Problem 12-3 Reif

(a) With the electric field $E$ acting only in the $x$ direction, the equation of motion for the ion is

$$m \frac{dv_x}{dx} = eE,$$

where $e$ is the electric charge on the ion. Next, integrate twice to find $x$, the distance traveled in time $t$. First we get

$$v_x = \frac{eE}{m} t + v_x(0),$$

and with $v_x = dx/dt$ and $x(0) = 0$, we then find

$$x = \frac{eE}{2m} t^2 + v_x(0)t.$$

Now we take the mean value of this equation, assuming that $v_x(0) = 0$ due to randomization of the initial values by collisions. Note that the mean value $v_x(0)$ is an average over $v_x$ values at a single point in time, so it is qualitatively different from the averaging needed to obtain $\bar{x}$ and $\bar{t^2}$, shown just below. Let’s do the averaging in two steps. First, the velocity averaging gives us

$$x = \frac{eE}{2m} t^2.$$

Then from averaging over travel times, we obtain

$$\bar{x} = \frac{eE}{2m} \bar{t^2},$$

where

$$\bar{t^2} = \int_0^{\infty} t^2 \exp\left(-\frac{t}{\tau}\right) \frac{dt}{\tau},$$

and $\exp(-t/\tau)dt/\tau$ is the probability that the ion travels a time $t$ to $t+dt$ before its next collision. The mean time between collisions $\tau$ is defined as $w^{-1}$, where $w$ is the mean collision frequency for an ion with background gas molecules. An easy way to do the required integral is to introduce a dimensionless variable $y = t/\tau$. Then we have

$$\bar{t^2} = \tau^2 \int_0^{\infty} y^2 \exp(-y)dy.$$

The integral has the form shown in Eq. (A.3.3) with $n = 2$. The answer is $2!$. Hence we obtain

$$\bar{t^2} = 2\tau^2,$$

and finally we have

$$\bar{x} = \frac{eE}{m} \tau^2.$$
(b) To determine the fraction \( F \) of cases in which an ion travels a distance \( x < \bar{x} \), it is convenient to use the probability density for collision lifetimes \( P(t) = \exp(-t/\tau) / \tau \). If \( t_\bar{x} \) is the time needed for an ion to travel a distance \( \bar{x} \), then the integral of \( P(t) \) from 0 to \( t_\bar{x} \) will equal the fraction of ions whose travel times are less than \( t_\bar{x} \), which equals the fraction traveling a distance less than \( \bar{x} \). So what does \( t_\bar{x} \) equal? We can answer this using Eq.(4) and solving for \( t \) to find

\[
t = \sqrt{\frac{2m}{e\xi}} x. \tag{10}
\]

We then replace \( x \) by \( \bar{x} \) using Eq. (8) to obtain

\[
t_\bar{x} = \sqrt{2} \tau. \tag{11}
\]

With this value the desired fraction is easy to obtain as

\[
F = \int_0^{\sqrt{2} \tau} \exp\left(-\frac{t}{\tau}\right) \frac{dt}{\tau} = 1 - \exp(-\sqrt{2}) = 0.757. \tag{12}
\]