \tag{3.1}$$

where we have used identity (2.11) for the underbraced factor $\mathbf{U}^n\,\mathbf{V}^{m'}$.

Theorem 3.1. *A set of the N^3 unitary operators, defined as*

$$\mathfrak{u}(l;m,n) := \sqrt{N}\,q^l\,\mathbf{X}(n,m) = \,q^l\,\mathbf{V}^n\,\mathbf{U}^m\,, \qquad 0 \le l,m,n \le N-1\,, \tag{3.2}$$

form an irreducible unitary representation $\mathcal{U}(N)$ on $\mathbb{C}^N$ of the finite Heisenberg group $\mathcal{H}_N$, which is defined as the set of elements

$$w(m,n;k) := \begin{pmatrix} 1 & m & n \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, \qquad 0 \le m,n,k \le N-1\,, \tag{3.3}$$

with matrix multiplication (mod N):

$$w(m,n;k)\,w(m',n';k') \,=\, w(m+m',n+n';k+k'+mn')\,.$$

Proof. From (3.1) it follows at once that the group multiplication rule for any two unitary matrices of the type (3.2) is

$$\mathfrak{u}(l_1;m_1,n_1)\,\mathfrak{u}(l_2;m_2,n_2) \,=\, \mathfrak{u}(l_1+l_2+m_1n_2;\,m_1+m_2,\,n_1+n_2)\,. \tag{3.4}$$

The irreducibility of the representation $\mathcal{U}(N)$ is not hard to prove directly by using the standard technique of character theory of finite groups. Indeed, recall first that any representation (π,V) of a group G is determined by its character $\chi(g) := \mathrm{trace}\Big(\pi(g)\Big)$ and this representation is irreducible if and only if its character χ has the property

$$\langle\,\chi,\chi\,\rangle := \frac{1}{|\,G\,|}\sum_{g\in G}\chi(g)^*\chi(g) = 1\,, \tag{3.5}$$

where $|\,G\,|$ is the order of G. These two assertions are usually formulated as corollaries to the main theorem on the theory of characters for finite groups (see, for example, pages 11 and 12 in [7]). So to verify that the property (3.5) is valid for the character χ

of the representation $\mathcal{U}(N)$, let us evaluate characters $\chi(l;m,n) := \mathrm{trace}\Big(\mathfrak{u}(l;m,n)\Big)$ of all the elements $\mathfrak{u}(l;m,n) \in \mathcal{U}(N)$. Since the matrix form of the operators $\mathbf{V}$ and $\mathbf{U}$ is by definition $(\mathbf{V})_{kl} = \delta_{k,l+1}$ and $(\mathbf{U})_{kl} = q^k \delta_{k,l}$, respectively, one readily evaluates by induction that $(\mathbf{V}^n)_{kl} = \delta_{k,l+n}$ and $(\mathbf{U}^m)_{kl} = q^{mk}\delta_{k,l}$, where $0 \le n, m \le N-1$. Therefore the matrix form of an element $\mathfrak{u}(l;m,n)$, defined by the formula (3.2), is described by

$$\mathfrak{u}_{ij}(l;m,n) \; = \; q^{l+mj}\, \delta_{i,j+n}\,. \tag{3.6}$$

Consequently, from (3.6) it follows at once that

$$\chi(l;m,n) = \sum_{i=0}^{N-1} \mathfrak{u}_{ii}(l;m,n) = \sum_{i=0}^{N-1} q^{l+mi}\, \delta_{i,i+n} = N\, q^l\, \delta_{m,0}\, \delta_{n,0}\,. \tag{3.7}$$

Now taking into account that the order $|\,\mathcal{U}(N)\,| = N^3$, one thus concludes that

$$\langle\, \chi(l;m,n), \chi(l;m,n)\,\rangle = 1\,. \tag{3.8}$$

This completes the proof of the irreducibility of the unitary representation $\mathcal{U}(N)$ on the complex vector space $\mathbb{C}^N$, spanned by the ND DFT eigenvectors $\vec{f}_n^{(N)}$. □

Remark 3.2. In the case when N is not a prime number, the Schwinger's approach enables one to derive a more detailed description of the complete orthonormal basis (2.13) and, consequently, of the elements (3.2) of the unitary group $\mathcal{U}(N)$ by replacing the single pair of complementary operators $\mathbf{V}$, $\mathbf{U}$ by several such pairs, the individual members of which have smaller periods than $\mathbf{V}$ and $\mathbf{U}$. For example, let us assume first that $N = N_1 N_2$, where N_1 and N_2 are relatively prime. So in this case one defines

$$\mathbf{V}_1 = \mathbf{V}^{l_1 N_2}, \quad \mathbf{U}_1 = \mathbf{U}^{N_2}; \qquad\qquad \mathbf{V}_2 = \mathbf{V}^{l_2 N_1}, \quad \mathbf{U}_2 = \mathbf{U}^{N_1}, \tag{3.9}$$

with $l_1 = {N_2}^{\phi(N_1)-1}$ (*mod* N_1), $l_2 = {N_1}^{\phi(N_2)-1}$ (*mod* N_2), and $\phi(N)$ being the number of integers less than and relatively prime to N. It is clear that $\mathbf{V}_1, \mathbf{U}_1$ are of period N_1, while $\mathbf{V}_2, \mathbf{U}_2$ have the period N_2, and that the two pairs of operators are mutually commutative. Furthermore,

$$\mathbf{V}_1\mathbf{U}_1 = e^{\frac{2\pi\mathrm{i}}{N_1}}\, \mathbf{U}_1\mathbf{V}_1, \qquad \mathbf{V}_2\mathbf{U}_2 = e^{\frac{2\pi\mathrm{i}}{N_2}}\, \mathbf{U}_2\mathbf{V}_2, \tag{3.10}$$

so that $\mathbf{V}_1, \mathbf{U}_1$ and $\mathbf{V}_2, \mathbf{U}_2$ form two independent pairs of complementary operators associated with the respective periods N_1 and N_2. Thus the set of N^2 basis operators for $N = N_1 N_2$ can be represented as

$$\mathbf{X}(m_1, m_2; n_1, n_2) = \prod_{j=1}^{2} \mathbf{X}(m_j, n_j)\,, \qquad \mathbf{X}(m_j, n_j) = N_j^{-1/2}\, \mathbf{V}_j^{m_j}\, \mathbf{U}_j^{n_j}\,, \tag{3.11}$$

where $0 \le m_j, n_j \le N_j - 1$, $j = 1, 2$. Now it is not hard to verify, by using identities (3.9) and (3.10), that

$$\mathbf{X}(m_1, m_2; n_1, n_2)\, \mathbf{X}(m'_1, m'_2; n'_1, n'_2)$$

$$= \frac{1}{\sqrt{N}}\, q^{\,n_1 m'_1 N_2 + n_2 m'_2 N_1}\, \mathbf{X}(m_1 + m'_1, m_2 + m'_2; n_1 + n'_1, n_2 + n'_2)\,. \tag{3.12}$$

So if one defines

$$\mathfrak{u}(l; m_1, m_2; n_1, n_2) := \sqrt{N}\, q^l\, \mathbf{X}(n_1, n_2; m_1, m_2)\,, \tag{3.13}$$

then from (3.12) it follows at once that the multiplication rule for any two unitary matrices of this type is

$$\mathfrak{u}(l; m_1, m_2; n_1, n_2)\, \mathfrak{u}(l'; m'_1, m'_2; n'_1, n'_2)$$

$$= \mathfrak{u}(l + l' + m_1 n'_1 N_2 + m_2 n'_2 N_1;\ m_1 + m'_1, m_2 + m'_2;\ n_1 + n'_1, n_2 + n'_2)\,. \tag{3.14}$$

It is now obvious how to proceed with the factorization of the unitary operator $\mathfrak{u}(l; m, n)$ in a more general setting, when $N = \prod_{j=1}^{\nu} N_j$, where ν is the total number of prime factors in N, including repetitions.[3] The resulting commutatively factored basis

$$\mathbf{X}(m, n) = \prod_{j=1}^{\nu} \mathbf{X}(m_j, n_j), \quad \mathbf{X}(m_j, n_j) = N_j^{-1/2}\, \mathbf{V}_j^{m_j}\, \mathbf{U}_j^{n_j}, \quad 0 \le m_j, n_j \le N_j - 1\,,$$

can be thus constructed from the operator bases individually associated with the ν degrees of freedom. Finally, J. Schwinger formulates in [13, 15] how to investigate "the characteristics of a degree of freedom exhibiting an infinite number of states" $\mathfrak{u}(l; m, n)$. Naturally, in the limit as $\nu \to \infty$ the canonical Heisenberg–Weyl algebra Lie $\mathcal{W}_1$, associated with the Heisenberg–Weyl group W_1, is reproduced [10].

Turning to the lowering and raising difference operators $\mathbf{b}_N$ and $\mathbf{b}_N^{\dagger}$, the next logical step is to show that the knowledge of an explicit form of a complementary pair of unitary operators $\mathbf{V}$ and $\mathbf{U}$ on $\mathbb{C}^N$ actually enables one to establish the group-theoretic interpretation of the operators $\mathbf{b}_N$ and $\mathbf{b}_N^{\dagger}$. So we recall that explicit forms of these operators $\mathbf{b}_N$ and $\mathbf{b}_N^{\dagger}$,

$$\mathbf{b}_N = \frac{1}{2}\sqrt{\frac{N}{\pi}}\,\left[\mathbf{S} + \frac{1}{2}\Big(\mathbf{T}^{(+)} - \mathbf{T}^{(-)}\Big)\right],$$

$$\mathbf{b}_N^{\dagger} = \frac{1}{2}\sqrt{\frac{N}{\pi}}\,\left[\mathbf{S} - \frac{1}{2}\Big(\mathbf{T}^{(+)} - \mathbf{T}^{(-)}\Big)\right], \tag{3.15}$$

[3] Note that J. Schwinger calls this characteristic property of N *the number of degrees of freedom for a system possessing* N *states* [13].

have been found in [1] by analytically solving the intertwining relations (1.1). The operator $\mathbf{S}$ in (3.15) is represented by a diagonal matrix with the matrix elements $S_{kl} = \sin k\theta\, \delta_{kl}$ and $\mathbf{T}^{(\pm)}$ are the shift operators with matrix elements $T^{(\pm)}_{kl} = \delta_{k\pm1,l}$, respectively. But the matrix elements of the complementary pair of the unitary operators $\mathbf{V}$ and $\mathbf{U}$ on $\mathbb{C}^N$, defined above by (2.3) and (2.9), are of the form $V_{kl} = \delta_{k,l+1}$ and $U_{kl} = q^k\, \delta_{k,l}$, respectively. Therefore

$$\mathbf{S} = \frac{1}{2\mathrm{i}}\Big(\mathbf{U} - \mathbf{U}^{\dagger}\Big), \qquad \mathbf{T}^{(+)} = \mathbf{V}^{\dagger}, \qquad \mathbf{T}^{(-)} = \mathbf{V},$$

and the lowering and raising operators $\mathbf{b}_N$ and $\mathbf{b}^{\dagger}_N$ can be consequently written in terms of the unitary operators $\mathbf{V}$ and $\mathbf{U}$ as their linear combinations of the form

$$\mathbf{b}_N = \frac{1}{4}\sqrt{\frac{N}{\pi}}\,\Big[\mathbf{V}^{\dagger} - \mathbf{V} + \mathrm{i}\Big(\mathbf{U}^{\dagger} - \mathbf{U}\Big)\Big],$$

$$\mathbf{b}^{\dagger}_N = \frac{1}{4}\sqrt{\frac{N}{\pi}}\,\Big[\mathbf{V} - \mathbf{V}^{\dagger} - \mathrm{i}\Big(\mathbf{U} - \mathbf{U}^{\dagger}\Big)\Big]. \tag{3.16}$$

Notice that the intertwinning relations for the lowering and raising operators $\mathbf{b}_N$ and $\mathbf{b}^{\dagger}_N$ in the form (3.16) can be readily verified to hold by using two identities

$$\mathbf{V}\,\mathbf{\Phi}^{(\mathbf{N})} = \mathbf{\Phi}^{(\mathbf{N})}\,\mathbf{U}^{\dagger}, \qquad \mathbf{U}\,\mathbf{\Phi}^{(\mathbf{N})} = \mathbf{\Phi}^{(\mathbf{N})}\,\mathbf{V},$$

which are direct consequences of the fact that by (14) the operators $\mathbf{V}$ and $\mathbf{U}$ are unitarily similar to each other with the transforming operator $\mathbf{\Phi}^{(N)}$.

Thus from the algebraic point of view the lowering and raising operators $\mathbf{b}_N$ and $\mathbf{b}^{\dagger}_N$ turn out to be elements of the *group algebra* $\mathbb{C}\,[\mathcal{U}(N)]$ over the group $\mathcal{U}(N)$ of unitary transformations of the N-dimensional complex vector space $\mathbb{C}^N$, associated with the eigenvectors $\vec{f}^{(N)}_n$.[4]

It remains only to emphasize that contrary to the continuous case, when the lowering and raising operators $\mathbf{a}$ and $\mathbf{a}^{\dagger}$ obey the Heisenberg commutation relation $[\mathbf{a}, \mathbf{a}^{\dagger}] := \mathbf{a}\mathbf{a}^{\dagger} - \mathbf{a}^{\dagger}\mathbf{a} = \mathbf{I}$, a commutator between the lowering and raising difference operators $\mathbf{b}_N$ and $\mathbf{b}^{\dagger}_N$ turns out to be equal to

$$\mathcal{C}_N := \Big[\mathbf{b}_N, \mathbf{b}^{\dagger}_N\Big] = \frac{N\mathrm{i}}{8\pi}(1-q)\Big[\mathbf{V}\mathbf{U} + \mathbf{V}^{\dagger}\mathbf{U}^{\dagger} + \mathbf{U}\mathbf{V}^{\dagger} + \mathbf{U}^{\dagger}\mathbf{V}\Big]. \tag{3.17}$$

[4]Recall that the group algebra is defined as follows [9]. Let F be a field and G a finite group. Then $F[G]$ is the set of sums $\Big\{\sum_{g\in G} c_g\, g \,|\, c_g \in F\Big\}$. The structure of an F-vector space is given to $F[G]$ by identifying with g the element of $F[G]$ for which $c_g = 1$ and $c_h = 0$ if $h \neq g$. This identification embeds G into $F[G]$ and G is a basis for $F[G]$. Finally, to define multiplication on $F[G]$, one multiplies the basis vectors according to their group multiplication and extends linearly to all of $F[G]$. This defines the structure of an F-algebra on $F[G]$.

The matrix form of this *almost two-diagonal* symmetric matrix can be compactly expressed in terms of the Chebyshev polynomials of the first kind $T_n(x) = \cos(n\theta),\ x = \cos\theta$, as

$$\Big(\mathcal{C}_N\Big)_{kl} = \frac{N}{2\pi}\sqrt{1-x_N^2}\,\Big[T_{2k+1}(x_N)\,\delta_{k+1,l} \,+\, T_{2k-1}(x_N)\,\delta_{k-1,l}\Big] =$$

$$= \frac{N}{2\pi}\sqrt{1-x_N^2}\begin{pmatrix} 0 & T_1(x_N) & 0 & \cdots & 0 & 0 & T_1(x_N) \\ T_1(x_N) & 0 & T_3(x_N) & \cdots & 0 & 0 & 0 \\ 0 & T_3(x_N) & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & T_5(x_N) & 0 \\ 0 & 0 & 0 & \cdots & T_5(x_N) & 0 & T_3(x_N) \\ T_1(x_N) & 0 & 0 & \cdots & 0 & T_3(x_N) & 0 \end{pmatrix},$$

where $x_N = \cos(\pi/N),\ \delta_{N,l} = \delta_{0,l},\ \delta_{-1,l} = \delta_{N-1,l}$ and $T_{2N-1}(x_N) = T_{-1}(x_N) = T_1(x_N)$. Note that this explicit matrix form of the commutator operator $\mathcal{C}_N$ is valid for both the odd and even dimensions N: the only distinction between these two cases is that the superdiagonal and subdiagonal elements in the odd case are of the form

$$\{T_1(x_N), T_3(x_N), ..., T_{N-2}(x_N), T_N(x_N), T_{N-2}(x_N), ..., T_3(x_N)\},$$

whereas in the even case they are given as

$$\{T_1(x_N), T_3(x_N), ..., T_{N-1}(x_N), T_{N-1}(x_N), T_{N-3}(x_N), ..., T_3(x_N)\};$$

so that only in latter case the middle element $T_{N-1}(x_N)$ appears twice.

As a consequence of this property of the lowering and raising operators $\mathbf{b}_N$ and $\mathbf{b}_N^\dagger$, the intertwinning relations between these operators and the discrete number operator

$$\mathcal{N}^{(N)} = \mathbf{b}_N^\dagger\,\mathbf{b}_N = \frac{N}{16\pi}\Big[4\,\mathbf{I} - \mathbf{V}^2 - \mathbf{U}^2 - (\mathbf{V}^\dagger)^2 - (\mathbf{U}^\dagger)^2\Big] \,-\, \frac{1}{2}\,\mathcal{C}_N \tag{3.18}$$

have the more complicated form

$$\Big(\mathcal{N}^{(N)} + \mathcal{C}_N\Big)\,\mathbf{b}_N = \mathbf{b}_N\,\mathcal{N}^{(N)}, \qquad \mathcal{N}^{(N)}\,\mathbf{b}_N^\dagger = \mathbf{b}_N^\dagger\,\Big(\mathcal{N}^{(N)} + \mathcal{C}_N\Big), \tag{3.19}$$

than in the continuous case, when

$$\Big(\mathbf{N} + \mathbf{I}\Big)\,\mathbf{a} \,=\, \mathbf{a}\,\mathbf{N}, \qquad \mathbf{N}\,\mathbf{a}^\dagger \,=\, \mathbf{a}^\dagger\,\Big(\mathbf{N} + \mathbf{I}\Big).$$

Nevertheless, the eigenvectors $\vec{f}_n^{(N)}$ of the ND DFT operator $\mathbf{\Phi}^{(N)}$ can be explicitly written in terms of the lowering and raising operators $\mathbf{b}_N$ and $\mathbf{b}_N^\dagger$ as

$$\vec{f}_n^{(N)} \,=\, c_n\,\Big(\mathbf{b}_N^\dagger\Big)^n\,\vec{f}_0^{(N)}, \tag{3.20}$$

where $\vec{f}_0^{(N)}$ is a solution of the difference equation $\mathbf{b}_N \vec{f}_0^{(N)} = 0$.

However, what is still missing in the approach, formulated in [1], is the explicit form of the spectrum of the discrete number operator $\mathcal{N}^{(N)} = \mathbf{b}_N^{\dagger}\, \mathbf{b}_N$, which actually unambiguously governs the eigenvectors $\vec{f}_n^{(N)}$ of the ND DFT operator $\Phi^{(N)}$. We hope that this particular study, limited to the understanding of the algebraic properties of the lowering and raising difference operators $\mathbf{b}_N$ and $\mathbf{b}_N^{\dagger}$ and, consequently, of their product, the discrete number operator $\mathcal{N}^{(N)} = \mathbf{b}_N^{\dagger}\, \mathbf{b}_N$, will help us to resolve this final fragment of the puzzle, which has for a long time surrounded the explicit form of the eigenvectors of the discrete (finite) Fourier transform.

Acknowledgements

We are grateful to José Antonio de la Peña, Timothy Gendron, Rolando Jiménez and Vladimir Matveev for illuminating discussions. The participation of MKA in this work was partially supported by the SEP-CONACyT project 168104 "Operadores integrales y pseudodiferenciales en problemas de física matemática" and by the *Proyecto de Redes* PROMEP (México). NMA has been supported by the Universidad Nacional Autónoma de México through the PAPIIT-DGAPA project IN101115 *Óptica Matemática*.

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