Problem 1.22 (Reif) Modified

We are considering a 1-d random walk with $N$ steps. The probability that a step has a displacement between $s$ and $s + ds$ is given by

$$w(s) \, ds = l^{-1} \exp(-s/l) \, ds .$$

(1)

The possible values for $s$ span the range $0$ to $\infty$. The total displacement $x$ of a particular walk is

$$x = \sum_{i=1}^{N} s_i .$$

(2)

(a) Find the mean total displacement $\bar{x}$. Taking the mean of Eq.(2) gives

$$\bar{x} = \sum_{i=1}^{N} \frac{s_i}{N} = N \bar{s} = N l .$$

(3)

It might not be obvious from the form of Eq.(1) that $\bar{s} = l$, so let’s verify it directly. We have

$$\bar{s} = \int_{0}^{\infty} s w(s) \, ds = l^{-1} \int_{0}^{\infty} s \exp(-s/l) \, ds .$$

(4)

Now let

$$y = s/l ,$$

(5)

from which it follows that

$$s = y l ,$$

(6)

and

$$ds = ldy .$$

(7)

With Eqs.(5)-(7), we can rewrite Eq.(4) as

$$\bar{s} = l \int_{0}^{\infty} ye^{-y} \, dy = l .$$

(8)

The integral is evaluated in Appendix A.3 of Reif (or integrate by parts).

(b) Find the dispersion $(x - \bar{x})^2$: First, we square Eq.(2) to get

$$x^2 = \sum_{i=1}^{N} s_i \sum_{j=1}^{N} s_j = \sum_{i=1}^{N} s_i^2 + \sum_{i=1}^{N} \sum_{j \neq i}^{N} s_j s_i ,$$

(9)

and then we take the mean of Eq.(9) to get

$$\bar{x^2} = \sum_{i=1}^{N} \frac{s_i^2}{N} + \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{s_j s_i}{N} .$$

(10)
When \(i\) and \(j\) are different, \(s_j\) and \(s_i\) are statistically independent (or uncorrelated), and \(\overline{s_j s_i} = \overline{s_i s_j} = \overline{s^2} = l^2\). We can then reduce Eq.(10) to
\[
\overline{x^2} = N\overline{s^2} + N(N - 1)l^2, \tag{11}
\]
and with the use of Eq.(3) the dispersion becomes
\[
\overline{(x - \overline{x})^2} = \overline{x^2} - \overline{x}^2 = N\overline{s^2} + N(N - 1)l^2 - (Nl)^2 = N(\overline{s^2} - l^2) = N(\overline{s} - l)^2. \tag{12}
\]
You can also find this result in Eq.(1.9.12) of Reif by following the development that starts with Eq.(1.9.5). To finish the problem all we need is to evaluate \(\overline{s^2}\)
\[
\overline{s^2} = \int_0^\infty s^2 w(s) \, ds = \ln^{-1} \int_0^\infty s^2 \exp(-s/l) \, ds \tag{13}
\]
Change variables as in Eq.(5)-(7) to find
\[
\overline{s^2} = l^2 \int_0^\infty y^2 \exp(-y) \, dy = 2l^2. \tag{14}
\]
Again, the integral is evaluated in Appendix A.3 of Reif (or integrate by parts). Now we simply substitute this result into the appropriate portion of Eq.(12) to get
\[
\overline{(x - \overline{x})^2} = N(\overline{s^2} - l^2) = N(2l^2 - l^2) = Nl^2, \tag{15}
\]
for our final answer.