**SIMPLE HARMONIC MOTION**

**Learning Objectives**
After you complete the homework associated with this lecture, you should be able to:

- Use the formula for the position of a simple harmonic oscillator (SHO) to analyze its position, velocity and acceleration at any time;
- Use system information to determine the values of parameters for to describe simple harmonic motion;
- Describe the cyclic transformations of kinetic and potential energies, including their mathematical property of being 90° out of phase with respect to one another.

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Consider a mass $m$ on a frictionless surface that is connected to a spring with spring constant $k$.

$F_{spring}$ is opposite to the displacement $\vec{D}$

$F_{spring}$ is a linear restoring force

$\Rightarrow \quad F_x = -kx$

- $F_x$ is negative if $x$ is positive (extension)
- $F_x$ is positive if $x$ is negative (compression)

Using Newton’s Second Law for the spring

$ma_x = F_x$

$m \frac{d^2x}{dt^2} = -kx$

$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$

This equation is in the form of: $\frac{d^2x}{dt^2} = -\omega^2x$

where $\omega = [k/m]^{1/2}$ is the angular frequency.
Equation for SHO: \[ \frac{d^2x}{dt^2} = -\omega^2 x \]

The general solution to this equation is known:
\[ x(t) = A \cos(\omega t + \phi) \]

- \( A \) and \( \phi \) are the two “constants of integration” that arise from the solution of a second-order differential equation.
- They are determined by the initial conditions imposed on the motion \( x(t) \)

**Amplitude \( A \) of Simple Harmonic Oscillator**

For an SHO
\[ x(t) = A \cos(\omega t + \phi) \]

- Because the range of cosine is \([-1, +1]\]
  \( A \) = the amplitude of the motion of \( x(t) \)
  \[ \Rightarrow |x(t)| \leq A \]
  \[ \Rightarrow x_{\text{max}} = +A \quad \& \quad x_{\text{min}} = -A \]

**Phase Constant \( \phi \) Set by Initial Condition**

How do we describe \( x(t) = A \cos(\omega t + \phi) \) motions that have different starting points \( x_0 \equiv x(t=0) = x(0) \)?

**Amplitude and Force in SHO**

<table>
<thead>
<tr>
<th>Case 1 at ( t=0 )</th>
<th>Case 2 at ( t=0 )</th>
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- \( x(t = 0) = A \cos (0 + \phi) = A \cos(\phi) \)
  \[ \Rightarrow \cos(\phi) = x(0)/A \]
  \[ \Rightarrow \text{phase constant } \phi \text{ determined by initial value } x_0 \equiv x(t=0) \]
x(t) = A \cos(\omega t + \phi)

Initial condition: x(0) = A \cos(\phi)

For example:

**Case 1:** mass released at x = -A at t = 0,
\[
\cos(\phi_1) = x(0)/A = -A/A = -1 \quad \Rightarrow \quad \phi_1 = \pi
\]

**Case 2:** mass released at x = +A at t = 0,
\[
\cos(\phi_2) = x(0)/A = +A/A = +1 \quad \Rightarrow \quad \phi_2 = 0
\]

**Why Is \( \omega \) an “Angular Frequency”?**

- How long does it take for a complete cycle?
  \[
x(0) \rightarrow +A \rightarrow -A \rightarrow 0 \rightarrow +A
\]
  Ans: Time \( T \) (the period) for \( x(t) = A \cos(\omega t + \phi) \) to go through one cycle

  \[
  \Delta(\omega t + \phi) = \Delta(\omega t) = \omega(\Delta t)_{\text{cycle}} = \omega T = 2\pi
  \]

  \( \omega = 2\pi / T = 2\pi \text{ radians per time period} \)

  (an angular frequency!)

  Tells how “fast” \( \cos(\omega t + \phi) \) goes through a cycle.

**Effects of Mass and Amplitude on Period**

\[
\omega T = 2\pi
\]

\[
T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}} = \frac{2\pi}{\sqrt{\frac{m}{k}}}
\]

As \( m \) increases, \( T \) increases

But amplitude \( A \) does not appear \( \rightarrow \) \( A \) does not effect \( T \)!!

**DEMO:** Vertical and horizontal springs showing effect of \( m \) and \( A \)

**Hanging Weight on Spring**

\[
m \frac{d^2s}{dt^2} = F_{\text{spring},s} + F_{\text{grav},s} = -ks + mg
\]

\[
\frac{d^2s}{dt^2} = -\left(\frac{k}{m}\right)\left(\frac{s - mg}{k}\right)
\]

change of variables: \( x = s - mg/k \rightarrow dx = ds \)

also \( \omega^2 = \frac{k}{m} = \frac{d^2x}{dt^2} = -\omega^2 x \) = same SHO freq

SHO oscillations about new equilibrium position of \( x_{eq} = 0 \) with ALL grav effects already accounted for.
**Velocity of Motion**

- $v_x(t) = \frac{dx}{dt} = \frac{d[A \cos(\omega t + \phi)]}{dt}$
  
  \[ v_x(t) = -A \sin(\omega t + \phi) \]

- $x = \pm A$ at time $t_m$ such that $\cos(\omega t_m + \phi) = \pm 1$
  
  \[ (\omega t_m + \phi) = 0 \text{ or } \pi \]

\[ v_x(t_m) = -\omega A \sin(0 \text{ or } \pi) = 0 \]

$\Rightarrow$ the mass stops and “turns around” when it reaches its maximum displacement

**Velocity in SHO**

- $v_x(t) = -\omega A \sin(\omega t + \phi)$

- When is the speed $|v_x|$ a maximum?
  
  \[ | -\omega A \sin(\omega t + \phi) | = \text{ maximum} \]

\[ |\sin(\omega t + \phi)| = \text{ maximum} = 1 \]

\[ |\cos(\omega t + \phi)| = 0 \Rightarrow x = 0 \]

\[ \Rightarrow \text{ max speed} = \omega A \text{ occurs at equilib } x = 0 \]

**Sinusoidal Energies in Harmonic Motion**

\[ E_{\text{total}} = K(t) + U(t) = \text{ independent of time} \]

\[ K(t) = \frac{1}{2} m v^2(t) = \frac{1}{2} m [ -\omega A \sin(\omega t + \phi) ]^2 \]

\[ = (\frac{1}{2} m \omega^2 A^2) \sin^2(\omega t + \phi) = K_{\text{max}} \sin^2(\omega t + \phi) \]

where $K_{\text{max}} = \frac{1}{2} m (v_{\text{max}})^2 = \frac{1}{2} m (\omega A)^2$

\[ U(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k [ A \cos(\omega t + \phi) ]^2 \]

\[ = (\frac{1}{2} k A^2) \cos^2(\omega t + \phi) = U_{\text{max}} \cos^2(\omega t + \phi) \]

where $U_{\text{max}} = \frac{1}{2} k (x_{\text{max}})^2 = \frac{1}{2} k A^2$
Interplay of Kinetic and Potential Energies

\[ E = K_{\text{max}} \sin^2(\omega t + \phi) + U_{\text{max}} \cos^2(\omega t + \phi) \]

\( K(t) \) and \( U(t) \) are both sinusoidal functions of time, but are \( \frac{1}{2} \pi \) radians (90°) out of phase.

\[ \Rightarrow \quad \text{When } K \text{ is maximum, } U \text{ is minimum} \]

\[ \Rightarrow \quad \text{When } K \text{ is minimum, } U \text{ is maximum} \]

\[ E = K_{\text{max}} + U_{\text{min}} = K_{\text{min}} + U_{\text{max}} \]

\( K_{\text{min}} = U_{\text{min}} = 0 \), BUT they occur at different times!

Example: A block of mass \( M \) is attached to a spring. It executes simple harmonic motion of amplitude \( A \). At what displacement(s) \( X \) from equilibrium does its kinetic energy equal twice its potential energy?

\( \Rightarrow \quad K(X) = 2U(X) \)

We can solve this "elegantly" using our Energy Toolbox:

\[ E(X) = K(X) + U(X) = E_{\text{tot}} = \text{always constant} \]

\[ = K(A) + U(A) = 0 + \frac{1}{2} k A^2 = \frac{1}{2} k A^2 \]

\[ 2U(X) + U(X) = \frac{1}{2} k A^2 \]

\[ 3 \left[ \frac{1}{2} k X^2 \right] = \frac{1}{2} k A^2 \quad \Rightarrow \quad X = \pm \sqrt{\frac{1}{3}} A \]