Problem 7-31 Reif

Look at the picture in the book. Pick the coordinate system so the $z$ axis runs from the center of the hole in the enclosure to the center of the disk suspended a distance $L$ from the hole. Each molecule that hits the disk has diffused out of the container with a particular value of $v_z$, the $z$ component of the velocity. When this molecule hits the disk, the disk experiences a net momentum transfer of $2mv_z$. The momentum transfer to the disk per unit time by molecules with a particular value of $v_z$ is given by the product of $2mv_z$ with the number of molecules hitting the disk per unit time with velocity in the narrow range $v$ to $v+dv$. This number of molecules hitting per unit time is equal to $A\Phi(v)dv$, where $\Phi(v)dv$ is the differential flux of molecules leaving the enclosure and $A$ is the area of the hole. (The area $A$ controls the number leaving and, hence, the number hitting.) A momentum transfer per unit time is a force. Let’s denote the contribution to the force exerted on the disk by molecules with velocity in the range $v$ to $v+\delta v$ as $\delta\mathbf{F}(v)$. Thus, we have

$$d\mathbf{F}(v) = 2mv_z\Phi(v)dv,$$

where the differential flux is given by

$$\Phi(v)dv = nv_z\hat{f}(v)dv,$$

and $\hat{f}(v)$ is the 3-d Maxwellian velocity probability density

$$\hat{f}(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{mv_z^2}{2kT}\right].$$

To get the total force we must integrate Eq.(1) over the correct velocity range. All positive values of $v_z$ will contribute, but the $v_x$ and $v_y$ integrations must be restricted to cover only those values that permit a molecule to hit the disk. The easiest way to do this is to work in spherical coordinates $v, \theta$, and $\phi$, where $\theta$ is the polar angle measured from the $z$ axis and $\phi$ is the azimuthal angle measured from the $x$ axis in the $x-y$ plane. Noting that $v_z = v\cos \theta$ and that $dv = v^2 dv \sin \theta d\theta d\phi$, from Eqs.(1) and (2) we may write the total force as

$$\mathbf{F} = 2nmA\int_0^\infty dv v^4 \hat{f}(v) \int_0^\Theta \cos^2 \theta \sin \theta d\theta \int_0^{2\pi} d\phi,$$

where $\Theta$ is the maximum value of $\theta$ for which a molecule just hits the outer edge of the disk,

$$\Theta = \tan^{-1}\left(\frac{R}{L}\right) = \sin^{-1}\left(\frac{R}{\sqrt{R^2 + L^2}}\right).$$

The $\phi$ integral is obviously $2\pi$. The other two integrals are

$$\int_0^\Theta \cos^2 \theta \sin \theta d\theta = \frac{-1}{3} \int_0^\Theta d\cos^3 \theta = \frac{1 - \cos^3 \Theta}{3},$$

where $\Theta = \tan^{-1}\left(\frac{R}{L}\right) = \sin^{-1}\left(\frac{R}{\sqrt{R^2 + L^2}}\right).$
and
\[ \int_0^\infty dv v^4 \tilde{f}(v) = \left( \frac{m}{2\pi kT} \right)^{3/2} \int_0^\infty dv v^4 \exp\left(-\frac{mv^2}{2kT}\right) = \frac{3}{4\pi} \left( \frac{kT}{m} \right). \] (7)

With these results, Eq. (4) becomes
\[ F = nkTA(1 - \cos^3 \Theta), \] (8)

and if we use the ideal gas law in the form \( \bar{p} = nkT \) to replace \( nkT \), we finally obtain the force on the disk as
\[ F = \bar{p}A(1 - \cos^3 \Theta). \] (9)

In the limit that \( R/L << 1 \), the angle \( \Theta \) becomes very small, and we may use a Taylor series expansion of the cosine to obtain
\[ \cos^3 \Theta = \left( 1 - \frac{1}{2} \Theta^2 + \cdots \right)^3 = 1 - \frac{3}{2} \Theta^2 + \cdots, \] (10)

and since, in this limit,
\[ \Theta \approx \frac{R}{L}, \] (11)

we finally arrive at
\[ F = \frac{3}{2} \bar{p}A \left( \frac{R}{L} \right)^2. \] (12)

If the molecules stuck to the disk, the momentum transfer would be half of the value for an elastic collision, so the force would now be \( 1/2 \) of the values shown in Eqs. (8), (9), and (12).