Problem 7.20 Reif

(a) To compute this average, use the 3-d Maxwellian speed probability density, \( R(v) \),

\[
R(v) = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} v^2 \exp\left[ -\frac{mv^2}{2kT} \right].
\]  

(1)

Recall that \( R(v)dv \) is the probability of finding a molecule with a speed in the range \( v \) to \( v + dv \). The mean value of \( \frac{1}{v} \) is defined as

\[
\langle \frac{1}{v} \rangle = \int_0^\infty \frac{1}{v} R(v)dv = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} \int_0^\infty v \exp\left[ -\frac{mv^2}{2kT} \right] dv.
\]  

(2)

To do the integral, make a simple variable substitution, \( u^2 = \frac{mv^2}{2kT} \), and also note \( udu = \frac{mvdv}{2kT} \). Equation (2) then transforms into

\[
\langle \frac{1}{v} \rangle = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} \frac{kT}{m} \int_0^\infty 2u \exp\left[ -u^2 \right] du = \left( \frac{2m}{\pi kT} \right)^{1/2} \int_{u=0}^{u=\infty} d \exp\left[ -u^2 \right],
\]  

(3)

which finally yields

\[
\langle \frac{1}{v} \rangle = \left( \frac{2m}{\pi kT} \right)^{1/2}.
\]  

(4)

If we compare this result with \( \frac{1}{\bar{v}} \), where the mean speed is \( \bar{v} = (8kT/\pi m)^{1/2} \), we see that

\[
\langle \frac{1}{v} \rangle = \frac{4}{\pi} \frac{1}{\bar{v}}.
\]  

(5)

There is about a 27% difference between the two quantities.

(b) The translational kinetic energy \( \epsilon \) is uniquely determined by the molecular speed \( v \), \( \epsilon = \frac{mv^2}{2} \). Thus the fraction of molecules with speed in the range \( v \) to \( v + dv \) must be the same as the fraction with energy in the range \( \epsilon \) to \( \epsilon + d\epsilon \). Let’s call this fraction \( \psi(\epsilon)d\epsilon \), and we thus have

\[
\psi(\epsilon)d\epsilon = R(v)dv,
\]  

(6)

where \( R(v) \) is the 3-d Maxwellian speed probability density defined above. Now we simply convert the right side of Eq.(6) into energy terms by substituting

\[
v^2 = 2\epsilon/m,
\]  

(7)

and

\[
dv = (2m\epsilon)^{-1/2}d\epsilon,
\]  

(8)

to obtain

\[
\psi(\epsilon)d\epsilon = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} \frac{2\epsilon}{m} \exp\left[ -\frac{\epsilon}{kT} \right] \frac{d\epsilon}{\sqrt{2m\epsilon}},
\]  

(9)

which simplifies to

\[
\psi(\epsilon)d\epsilon = 2\pi \left( \frac{1}{\pi kT} \right)^{3/2} \sqrt{\epsilon} \exp\left[ -\frac{\epsilon}{kT} \right] d\epsilon.
\]  

(10)