LINEAR MOTION OF A SYSTEM OF PARTICLES

Learning Objectives
After you complete the homework associated with this lecture, you should be able to:

• Decide when to use a momentum viewpoint and when to use an energy viewpoint in mechanical processes.
• Use this insight to analyze collisions of two objects.
• Locate a system’s center-of-mass and describe how its motion is affected by forces.
• Describe how the motion of a rocket is an example of conservation of linear momentum of a system that includes the rocket’s exhaust.

Example: A wooden block of mass \( M \) is attached to a spring (constant \( k \)), initially compressed distance \( D \). The block is released from rest. After traveling distance \( \frac{1}{2} D \) on a level frictionless surface, the block is hit head on by a bullet of mass \( m \) and speed \( v \). The bullet passes straight through the block, emerging with a speed of \( \frac{1}{2}v \). What is the block’s speed just after the bullet emerges from it?

1) Between states ① [initial] & ② [before the hit], mechanical energy is conserved for block-bullet system (but not momentum):

\[
\begin{align*}
\text{Initial:} & \quad k \begin{cases} M & V_1 = 0 \\ s_1 = -D \end{cases} \\
\text{Before hit:} & \quad \begin{cases} M & \quad 0 \quad V_2 = \frac{1}{2}v \\ s_2 = -D + \frac{1}{2}D = \frac{1}{2}D \end{cases}
\end{align*}
\]

\[
E_2 - E_1 = W_{\text{other}, 1 \rightarrow 2}
\]

\[
(K_2 + U_{\text{spring}, 2}) - (K_1 + U_{\text{spring}, 1}) = W_{\text{by forces other than spring}} = 0
\]

⇒ Solve for block’s speed \( V_2 \) just before bullet hits it:

\[
E_2 = E_1 \Rightarrow \frac{1}{2}MV_2^2 + \frac{1}{2}k(\frac{1}{2}D)^2 = \frac{1}{2}MV_1^2 + \frac{1}{2}k(-D)^2
\]

\[
\frac{1}{2}MV_2^2 = \frac{1}{2}kD^2 - \frac{1}{2}kD^2 = \frac{1}{2}kD^2
\]

Caution: Momentum of block-bullet system is NOT conserved:

\[
\vec{p}_2 - \vec{p}_1 = \vec{J}_{\text{net external, 1 \rightarrow 2}} = \vec{J}_{\text{spring, 1 \rightarrow 2}} \neq 0 \quad (\text{substantial time})
\]
2) Between states \(\text{② [just before bullet hits]} \& \text{③ [right after collision]}\), linear momentum of block-bullet system is conserved

\[
\mathbf{P}_2 - \mathbf{P}_3 = \mathbf{J}_{\text{net external, 2-3}} = \mathbf{J}_{\text{spring, 2-3}} \neq 0 \quad \text{(short collision time)}
\]

⇒ Solve for \(V_{3x}\) (block’s velocity component just after collision)

\[
\mathbf{P}_{3x} = \mathbf{P}_{2x} \implies MV_{3x} + m(-\frac{1}{2}v) = M(V_2) + m(-v)
\]

\[
MV_{3x} = MV_2 - m v + \frac{1}{2} m v = M\frac{M_2^2}{2M} - \frac{1}{2} m v
\]

**Caution:** Mechanical E of block-bullet system is NOT conserved:

\[
(K_{3x} + U_{spring, 3}) - (K_2 + U_{spring, 2}) = W_{\text{by forces other than spring, 2-3}}
\]

\[
= W_{\text{friction/disruption during collision}} \neq 0
\]

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**EXAMPLE:** Colliding blocks

Consider a simple one-dimensional collision of two air-track carts, each mass \(m\), with the blue one moving initially to the right at speed \(v_{bi}\) and the red one to the left at speed \(v_{ri}\). A spring is attached to the end of the collision end of the blue cart. What is the maximum compression \(d\) of the spring during the collision?

At maximum compression, both carts will have the same velocity (compressing \(\rightarrow\) coming together, while extending \(\rightarrow\) going apart). Thus \(v_{bf} = v_{rf} = V\).

Can we get by with just Energy Conservation? \(E_i = E_f\)

\[
\frac{1}{2}mv_{bi}^2 + \frac{1}{2}mv_{ri}^2 + \frac{1}{2}k(0)^2 = \frac{1}{2}mv_{bf}^2 + \frac{1}{2}mv_{rf}^2 + \frac{1}{2}k(-d)^2
\]

\[
\frac{1}{2}mv_{bi}^2 + \frac{1}{2}mv_{ri}^2 = \frac{1}{2}mV^2 + \frac{1}{2}mV^2 + \frac{1}{2}kd^2 = mV^2 + \frac{1}{2}kd^2
\]

Both \(V\) and \(d\) are unknowns \(\rightarrow\) need another equation!

Use conservation of momentum to get second equation:

\[\text{if } \sum \mathbf{F}_{\text{external}} = 0 \implies \mathbf{P}_i = \mathbf{P}_f\]

\(x\)-component of Momentum Conservation:

\[
mv_{bi} + mv_{ri} = (2m)V_x
\]

\[
p(+v_{bi}) + p(-v_{ri}) = (2p)V_x \quad \text{(Note speeds \(v\))}
\]

\[
V_x = \frac{1}{2} (v_{bi} - v_{ri})
\]

\[
\frac{1}{2}mv_{bi}^2 + \frac{1}{2}mv_{ri}^2 = mV^2 + \frac{1}{2}kd^2 = m[\frac{1}{2}(v_{bi} - v_{ri})]^2 + \frac{1}{2}kd^2
\]

One equation with one unknown: \(d\)!
**Example of Quick and Slow Impulses**

A thrown ball of mass $m_1 = m$ has speed of $v$ when it is at the top of its trajectory. At the top, it hits and sticks to a ball of mass $m_2 = 2m$ that is balanced precariously at the top of a pole.

Use impulse methods to determine the balls' common velocity at any time $T$ after the collision.

The collision is quick, so we neglect gravitational impulse to determine the momentum $\vec{p}_c$ from just before to just after the collision.

$$\vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} \vec{F}_{\text{grav}}(t) \, dt$$

$$(m + 2m)\vec{V} - m\vec{v} = \int (m + 3m)\vec{g}(T - 0) \, dt$$

$$\vec{V}(T) = \frac{1}{2} \vec{v} - gT \mathbf{\hat{j}} + \vec{v}^2 = \frac{1}{9} v^2 + g^2 T^2$$

If we just wanted to know the speed of the combined object at distance $D (= \frac{1}{2} gT^2)$ below the stick’s top, we could use energy methods instead to describe the state of the system after the collision is over.

1st momentum conservation

$$\vec{P}_i = m \vec{v}_i = \vec{P}_c = (m + 2m)\vec{V}_c$$

$$\vec{V}_c = \frac{1}{3} \vec{v}$$

2nd use energy conservation

$$E_f = E_i \quad \text{because} \quad W_{\text{other}} = 0$$

$$\frac{1}{2} M \vec{V}_f^2 + Mg_0 \frac{1}{2} M \vec{v}^2 + Mgy_0$$

$$\frac{1}{2} \vec{V}_f^2 + g(-D) = \frac{1}{2} (\frac{1}{3} \vec{v})^2 + g(0) = \vec{V}_f^2 = \frac{1}{9} v^2 + 2gD$$

**Center of Mass of a System**

We define a quantity called the center of mass $\vec{R}_{cm}$ of a system of $\{m_n\}$ particles located at position vectors $\{\vec{r}_n\}$.

To the “outside” world, it looks like all the system’s mass is concentrated at $\vec{R}_{cm}$:

$$\sum m_n \vec{r}_n = (M_{\text{tot}}) \vec{R}_{cm} \quad \text{where} \quad M_{\text{tot}} = \sum m_n = m_1 + m_2 + \ldots$$

A vector equation! → $(M_{\text{tot}}) X_{cm} = \sum (m_n X_n)$

$(M_{\text{tot}}) Y_{cm} = \sum (m_n y_n)$

Caution: $(M_{\text{tot}})_x$ & $(M_{\text{tot}})_y$ have no meaning

By taking time derivative:

$$\sum (m_n \vec{v}_n) = \vec{p}_{\text{system}} = M_{\text{tot}} \vec{V}_{cm}$$
SYMMETRY AND CENTER OF MASS

If an object (or a portion of an object) has a line of symmetry, the center of mass (cm) will lie on it.

Insight: One can concentrate the entire mass of an object at its center of mass in Statics and in some Dynamics.

CENTER-OF-MASS MOTION

Because the sum of internal forces acting between particles in a system is zero, the motion of the center of mass is only affected by forces arising from interactions of particles in a system with agents external to the system:

\[ \sum \vec{F}_{\text{system}} = \vec{F}_{\text{net external}} + \vec{F}_{\text{net internal}} = \vec{F}_{\text{net ext}} + 0 = \vec{F}_{\text{net ext}} \]

This yields a dynamical equation for center-of-mass motion:

\[ \sum \vec{F}_{\text{external}} = \frac{d\vec{P}}{dt} = \frac{d(M_{\text{tot}} \vec{V}_{CM})}{dt} = M_{\text{tot}} \frac{d\vec{V}_{CM}}{dt} = M_{\text{tot}} \vec{a}_{CM} \]

Demo: CM of irregularly shaped object.

Video: Center of mass motion of objects connected by springs

Web: Parabolic trajectory of extended objects

http://www.surendranath.org/Applets/Dynamics/CMECMApplet.html
http://techtv.mit.edu/videos/3052-span-classhighlightcenterspan-span-classhighlightofspan-mass-trajectory

ROCKET MOTION

For a rocket in deep space, its acceleration due to firing its engines is an example of conservation of linear momentum, because with no external forces

\[ \sum \vec{F}_{\text{external}} = 0 \rightarrow \vec{J}_{\text{net}} = 0 \rightarrow \vec{P}_{i} = \vec{P}_{f} \]

If it starts from rest, \( \vec{P}_{xi} = 0 = \vec{P}_{xf} \) (at all times!)
ROCKET AND EXHAUST CENTERS OF MASS

subscript $e$ for exhaust $\mathbf{P}_e = M_{\text{ext}}(\mathbf{V}_{CM}) = 0$

\[ t_1 \frac{dM}{dt} = m_e(t_1) \quad \mathbf{V}_e(t_1) = \mathbf{V}_{CM}(t_1) \]

subscript $R$ for Rocket $M_R(t) = M_{\text{init}} - m_e(t_1)$

\[ \mathbf{V}_R(t_1) = \mathbf{V}_{CM}(t_1) \]

\[ \mathbf{P}_e = m_e(t) \mathbf{V}(t)_{CM} \]

\[ \mathbf{P}_R = M_R(t) \mathbf{V}_R(t) \]

\[ \mathbf{P}_{\text{tot}} = \mathbf{P}_e + \mathbf{P}_R \]

\[ \mathbf{P}_{\text{tot}} = M_{\text{tot}} \mathbf{V}_{CM} \]

\[ \mathbf{P}_e = \mathbf{P}_{ex} + \mathbf{P}_{Rx} = |\mathbf{P}_R| - |\mathbf{P}_e| \]

Demo: S&T Physics Rocket Tricycle