Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 11, Number 1, pp. 67–80 (2016)
http://campus.mst.edu/adsa

Existence of Periodic Solutions for a Quantum Volterra Equation

Muhammad N. Islam
University of Dayton
Department of Mathematics
Dayton, OH 45469, USA
mislam1@udayton.edu

Jeffrey T. Neugebauer
Eastern Kentucky University
Department of Mathematics and Statistics
Richmond, KY 40475, USA
jeffrey.neugebauer@eku.edu

Abstract

The objective of this paper is to study the periodicity properties of functions that arise in quantum calculus, which has been emerging as an important branch of mathematics due to its various applications in physics and other related fields. The paper has two components. First, a relation between two existing periodicity notions is established. Second, the existence of periodic solutions of a q-Volterra integral equation, which is a general integral form of a first order q-difference equation, is obtained. At the end, some examples are provided. These examples show the effectiveness of the relation between the two periodicity notions that is established in this paper.

AMS Subject Classifications: 39A13, 39A20, 39A23, 39A12.
Keywords: Periodic solution, q-Volterra integral equation, fixed point, Krasnosel'skii.

1 Introduction

In the last decades, there has been a remarkable interest on periodicity notion on time scales (nonempty closed subsets of reals), due to its tremendous applications in engineering, biology, biomathematics, chemistry etc. There is a vast literature on periodic

Received October 15, 2015; Accepted January 23, 2016
Communicated by Jaqueline G. Mesquita

solutions of several types of dynamic and integro-dynamic equations on additively periodic domains (see [2,6,7,9,12,13,16] and the references therein). A time scale, denoted by $\mathbb{T}$, is said to be additively T-periodic if there exists a $T > 0$ such that $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [11]). A function f on additively periodic domain $\mathbb{T}$ with period T is said to be periodic with period P if there exists an $n \in \mathbb{N}$ such that $P = nT$, $f(t \pm P) = f(t)$ for all $t \in \mathbb{T}$, and P is the smallest number such that $f(t \pm P) = f(t)$. Notice that the set of reals $\mathbb{R}$, the set of integers $\mathbb{Z}$ and the set $h\mathbb{Z} := \{ht : t \in \mathbb{Z}\}$ are all examples of additively periodic time scales, while the set $q^{\mathbb{N}_0} := \{q^n : n = 0, 1, 2...\}$ is not. Hence, we cannot define periodicity on $q^{\mathbb{N}_0}$ in a same way we do on additively periodic domains.

But it is very important to develop a theory on the periodicity on $q^{\mathbb{N}_0}$ because the two important fields of mathematical physics, namely, quantum calculus and the theory of q-difference equations, are constructed on a class of functions defined on the set $q^{\mathbb{N}_0}$. For a comprehensive review of basic theory on quantum calculus, we refer to [10]. Theory of q-difference equations based on q-derivative has became a popular research field recently due to the close relationship between q-difference equations and differential equations. It is stated in [14] that "*in the p-adic context, q-difference equations are not simply a discretization of solutions of differential equations, but they are actually equal.*" We may also refer to [4] for further discussion about the equivalence between q-difference equations and differential equations and [8] for a study on the existence of q-difference equations.

Because of the close relationship between conventional calculus and quantum calculus, it is reasonable to ask for the availability of periodicity notion on $q^{\mathbb{N}_0}$. To the best of our knowledge, the first periodicity notion on $q^{\mathbb{N}_0}$ has been introduced by Bohner and Chieochan in [5]. They defined a P-periodic function on $q^{\mathbb{N}_0}$ as follows:

Definition 1.1 (See [5])**.** Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is said to be P-periodic if

$$f(t) = q^P f\left(q^P t\right) \text{ for all } t \in q^{\mathbb{N}_0}. \tag{1.1}$$

Afterwards, Adivar [1] (see also [3]) introduced a more general periodicity notion on time scales that are not necessarily additively periodic. On $q^{\mathbb{N}_0}$, this is defined as follows:

Definition 1.2 (See [1])**.** Let $P \in \mathbb{N}$. A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is said to be P-periodic if

$$f\left(q^P t\right) = f(t) \text{ for all } t \in q^{\mathbb{N}_0}.$$

In accordance with the periodicity notion on the continuous domain $\mathbb{R}$, Definition 1.2 regards a periodic function to be the one repeating its values after a certain number of steps on $q^{\mathbb{N}_0}$. On the other hand, Definition 1.1 resembles the periodicity on $\mathbb{R}$ in geometric meaning. Periodicity in Definition 1.1 is based on the equality of areas lying below the graph of the function at each period. For example, the function $h(t) = (-1)^{\frac{\ln t}{\ln q}}$ on $q^{\mathbb{N}_0}$ is a 2-periodic function according to Definition 1.2, since $h(q^2 t) = h(t)$ holds. On the other hand, the function $g(t) = 1/t$ is 1-periodic with respect to Definition 1.1 since it satisfies $qg(qt) = g(t)$.

First, we show, in Section 2, that the periodicity notions introduced in Definitions 1.1 and 1.2 are closely related. This linkage provides an easy way for the construction of relationship between coefficients of equations whose solutions are periodic with respect to Definitions 1.1 and 1.2, respectively. Two examples that we have provided at the end of the paper show the effectiveness of this linkage.

Then, in Section 3, we study the existence of P-periodic solutions of the q-Volterra integral equation

$$\tilde{x}(t) = \tilde{g}\left(t, \tilde{x}\left(t\right)\right) + \int_{t}^{q^P t} \tilde{C}\left(t, s\right) \tilde{f}\left(s, \tilde{x}\left(s\right)\right) d_q s,$$

which is a general integral form of the first order q-difference equation (2.1) given below. In the process of obtaining the periodic solutions of the above q-Volterra integral equation, we employed Banach's contraction mapping principle and Krasnosel'skii's fixed point theorem.

2 Comparison of two Periodicity Notions

This section is devoted to a comparison of two q-periodicity notions given in Definitions 1.1 and 1.2. Now, we list the following observations establishing a linkage between these two periodicity definitions.

Proposition 2.1. *Let $f : q^{\mathbb{N}_0} \to \mathbb{R}$. Then f is periodic with respect to of Definition 1.1 if and only if $\tilde{f}\left(t\right) = tf\left(t\right)$ is periodic with respect to Definition 1.2 with the same period.*

Proposition 2.2. *The function x is a P-periodic solution of the following first order q-difference equation*

$$D_q x(t) + a(t) x^{\sigma}(t) = f\left(t, tx\left(t\right)\right), \ t \in q^{\mathbb{N}_0}, \tag{2.1}$$

with respect to Definition 1.1 if and only if $\tilde{x}(t) := tx(t)$ is a P-periodic solution of the first order q-difference equation

$$D_q \tilde{x}(t) + \tilde{a}(t) \tilde{x}^{\sigma}(t) = \tilde{f}\left(t, \tilde{x}\left(t\right)\right), \ t \in q^{\mathbb{N}_0}, \tag{2.2}$$

where

$$\tilde{a}(t) := \frac{ta\left(t\right) - 1}{qt},$$

and $\tilde{f}\left(t, \tilde{x}\left(t\right)\right) = tf\left(t, tx\left(t\right)\right)$, with respect to Definition 1.2. Here, $D_q f$ represents the q-derivative of f defined by

$$\left(D_q f\right)(t) = \frac{f(qt) - f(t)}{(q-1)t}, \ t \in q^{\mathbb{N}_0}. \tag{q-derivative}$$

Proof. Let x be a solution of (2.1). Then

$$tD_qx(t) + x^\sigma(t) - x^\sigma(t) + ta(t)x^\sigma(t) = tf\left(t, tx\left(t\right)\right),$$

which implies

$$D_q(tx(t)) + \frac{ta(t) - 1}{qt} qtx^\sigma(t) = tf(t, tx(t)).$$

This implies $\tilde{x}$ solves (2.2). The proof that $\tilde{x}$ solves (2.2) implies x solves (2.1) is similar. Proposition 2.1 implies that x is P-periodic with respect to Definition 1.1 if and only if $\tilde{x}$ is P-periodic with respect to Definition 1.2. □

Suppose that $a : q^{\mathbb{N}_0} \to \mathbb{R}$ is a function with $(1 + (q-1)ta(t)) \neq 0$ for all $t \in q^{\mathbb{N}_0}$. Based on the function a, we define the functions

$$e_a(q^n, q^m) := \prod_{k=m}^{n-1} (1 + (q-1)q^k a(q^k))$$

and

$$e_{\ominus a}\left(q^n, q^m\right) := e_a(q^n, q^m)^{-1}.$$

Multiplying both sides of equation (2.1) by $e_a\left(t, 1\right)$ gives

$$e_a(t, 1)D_qx(t) + a(t)e_a(t, 1)x^\sigma(t) = e_a(t, 1)f(t, tx(t)). \tag{2.3}$$

But

$$\begin{aligned} D_q[e_a(\cdot, 1)x](t) &= e_a(t, 1)D_qx(t) + a(t)e_a(t, 1)x^\sigma(t) \\ &= e_a(t, 1)[D_qx(t) + a(t)x^\sigma(t)] \\ &= e_a(t, 1)f(t, tx(t)). \end{aligned}$$

We integrate (2.3) from t to $q^P t$ to obtain

$$e_a(q^P t, 1)x(q^P t) - e_a(t, 1)x(t) = \int_t^{q^P t} e_a(s, 1)f(s, sx(s))d_qs,$$

which implies

$$e_a(q^P t, 1)x(q^P t) = e_a(t, 1)x(t) + \int_t^{q^P t} e_a(s, 1)f(s, sx(s))d_qs.$$

Multiplying both sides of the equation by q^P yields

$$e_a(q^P t, 1)q^P x(q^P t) = q^P e_a(t, 1)x(t) + q^P \int_t^{q^P t} e_a(s, 1)f(s, sx(s))d_qs.$$

Assuming $q^P x(q^P t) = x(t)$ and then dividing by $e_a(q^P t, 1)$, we arrive at

$$x(t) = q^P e_{\ominus a}\left(q^P t, t\right) x(t) + q^P \int_t^{q^P t} e_{\ominus a}\left(q^P t, s\right) f\left(s, sx(s)\right) d_q s. \tag{2.4}$$

Similarly, multiplying both sides of equation (2.2) by $e_{\tilde{a}}(t, 1)$, integrating from t to $q^P t$ and assuming $x(q^P t) = x(t)$ gives

$$\tilde{x}(t) = e_{\ominus \tilde{a}}\left(q^P t, t\right) \tilde{x}(t) + \int_t^{q^P t} e_{\ominus \tilde{a}}\left(q^P t, s\right) f\left(s, \tilde{x}(s)\right) d_q s, \tag{2.5}$$

for $t \in q^{\mathbb{N}_0}$. Here the q-integral is defined by

$$\int_{q^m}^{q^n} f(s) d_q s := (q-1) \sum_{k=m}^{n-1} q^k f(q^k). \tag{q-integral}$$

Next, the generalizations of (2.4) and (2.5) have the form of q-Volterra integral equations as follows:

$$x(t) = g\left(t, tx(t)\right) + \int_t^{q^P t} C(t, s) f\left(s, sx(s)\right) d_q s, \tag{2.6}$$

and

$$\tilde{x}(t) = \tilde{g}\left(t, \tilde{x}(t)\right) + \int_t^{q^P t} \tilde{C}(t, s) \tilde{f}\left(s, \tilde{x}(s)\right) d_q s, \tag{2.7}$$

where $g, \tilde{g}, f, \tilde{f} : q^{\mathbb{N}_0} \times \mathbb{R} \to \mathbb{R}$ are continuous in their second variable and $C, \tilde{C} : q^{\mathbb{N}_0} \times q^{\mathbb{N}_0} \to \mathbb{R}$,

$$\tilde{C}(t, s) = \frac{t}{s} C(t, s), \tag{2.8}$$

$$\tilde{f}\left(t, \tilde{x}(t)\right) = t f\left(t, tx(t)\right), \tag{2.9}$$

and

$$\tilde{g}\left(t, \tilde{x}(t)\right) = t g\left(t, tx(t)\right). \tag{2.10}$$

Similar to the Proposition 2.2, we can establish a linkage between two periodicity notions in terms of periodic solutions of the integral equations (2.6) and (2.7).

Proposition 2.3. *Assume that C, f, g and x satisfy*

$$C\left(q^P t, q^P s\right) = C(t, s), \tag{2.11}$$

$$q^P f\left(q^P t, q^P tx\left(q^P t\right)\right) = f\left(t, tx(t)\right), \tag{2.12}$$

$$q^P g\left(q^P t, q^P tx\left(q^P t\right)\right) = g\left(t, tx(t)\right). \tag{2.13}$$

Then $x(t)$ is a P-periodic solution of (2.6) *with respect to Definition 1.1 if and only if* $\tilde{x}(t) = tx(t)$ *is a P-periodic solution of* (2.7) *with respect to Definition 1.2.*

Proof. Assume (2.11)-(2.13) hold and suppose that $x(t)$ solves (2.6) and is P-periodic with respect to Definition 1.1. Let us multiply both sides of (2.6) by t, i.e.,

$$tx(t) = tg(t, tx(t)) + \int_t^{q^P t} tC(t,s) f(s, sx(s))\, d_q s,$$

or

$$tx(t) = tg(t, tx(t)) + \int_t^{q^P t} \frac{t}{s} C(t,s)\, sf(s, sx(s))\, d_q s.$$

By employing (2.8)-(2.10), we get

$$\tilde{x}(t) = \tilde{g}(t, \tilde{x}(t)) + \int_t^{q^P t} \tilde{C}(t,s) \tilde{f}(s, \tilde{x}(s))\, d_q s. \tag{2.14}$$

Notice that $\tilde{x}(t)$ is a P-periodic solution of (2.14) with respect to Definition 1.2. To show this, consider

$$\begin{aligned}
\tilde{x}(q^P t) = q^P tx\left(q^P t\right) &= \tilde{g}\left(q^P t, \tilde{x}\left(q^P t\right)\right) + \int_{q^P t}^{q^{2P} t} \tilde{C}\left(q^P t, s\right) \tilde{f}(s, \tilde{x}(s))\, d_q s \\
&= \tilde{g}\left(q^P t, \tilde{x}\left(q^P t\right)\right) + \int_t^{q^P t} \tilde{C}\left(q^P t, q^P s\right) \tilde{f}\left(q^P s, \tilde{x}\left(q^P s\right)\right) d_q s \\
&= q^P tg\left(q^P t, q^P tx\left(q^P t\right)\right) + \int_t^{q^P t} \frac{q^P t}{q^P s} C\left(q^P t, q^P s\right) q^P s\, f\left(q^P s, q^P sx\left(q^P s\right)\right) d_q s.
\end{aligned}$$

Using (2.11)-(2.13) we get

$$\begin{aligned}
\tilde{x}(q^P t) &= tg(t, tx(t)) + \int_t^{q^P t} \frac{t}{s} C(t,s)\, s\, f(s, sx(s))\, d_q s \\
&= \tilde{g}(t, \tilde{x}(t)) + \int_t^{q^P t} \tilde{C}(t,s) \tilde{f}(s, \tilde{x}(s))\, d_q s.
\end{aligned}$$

The proof of the necessity part can be done by following a similar procedure used in the sufficiency part, hence, we omit it. □

Remark 2.4. The above results reveal the basic linkage between two periodicity given in Definitions 1.1 and 1.2. In particular, Proposition 2.2 and Proposition 2.3 show the relationship between q-difference equations having periodic solutions with respect to Definitions 1.1 and 1.2. This provides a procedure for the rearrangement of an existence result based on Definition 1.2 to obtain an existence result based on Definition 1.1, and vice versa.

3 Existence Results

In this section, we study the existence of periodic solutions of the following type q-Volterra integral equations

$$\tilde{x}(t) = \tilde{g}\left(t, \tilde{x}\left(t\right)\right) + \int_{t}^{q^P t} \tilde{C}\left(t,s\right) \tilde{f}\left(s, \tilde{x}\left(s\right)\right) d_q s, \tag{3.1}$$

which is a general integral form of the first order q-difference equation (2.1). By taking the advantage of relationship between two periodicity notions, hereafter we adopt Definition 1.2 to our analysis for convenience. Let $\mathbb{P}$ be the set of all functions defined on $q^{\mathbb{N}_0}$ which are P-periodic. Then $(\mathbb{P}, \|\cdot\|)$ is a Banach space endowed with the norm

$$\|x\| = \max_{t \in [1,q^P]_{q^{\mathbb{N}_0}}} |x(t)|,$$

where $\left[1, q^P\right]_{q^{\mathbb{N}_0}} := \left[1, q^P\right] \cap q^{\mathbb{N}_0}$ and the set

$$\Pi_M := \{\varphi \in \mathbb{P} : \|\varphi\| \leq M\} \tag{3.2}$$

is a bounded, closed and convex subset of $\mathbb{P}$ for a positive constant M.

Since we are dealing with the existence of periodic solutions of (3.1), it is natural to have the following periodicity assumptions:

$\mathcal{K}1$ $\tilde{g}$ satisfies

$$\tilde{g}\left(q^P t, \tilde{x}\right) = \tilde{g}\left(t, \tilde{x}\right),$$

for all $t \in q^{\mathbb{N}_0}$;

$\mathcal{K}2$ $\tilde{C}$ satisfies

$$\tilde{C}\left(q^P t, q^P s\right) = \tilde{C}\left(t, s\right),$$

for all $(t,s) \in q^{\mathbb{N}_0} \times q^{\mathbb{N}_0}$; and

$\mathcal{K}3$ $\tilde{f}$ satisfies

$$q^P \tilde{f}\left(q^P t, \tilde{x}\right) = \tilde{f}\left(t, \tilde{x}\right),$$

for all $t \in q^{\mathbb{N}_0}$.

Lemma 3.1. *Assume ($\mathcal{K}1$-$\mathcal{K}3$) and for $\tilde{\varphi} \in \mathbb{P}$ define the operator Q as*

$$(Q\tilde{\varphi})(t) := \tilde{g}(t, \tilde{\varphi}(t)) + \int_t^{q^P t} \tilde{C}(t,s)\, \tilde{f}(s, \tilde{\varphi}(s))\, d_q s. \tag{3.3}$$

Then $Q : \mathbb{P} \to \mathbb{P}$.

Proof. Suppose that ($\mathcal{K}1$-$\mathcal{K}3$) hold and $\tilde{\varphi} \in \mathbb{P}$. Then

$$\begin{aligned}(Q\tilde{\varphi})\left(q^P t\right) &= \tilde{g}\left(q^P t, \tilde{\varphi}\left(q^P t\right)\right) + \int_{q^P t}^{q^{2P} t} \tilde{C}\left(q^P t, s\right) \tilde{f}(s, \tilde{\varphi}(s))\, d_q s \\ &= \tilde{g}(t, \tilde{\varphi}(t)) + \int_t^{q^P t} \tilde{C}\left(q^P t, q^P s\right) q^P \tilde{f}\left(q^P s, \tilde{\varphi}\left(q^P s\right)\right) d_q s \\ &= \tilde{g}(t, \tilde{\varphi}(t)) + \int_t^{q^P t} \tilde{C}(t,s)\, \tilde{f}(s, \tilde{\varphi}(s))\, d_q s \\ &= (Q\tilde{\varphi})(t),\end{aligned}$$

and this completes the proof. □

Next, we use the contraction mapping principle to show the existence of a unique solution of (3.1). We assume for every $\tilde{x}, \tilde{y} \in \mathbb{P}$ and $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$\mathcal{K}4$ $|\tilde{g}(t, \tilde{x}) - \tilde{g}(t, \tilde{y})| \le a_1 |\tilde{x} - \tilde{y}|$, $a_1 \in (0, 1)$; and

$\mathcal{K}5$ $\left|\tilde{f}(t, \tilde{x}) - \tilde{f}(t, \tilde{y})\right| \le a_2 |\tilde{x} - \tilde{y}|$, $a_2 \in \mathbb{R}_+$.

We define

$$\bar{C} := \max_{(t,s) \in [1,q^P]_{q^{\mathbb{N}_0}} \times [t, q^P t]_{q^{\mathbb{N}_0}}} \left|\tilde{C}(t,s)\right|. \tag{3.4}$$

Theorem 3.2. *Assume ($\mathcal{K}1$-$\mathcal{K}5$). If*

$$a := a_1 + q^P(q^P - 1)\bar{C} a_2 < 1, \tag{3.5}$$

then (3.1) *has a unique solution.*

Proof. Let $\tilde{\varphi}, \tilde{\psi} \in \mathbb{P}$. Then for every $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$\begin{aligned}|Q\tilde{\varphi} - Q\tilde{\psi}|(t) &\le |\tilde{g}(t, \tilde{\varphi}(t)) - \tilde{g}(t, \tilde{\psi}(t))| + \int_t^{q^P t} |\tilde{C}(t,s)||\tilde{f}(s, \tilde{\varphi}(s)) - \tilde{f}(s, \tilde{\psi}(s))| d_q s \\ &\le a_1 |\tilde{\varphi}(t) - \tilde{\psi}(t)| + q^P(q^P - 1)\bar{C} a_2 \|\tilde{\varphi} - \tilde{\psi}\| \\ &\le (a_1 + q^P(q^P - 1)\bar{C} a_2) \|\tilde{\varphi} - \tilde{\psi}\|.\end{aligned}$$

So $\|Q\tilde{\varphi} - Q\tilde{\psi}\| < a\|\tilde{\varphi} - \tilde{\psi}\|$ with $a < 1$. Thus Q has a unique fixed point in P, and (3.1) has a unique P-periodic solution with respect to Definition 1.2. □

Next we state Krasnosel'skii's fixed point theorem which we employ for showing the existence of a periodic solution of (3.1).

Theorem 3.3 (Krasnosel'skii, [15])**.** *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{M}$ implies $Ax + By \in \mathbb{M}$,

(ii) A is a contraction mapping, and

(iii) B is a compact and continuous mapping.

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

Now, the operator Q given in (3.3) can be written as

$$(Q\tilde{\varphi})(t) := (A\tilde{\varphi})(t) + (B\tilde{\varphi})(t),$$

where

$$(A\tilde{\varphi})(t) := \tilde{g}(t, \tilde{\varphi}(t)), \tag{3.6}$$

and

$$(B\tilde{\varphi})(t) := \int_t^{q^P t} \tilde{C}(t,s)\, \tilde{f}(s, \tilde{\varphi}(s))\, d_q s. \tag{3.7}$$

Lemma 3.4. *Suppose ($\mathcal{K}4$) holds. Then $A : \mathbb{P} \to \mathbb{P}$ is a contraction mapping.*

Define the function $\bar{F} : \mathbb{R} \to \mathbb{R}$ by

$$\bar{F}(m) = \sup_{(t,x)\in[1,q^P]_{q^{\mathbb{N}_0}}\times[-m,m]} |\tilde{f}(t,x)|. \tag{3.8}$$

Lemma 3.5. *Assume ($\mathcal{K}1$-$\mathcal{K}3$) hold. Then $B : \mathbb{P} \to \mathbb{P}$ is a continuous compact mapping.*

Proof. Let B be defined as in (3.7). Since $\tilde{f}$ is continuous in the second variable, given $\epsilon > 0$, there exists a $\delta > 0$ such that $\|\tilde{\varphi} - \tilde{\psi}\| < \delta$ implies $|\tilde{f}(\tau, \tilde{\varphi}(\tau)) - \tilde{f}(\tau, \tilde{\psi}(\tau))| < \dfrac{\epsilon}{\bar{C}q^P(q^P-1)}$ for each $\tau \in [1, q^{2P}]_{q^{\mathbb{N}_0}}$. Thus for $\|\tilde{\varphi} - \tilde{\psi}\| < \delta$ and for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$, $s \in [1, q^{2P}]$, we have

$$\begin{aligned}|B\tilde{\varphi} - B\tilde{\psi}|(t) &\le \int_t^{q^P t} |\tilde{C}(t,s)||\tilde{f}(s,\tilde{\varphi}(s)) - \tilde{f}(s,\tilde{\psi}(s))| d_q s \\ &< \frac{\epsilon}{\bar{C}q^P(q^P-1)} \bar{C} \int_t^{q^P t} d_q s \le \epsilon.\end{aligned}$$

Thus for $\|\tilde{\varphi} - \tilde{\psi}\| < \delta$, $\|B\tilde{\varphi} - B\tilde{\psi}\| < \epsilon$. Therefore B is continuous.

In order to show that B is compact, consider the set Π_M which is defined in (3.2). For compactness of $B\Pi_M$, we will use the sequential criterion, which states that the metric space $B\Pi_M$ is compact if and only if every sequence in $B\Pi_M$ has a subsequence converging to an element in $B\Pi_M$. Consider a sequence of P-periodic functions $\left\{\tilde{\xi}_n\right\} \subset \Pi_M$. At first, we will show $\left\{B\tilde{\xi}_n\right\}$ is uniformly bounded.

To show that B is uniformly bounded, we obtain, for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$
\begin{aligned}
|B\tilde{\xi}_n(t)| &\leq \int_t^{q^P t} |\tilde{C}(t,s)||\tilde{f}(s, \tilde{\xi}_n(s))| d_q s \\
&\leq q^P(q^P - 1)\bar{C}\bar{F}(M).
\end{aligned} \tag{3.9}
$$

Therefore

$$
\|B\tilde{\xi}_n\| \leq q^P(q^P - 1)\bar{C}\bar{F}(M),
$$

which implies that $\left\{B\tilde{\xi}_n\right\}$ is uniformly bounded.

Notice that for each $t \in q^{\mathbb{N}_0}$, since $\left\{B\tilde{\xi}_n\right\}$ is uniformly bounded, $\left\{(B\tilde{\xi}_n)(t)\right\}$ is a bounded sequence of real numbers. Therefore, at the first point, $q^0 := t_1 \in q^{\mathbb{N}_0}$, by the Bolzano–Weierstrass theorem, there is a subsequence $\{(B\tilde{\xi})_{n_{1_j}}\}$ such that $\{(B\tilde{\xi})_{n_{1_j}}(t_1)\}$ converges as $j \to \infty$. Again, we choose a subsequence of n_{1_j} which we call n_{2_j} such that sequence $\{(B\tilde{\xi})_{n_{2_j}}(t_2)\}$ converges at the next point, $q^1 := t_2 \in q^{\mathbb{N}_0}$. Proceeding with this method inductively, we obtain a subsequence n_{k_j} of the sequence $n_{(k-1)_j}$ such that $\{(B\tilde{\xi})_{n_{k_j}}(t_k)\}$ converges at the next point, $q^{k-1} := t_k \in q^{\mathbb{N}_0}$. Note that by our construction process, it is guaranteed that $\{(B\tilde{\xi})_{n_{k_j}}(r)\}$ converges for all $r \in [1, t_k]_{q^{\mathbb{N}_0}}$. Now let $n_j = n_{j_j}$, i.e., we pick the 'diagonal sequence'. Clearly, $\{(B\tilde{\xi})_{n_j}(r)\}$ converges for all $r \in [1, t_k]_{q^{\mathbb{N}_0}}$. Therefore, we obtain the required subsequence $\{(B\tilde{\xi})_{n_j}\} \subset B\Pi_M$ such that $(B\tilde{\xi})_{n_j}(t) \to B\tilde{\xi}(t)$ as $j \to \infty$, uniformly for all $t \in q^{\mathbb{N}_0}$. But $B\tilde{\xi} = \{B\tilde{\xi}(t_1), B\tilde{\xi}(t_2), ..., B\tilde{\xi}(t_k), ...\}$, where $B\tilde{\xi}(t_k)$ is the limit of $(B\tilde{\xi})_{n_j}(t_k)$ as $j \to \infty$. Since the operator B is continuous, and each $\{(B\tilde{\xi})_{n_j}\}$ is in $B\Pi_M$, then the limit $B\tilde{\xi} \in B\Pi_M$. This concludes that $B\Pi_M$ is compact. □

Theorem 3.6. *Assume ($\mathcal{K}1$-$\mathcal{K}4$). If there exists a positive constant M_0 such that*

$$
\frac{\alpha + \bar{C}\bar{F}(M_0)q^P(q^P - 1)}{1 - a_1} \leq M_0, \tag{3.10}
$$

where $\bar{C}$ as in (3.4) *and $\alpha = \|g(t,0)\|$, then equation* (3.1) *has a P-periodic solution in $\Pi_{M_0} := \{\varphi \in \mathbb{P} : \|\varphi\| \leq M_0\}$ with respect to Definition 1.2.*

Proof. For $\tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0}$, we have, for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$
|A\tilde{\varphi} + B\tilde{\psi}|(t) \leq |\tilde{g}(t, \tilde{\varphi}(t))| + \left| \int_t^{q^P t} \tilde{C}(t,s)\tilde{f}(s, \tilde{\phi}(s)) d_q s \right|. \tag{3.11}
$$

Notice that for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$\begin{aligned} |\tilde{g}(t, \tilde{\varphi}(t))| &\leq |\tilde{g}(t, \tilde{\varphi}(t)) - \tilde{g}(t, 0)| + |g(t, 0)| \\ &\leq a_1 |\tilde{\varphi}(t)| + \alpha \\ &\leq a_1 \|\tilde{\varphi}\| + \alpha \\ &\leq a_1 M_0 + \alpha. \end{aligned} \tag{3.12}$$

Therefore by (3.9), (3.10), (3.11), and (3.12), we obtain, for $t \in [1, q^P]_{q^{\mathbb{N}_0}}$,

$$|A\tilde{\varphi} + B\tilde{\psi}|(t) \leq a_1 M_0 + \alpha + \bar{C}\bar{F}(M_0) q^P (q^P - 1) \leq M_0.$$

Therefore, for $\tilde{\varphi}, \tilde{\psi} \in \Pi_{M_0}$, $\|A\tilde{\varphi} + B\tilde{\psi}\| \leq M_0$. So $A\tilde{\varphi} + B\tilde{\psi} \in \Pi_{M_0}$, which proves condition (i) of Theorem 3.3. Notice Lemma 3.4 and Lemma 3.5 prove conditions (ii) and (iii) of Theorem 3.3. Therefore there exists a P-periodic solution of (3.1) with respect to Definition 1.2. □

4 Examples

Example 4.1. Consider the following Volterra equation constructed on $2^{\mathbb{N}_0}$,

$$\tilde{x}(t) = \frac{1}{2} + \frac{1}{6}(-1)^{\frac{\ln t}{\ln 2}} \tilde{x}(t) + \int_t^{4t} \frac{1}{96t} \cos\left(\frac{\ln s}{\ln 2}\pi\right) \tilde{x}(s) d_2 s. \tag{4.1}$$

Comparing (4.1) with (3.1) gives

$$\begin{aligned} \tilde{g}(t, \tilde{x}(t)) &= \frac{1}{2} + \frac{1}{6}(-1)^{\frac{\ln t}{\ln 2}} \tilde{x}(t), \\ \tilde{C}(t, s) &= \frac{s}{t}, \end{aligned}$$

and

$$\tilde{f}(t, \tilde{x}(t)) = \frac{1}{96t} \cos\left(\frac{\ln t}{\ln 2}\pi\right) \tilde{x}(t).$$

Then, one may easily check that conditions ($\mathcal{K}1$-$\mathcal{K}5$) are satisfied with positive constants $a_1 = 1/6$, $a_2 = 1/96$ and $\bar{C} = 4$. Then, since $a_1 + q^P(q^P - 1)\bar{C}a_2 = 2/3 < 1$, by employing Theorem 3.2, we deduce that the equation (4.1) has a unique 2-periodic solution with respect to Definition 1.2.

Remark 4.2. Motivated by the discussions given in the first section and using Proposition 2.2 and (2.8)-(2.10), we have the following modified q-Volterra equation

$$x(t) = \frac{1}{2t} + \frac{1}{6}(-1)^{\frac{\ln t}{\ln 2}} x(t) + \int_t^{4t} \frac{1}{96t^2} \cos\left(\frac{\ln s}{\ln 2}\pi\right) s x(s) d_2 s. \tag{4.2}$$

Observe that (4.1) and (4.2) are correlated by Proposition 2.3. We emphasize that by proving the existence of a unique 2-periodic solution of (4.1) in the sense of Definition 1.2, we also prove the existence of a unique 2-periodic solution of (4.2) with respect to Definition 1.1.

Example 4.3. Consider the equation

$$\tilde{x}(t) = \frac{1}{2}\cos\left(\frac{\ln t}{\ln 2}\pi\right)\tilde{x}(t) + \frac{1}{48e^4}\int_t^{4t} \exp\left(\frac{s}{t}\right)\tilde{\Lambda}(s)\,\tilde{x}(s)d_2s,\ t \in 2^{\mathbb{N}_0}. \tag{4.3}$$

Comparing (4.3) to (3.1), we get that

$$\tilde{g}(t, \tilde{x}(t)) = \frac{1}{2}\cos\left(\frac{\ln t}{\ln 2}\pi\right)\tilde{x}(t),$$
$$\tilde{C}(t,s) = \exp\left(\frac{s}{t}\right),$$

and

$$\tilde{f}(t, \tilde{x}(t)) = \frac{1}{48e^4}\tilde{\Lambda}(t)\,\tilde{x}(t),$$

where

$$\tilde{\Lambda}(t) = \begin{cases} 1/t, & \text{if } \log_2 t \text{ is odd} \\ 2/t, & \text{if } \log_2 t \text{ is even} \end{cases}$$

is a function satisfying (1.1) [5, Example 3.5] with $P = 2$. Observe that assumptions ($\mathcal{K}1 - \mathcal{K}3$) are satisfied, and the function $\tilde{g}$ satisfies the Lipschitz condition ($\mathcal{K}4$) with constant $a_1 = 1/2$. From (3.4) and (3.8), we obtain $\bar{C} = e^4$ and $\bar{F}(M_0) = (1/e^4 24)M_0$, respectively. Also, $\alpha = 0$. Then, the inequality (3.10) is satisfied for any positive constant M_0. By Theorem 3.6, we conclude that the equation (4.3) has a 2-periodic solution with respect to Definition 1.2. Observe that Theorem 3.2 does not work for concluding existence of periodic solutions of (4.3) since $a_2 = 1/24e^4$ and

$$a_1 + q^P(q^P - 1)\bar{C}a_2 = 1$$

(i.e., condition (3.5) does not hold).

Remark 4.4. Since (4.3) has a 2-periodic solution with respect to Definition 1.2, then the integral equation

$$x(t) = \frac{1}{2}\cos\left(\frac{\ln t}{\ln 2}\pi\right)x(t) + \frac{1}{48e^4}\int_t^{4t}\frac{1}{t}\exp\left(\frac{s}{t}\right)\tilde{\Lambda}(s)sx(s)d_2s, \tag{4.4}$$

has a 2-periodic solution with respect to Definition 1.1.

Acknowledgements

We are very grateful to Professors Murat Adivar and H.Can Koyuncuoğlu for their contributions and suggestions in this paper. We thank Professor Adivar for bringing up to our attention that the two existing periodicity notions discussed in this paper might be related in some way, and we thank Professor Koyuncuoğlu for helping us construct the examples.

References

[1] Murat Adivar. A new periodicity concept for time scales. *Math. Slovaca*, 63(4):817–828, 2013.

[2] Murat Adivar, H. Can Koyuncuoğlu, and Youssef N. Raffoul. Periodic and asymptotically periodic solutions of systems of nonlinear difference equations with infinite delay. *J. Difference Equ. Appl.*, 19(12):1927–1939, 2013.

[3] Murat Adıvar, Halis Can Koyuncuoğlu, and Youssef N. Raffoul. Existence of periodic solutions in shifts $\delta_\pm$ for neutral nonlinear dynamic systems. *Appl. Math. Comput.*, 242:328–339, 2014.

[4] Yves André and Lucia Di Vizio. q-difference equations and p-adic local monodromy. *Astérisque*, (296):55–111, 2004. Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes. I.

[5] Martin Bohner and Rotchana Chieochan. Floquet theory for q-difference equations. *Sarajevo J. Math.*, 8(21)(2):355–366, 2012.

[6] Martin Bohner and Rotchana Chieochan. Positive periodic solutions for higher-order functional q-difference equations. *J. Appl. Funct. Anal.*, 8(1):14–22, 2013.

[7] T. A. Burton. *Stability and periodic solutions of ordinary and functional differential equations*. Dover Publications, Inc., Mineola, NY, 2005. Corrected version of the 1985 original.

[8] Moustafa El-Shahed and H. A. Hassan. Positive solutions of q-difference equation. *Proc. Amer. Math. Soc.*, 138(5):1733–1738, 2010.

[9] Muhammad N. Islam. Three fixed point theorems: periodic solutions of a Volterra type integral equation with infinite heredity. *Canad. Math. Bull.*, 56(1):80–91, 2013.

[10] Victor Kac and Pokman Cheung. *Quantum calculus*. Universitext. Springer-Verlag, New York, 2002.

[11] Eric R. Kaufmann and Youssef N. Raffoul. Periodic solutions for a neutral nonlinear dynamical equation on a time scale. *J. Math. Anal. Appl.*, 319(1):315–325, 2006.

[12] Yongkun Li and Lifei Zhu. Existence of positive periodic solutions for difference equations with feedback control. *Appl. Math. Lett.*, 18(1):61–67, 2005.

[13] Benedetta Lisena. Periodic solutions of logistic equations with time delay. *Appl. Math. Lett.*, 20(10):1070–1074, 2007.

[14] Andrea Pulita. p-adic confluence of q-difference equations. *Compos. Math.*, 144(4):867–919, 2008.

[15] D. R. Smart. *Fixed point theorems*. Cambridge University Press, London-New York, 1974. Cambridge Tracts in Mathematics, No. 66.

[16] Weibing Wang and Zhiguo Luo. Positive periodic solutions for higher-order functional difference equations. *Int. J. Difference Equ.*, 2(2):245–254, 2007.