Problem Set3 #1

(a) Problem 7.17 asks for the fraction of molecules whose $x$ velocity component lies between $-\bar{v}$ and $+\bar{v}$. Call this fraction $P_x$. Then we have

$$P_x = \int_{-\bar{v}}^{\bar{v}} f_x(v_x) dv_x ,$$

where $f_x(v_x)$ is the 1-d Maxwellian probability density for the $x$ velocity component,

$$f_x(v_x) = \left( \frac{m}{2\pi kT} \right)^{1/2} \exp\left[-\frac{mv_x^2}{2kT}\right] .$$

The most probable speed is $\bar{v} = \sqrt{2kT/m}$, and if we introduce the dimensionless variable $x = v_x/\bar{v}$, Eq.(1) can be easily rewritten in terms of the error function erf($x$)

$$P_x = \frac{1}{\sqrt{\pi}} \int_{-1}^{1} \exp(-x^2) dx = \frac{2}{\sqrt{\pi}} \int_{0}^{1} \exp(-x^2) dx = \text{erf}(1) = 0.843 .$$

The fraction of molecules $\tilde{P}$ whose $x, y, z$ velocity components simultaneously lie between $-\bar{v}$ and $+\bar{v}$ is the three dimensional generalization of Eq.(1)

$$\tilde{P} = \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{v}}^{\bar{v}} \int_{-\bar{v}}^{\bar{v}} f(v) dv ,$$

where $f(v)$ is the 3-d Maxwellian velocity probability density

$$f(v) = \left( \frac{m}{2\pi kT} \right)^{3/2} \exp\left[-\frac{mv^2}{2kT}\right] = f_x(v_x)f_y(v_y)f_z(v_z) .$$

Because of the independence of each direction of motion, Eq.(4) immediately factors into the product

$$\tilde{P} = \tilde{P}_x \tilde{P}_y \tilde{P}_z = [\text{erf}(1)]^3 = 0.599 .$$

(b) The fraction of molecules with speeds less than $\bar{v}$ is given by

$$\tilde{f} = \int_{0}^{\bar{v}} R(v) dv ,$$

where $R(v)$ is the normalized Maxwellian molecular speed probability distribution

$$R(v) = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} v^2 \exp\left[-\frac{mv^2}{2kT}\right] .$$

If we introduce the dimensionless variable $u = v/\bar{v}$, similar to $x$ above, we can immediately put Eq.(7) into the form

$$\tilde{f} = \frac{4}{\sqrt{\pi}} \int_{0}^{1} u^2 \exp(-u^2) du ,$$

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which is, unfortunately, not simply in terms of the error function. This integral

can be readily computed numerically on most hand calculators, but we can avoid

that exercise by instead doing an integration by parts to analytically simplify

Eq.(9). First, realize that

\[ u^2 \exp(-u^2)du = \frac{1}{2}ud\exp(-u^2) \quad . \]  

(10)

With Eq.(10), Eq.(9) becomes

\[ \tilde{f} = -\frac{2}{\sqrt{\pi}} \int_0^1 ud\exp(-u^2) = -\frac{2}{\sqrt{\pi}} \left\{ \left[u\exp(-u^2)\right]_0^1 - \int_0^1 \exp(-u^2)du \right\} , \]  

(11)

and after evaluating the term in square brackets at the upper and lower limits,

we see that

\[ \tilde{f} = \text{erf}(1) - \frac{2e^{-1}}{\sqrt{\pi}} = 0.428 . \]  

(12)

By comparing Eq.(6) with (12) it is obvious that \( \tilde{f} < \tilde{P} \). The explanation

for this is essentially geometric: \( \tilde{f} \) counts molecules whose speed lies within a

sphere of radius \( \tilde{v} \) centered at the origin; \( \tilde{P} \) counts molecules whose velocity

components lie within a cube of side length \( 2\tilde{v} \), also centered at the origin. The

sphere fits inside the cube, but in the corners of the cube beyond the sphere the

molecular \textit{speeds} are greater than \( \tilde{v} \). Thus, \( \tilde{P} \) counts more molecules than \( \tilde{f} \).