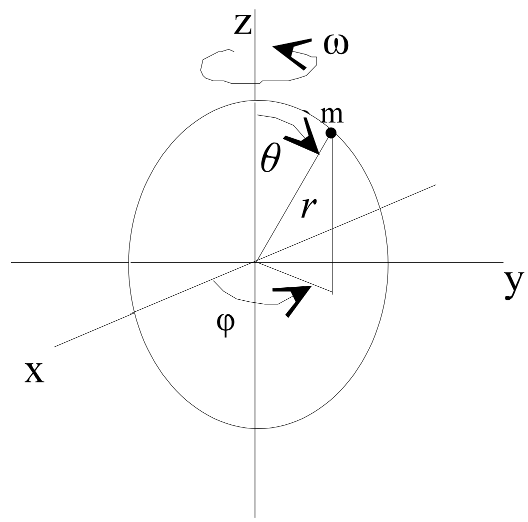
Goldstein Problem 2.17 (3rd ed. # 2.18)

The geometry of the problem:

A particle of mass *m* is constrained to move on a circular hoop of radius *a* that is vertically oriented and forced to rotate about the vertical symmetry axis with angular frequency ω. The only external force is that of gravity. The symmetry of this problem strongly suggests the use of spherical coordinates (r, θ, φ), defined in the figure, as the generalized coordinates: r is the radial displacement of the particle from the origin, θ is the polar angle, and φ is the azimuthal angle. You could also use cylindrical coordinates (ρ, φ, z), where $\rho^2 = x^2 + y^2$, but the analysis is then a little more complicated.



Goldstein 2-17 (3^{rd} ed. #2.18)

The Lagrangian of this system is essentially that of the spherical pendulum, which we worked out in Problem GPS 1.19. One difference is that here we will let the polar (z) axis point upwards (see the figure) so that (1) the polar angle θ differs by π from the earlier definition and (2) the potential energy V will be opposite in sign to that of the earlier problem. The radial coordinate r is, of course, constant and equals the radius of the hoop a. Hence, r is not a generalized coordinate, i.e., there is no radial degree of freedom. The other principal difference between this problem and the spherical pendulum is that the azimuthal angle φ is not a generalized coordinate in this problem because $\dot{\varphi}$ is prescribed as $\dot{\varphi} = \omega$. As a result, $\varphi = \omega t + \varphi_0$, and we lose another degree of freedom. Thus, we see that this system has only one degree of freedom that is most conveniently described by the angle θ.

The kinetic energy of the mass point is

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \ . \tag{1}$$

In the spherical coordinate system we have

$$x = a \sin\theta \cos\phi \,, \tag{2}$$

$$y = a \sin\theta \sin\phi \,, \tag{3}$$

$$z = a \cos\theta \,, \tag{4}$$

from which it follows that

$$\dot{x} = a[\dot{\theta} \cos\theta \cos\phi - \dot{\phi} \sin\theta \sin\phi] \,, \tag{5}$$

$$\dot{y} = a[\dot{\theta} \cos\theta \sin\phi + \dot{\phi} \sin\theta \cos\phi] \,, \tag{6}$$

$$\dot{z} = -a\dot{\theta} \sin\theta \,. \tag{7}$$

After substituting Eqs.(5)–(7) into Eq.(1) we obtain

$$T = \frac{1}{2} m a^2 [\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta] \ . \tag{8}$$

Since the z axis points up in our coordinate system, the potential energy is

$$V = mgz = mga \cos\theta \,. \tag{9}$$

Thus, the Lagrangian of the system is

$$L = \frac{1}{2} m a^2 [\dot{\theta}^2 + \omega^2 \sin^2\theta] - mga \cos\theta \,, \tag{10}$$

where we have replaced $\dot{\phi}$ by ω.

The θ Equation of Motion:

We have

$$\frac{\partial L}{\partial \theta} = ma^2\omega^2 \sin\theta\cos\theta + mga\sin\theta\,, \tag{11}$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}\,. \tag{12}$$

Thus, the only equation of motion is

$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta - (g/a)\sin\theta = 0. \tag{13}$$

A first integral of this EOM will yield a constant of the motion. [Aside: Constants of the motion can be thought of as expressions involving generalized coordinates and generalized velocities in some combination that is a constant. Thus, although $\dot{\phi}$ is constant, it isn't really a constant of the motion because φ isn't a generalized coordinate in this problem. Also, the z angular momentum, $ma^2\omega\sin^2\theta$, is not constant unless θ is constant.]

We can integrate Eq.(13) in the usual way by multiplying it by $\dot{\theta}dt$ and rearranging it, successively, into the forms

$$\frac{1}{2}d\dot{\theta}^2 - \omega^2\sin\theta\cos\theta d\theta - (g/a)\sin\theta d\theta = 0\ \ , \tag{14}$$

$$d\dot{\theta}^2 - \omega^2 d\sin^2\theta + 2(g/a)d\cos\theta = 0\ \ , \tag{15}$$

and

$$d[\dot{\theta}^2 - \omega^2\sin^2\theta + 2(g/a)\cos\theta] = 0\ \ . \tag{16}$$

From the result in Eq.(16) we see that the quantity in brackets is a constant of the motion. Does this quantity have a simple interpretation? To see, let's first multiply it by $\frac{1}{2}ma^2$ to obtain

$$\frac{1}{2}ma^2[\dot{\theta}^2 - \omega^2\sin^2\theta] + mga\cos\theta = C\,, \tag{17}$$

where C is a constant. Although each of the terms in this equation is either the total potential energy or part of the kinetic energy of the system, the combination is clearly *not* the total energy. (The minus sign prevents it.) Another possibility to consider is the energy function h, defined here as

$$h(\theta,\dot{\theta}) = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L\ \ . \tag{18}$$

After substituting Eqs.(10) and (12) into Eq.(18) we obtain

$$h(\theta, \dot{\theta}) = \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\omega^2 \sin^2\theta + mga\cos\theta \quad , \tag{19}$$

which we see is the same as Eq.(17). Thus, the energy function h is a constant of the motion, but it is not equal to the energy E, which is not conserved in this problem. You could easily have guessed that h would be conserved if you remembered from class (or easily rederived) the result

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} \ . \tag{20}$$

Because in this problem L does not depend explicitly on time, we know that $\partial L/\partial t = 0$, and thus h is a constant. Incidentally, we can also easily find out how the energy varies by noting that

$$E = h + ma^2\omega^2 \sin^2\theta \ , \tag{21}$$

which leads directly to the result

$$\frac{dE}{dt} = 2ma^2\omega^2 \sin\theta\cos\theta\dot{\theta} \ . \tag{22}$$

Thus, energy is conserved only when the particle is stationary on the hoop, i.e., when $\dot{\theta} = 0$. (Note that special values of θ do not imply energy conservation because these values are not maintained in time unless $\dot{\theta} = 0$.)

Conditions For Stationary Solutions:

The particle will be stationary on the hoop (although not in space) when $\dot{\theta} = 0$ and $\ddot{\theta} = 0$. If we use the latter condition with the EOM, Eq.(13), we see that aside from the special points at $\theta = 0,\ \pi$, stationary solutions are only possible if the following equation holds

$$\omega^2\cos\theta + (g/a) = 0 \ . \tag{23}$$

Because ω^2, g, and a are all positive, stable stationary solutions are possible only for $\pi/2 < \theta \leqslant \pi$, i.e., on the lower half of the hoop. We can rearrange Eq.(23) to see what value of ω is needed to produce a stationary solution for different values of θ:

$$\omega^2 = -(g/a\cos\theta) \ . \tag{24}$$

The minimum value of $\cos\theta$ is -1 at $\theta = \pi$. The corresponding value of ω at this angle is the critical value,

$$\omega_0 = \sqrt{g/a} \ . \tag{25}$$

To have stationary solutions for $\theta \neq \pi$, we clearly must have $\omega > \omega_0$. If the frequency is such that $\omega \leq \omega_0$, the only stable stationary point is at $\theta = \pi$, at the bottom of the hoop. The special stationary point at $\theta = 0$ is unstable; the slightest increase in θ leads to a positive value of $\ddot{\theta}$, driving the particle away from the top of the hoop.

Brief Consideration of Cylindrical Coordinates

In terms of the generalized coordinate z, the Lagrangian is

$$L = \frac{m}{2}\left(\frac{a^2}{a^2 - z^2}\dot{z}^2 + (a^2 - z^2)\omega^2\right) - mgz \ .$$

and the z EOM is

$$\ddot{z} + \frac{z}{a^2 - z^2}\dot{z}^2 + z(a^2 - z^2)\omega^2 + (a^2 - z^2)g = 0 \ .$$

If we put $\ddot{z} = 0$ and $\dot{z} = 0$ in this EOM, the condition for a stationary solution is

$$(a^2 - z^2)[z\omega^2 + g] = 0 \ ,$$

which leads to the same conclusions as above. You can also show that

$$h(z, \dot{z}) = \frac{m}{2}\left(\frac{a^2}{a^2 - z^2}\dot{z}^2 - (a^2 - z^2)\omega^2\right) + mgz$$

is a constant of the motion.