CONSERVATION OF ANGULAR MOMENTUM

**Learning Objectives**

After you complete the homework associated with this lecture, you should be able to:

- Explain the conditions under which the total angular momentum of a system is conserved.
- Use this to analyze the internal dynamics of rotating systems when the distances from the rotation axis change.
- Apply the conservation of angular momentum to predict the rotational motion of systems before and after collisions and other interactions.

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**Location of Forces and Changes in Momentum**

\[
\frac{d \mathbf{P}}{dt} = \mathbf{F}_{\text{ext}} = \mathbf{F}_1 + \mathbf{F}_2
\]

\[
\frac{d \mathbf{L}}{dt} = \mathbf{\tau}_{\text{ext}} = \mathbf{\tau}_1 + \mathbf{\tau}_2
\]

\[\mathbf{F}_{\text{ext}} \neq 0 \implies \text{linear momentum } \mathbf{P} \text{ changes and } \mathbf{a} \neq 0\]

\[\mathbf{\tau}_{\text{ext}} \neq 0 \implies \text{angular momentum } \mathbf{L} \text{ changes and } \mathbf{a} \neq 0\]

**DEMO:** Bicycle Wheel Gyroscope: \[d \mathbf{L} = \mathbf{\tau}_{\text{ext}} \, dt\]

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**Conservation of Angular Momentum**

- If the net external torque acting on a system is zero (\(\mathbf{\tau}_{\text{net}} = 0\)), then angular momentum is conserved.

\[\frac{d \mathbf{L}}{dt} = \mathbf{\tau}_{\text{net}} = 0 \implies \mathbf{L} \text{ is constant}\]

**Conservation of Angular Momentum!**

\[\mathbf{L}_i = \mathbf{L}_f \text{ if } \mathbf{\tau}_{\text{net}} = 0\]

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**Conservation of Angular Momentum explains many things:**

- Why a figure skater rotates faster when he pulls in his arms to his body.
- Why a collapsing massive star in its death throes rotates faster, producing a pulsar.
Angular Momentum $\vec{L}$ of Skater in a Spin

$$\vec{L} = I \vec{\omega} = (\text{rotational inertia about axis}) \ (\text{angular velocity})$$

No external torques $\Rightarrow \vec{L}_i = \vec{L}_f$

Assume arms massless

$$I_{\text{axis}} = I_{\text{body}} + m r_i^2 + m r_f^2$$

$$\vec{L}_i = \vec{L}_f \Rightarrow (I_{\text{axis,i}} \ \vec{\omega}_i)_z = (I_{\text{axis,f}} \ \vec{\omega}_f)_z$$

$$\vec{\omega}_{\text{f}} = \vec{\omega}_{\text{i}} \ (1 + 2 m d^2 / I_{\text{body}}) \Rightarrow \omega_f > \omega_i$$

Location of Forces and Changes in Momentum

- We know that: if $\sum \vec{F}_{\text{ext}} = 0$, then $\sum \vec{F}_f = \vec{F}_f$.
  
  if $\sum \vec{F}_{\text{ext}} = 0$, then $\vec{L}_i = \vec{L}_f$.

- But, depending upon WHERE forces are applied, we may have $\sum \vec{F}_{\text{ext}} = 0$, but $\sum \vec{r}_{\text{ext}} \neq 0$

or we may have $\sum \vec{F}_{\text{ext}} = 0$, but $\sum \vec{r}_{\text{ext}} \neq 0$

*Angular momentum can be conserved even when linear momentum is not conserved, and vice-versa.*

**Demo:** Prof Gets Sick on Rotating Chair

**Demo:** Rotating Chair and Bicycle Wheel
Angular Momentum in Linear Motion

Even an object traveling in a straight line can have angular momentum:

\[ \mathbf{\ell} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \left( m \mathbf{v}_{CM} \right) \]

or equivalently,

\[ |\mathbf{\ell}| = pr \sin \theta = pr_\perp = mv b \]

Example: A door of mass \( M \) and width \( W \) swings with angular speed \( \omega_i \). A ball with speed \( v_i \) and mass \( m \) strikes the door at incident angle \( \theta \) at distance \( d \) from the hinge. The ball bounces off at a right angle to the door with \( \frac{1}{4} \) its original speed. What is the final angular speed of the door?

When the ball hits, an impulsive external reaction force \( \mathbf{R} \) acts on the door-ball system at the hinge. This produces an external impulse \( \Rightarrow \) linear momentum not conserved. But:

Important: Forces supplied by supporting structures at pivot point have zero moment arms and therefore produce zero torques.

Although \( \mathbf{R} \) is unknown, \( \tau_R = 0 \) \( \Rightarrow \) \( (L_i)_z = (L_f)_z \)

\[
\sum \tau_z = \tau_{Rz} = 0 \quad \Rightarrow \quad \omega_f = \frac{\omega_i}{(\frac{1}{2} M W^2) + \frac{1}{4}mv_i d}
\]