**WORK OF THE PHYSICS KIND**

**Learning Objectives**
After you complete the homework associated with this lecture, you should be able to:

• Mathematically define kinetic energy.
• Calculate the dot product of two vectors.
• Define work in the physics sense, and explain how force and incremental displacements contribute to it.
• State the relationship between the change in an object’s kinetic energy and the work done on it.
• Use the Work-KE theorem to predict how an object’s speed changes when forces act on it over its path of motion.

**KINETIC ENERGY**

• A useful concept in the analysis of motion is Kinetic Energy (KE): \[ K = \frac{1}{2} m v^2 \]
• The greater an object’s speed, the greater its KE.

**CAUTION:**
• By its definition, Kinetic Energy is **NOT** a vector! There is NO such thing as \( K_x \) or \( K_y \).
  You have been hereby WARNED.

**FORCE & CHANGE IN KINETIC ENERGY**

• Forces cause acceleration and thus can cause change in speed \( \rightarrow \) change kinetic energy \( K = \frac{1}{2} m v^2 \).
• Component of force parallel to velocity \( \vec{v} \) changes \( K \).
• Kinetic energy changes are greater the longer the parallel force is applied.

We need a simple method of describing and determining how much of one vector (e.g., \( \vec{F} \)) is parallel to another vector (e.g., \( \vec{v} \)). This is done with the vector dot product.

**DOT PRODUCT OF TWO VECTORS**

The *Dot Product* of two vectors \( \vec{A} \) and \( \vec{B} \) is the *scalar* quantity:

\[ \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta_{AB} = A B \cos \theta_{AB} \]

where

\[ \theta_{AB} = \text{angle between vectors } \vec{A} \text{ & } \vec{B} \]
Equivalent view: \[ \mathbf{A} \cdot \mathbf{B} = AB \cos \Theta_{AB} = A \left[ B \cos \Theta_{AB} \right] = AB \text{dir} \mathbf{A} \]

where \( \text{dir} \mathbf{A} = B \cos \Theta_{AB} \) is the component of vector \( \mathbf{B} \) in the direction of the vector \( \mathbf{A} \).

We’re asking: How much of \( \mathbf{B} \) is in the direction of \( \mathbf{A} \)?

For \( \Theta \gt 90^\circ \), \( \text{dir} \mathbf{A} \lt 0 \)

\[ \mathbf{A} \cdot \mathbf{B} = AB \text{dir} \mathbf{A} \lt 0 \]

**SPECIAL DISPENSATIONS**

For perpendicular vectors, you are allowed to simply write

\[ \mathbf{A} \cdot \mathbf{B} = 0 \]

For similarly directed vectors, you are allowed to simply write

\[ \mathbf{A} \cdot \mathbf{B} = +AB \]

For oppositely directed vectors, you may write

\[ \mathbf{A} \cdot \mathbf{B} = -AB \]

**MATH PROPERTIES OF DOT PRODUCT**

The dot product satisfies the math properties:

\[ \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \] (commutative)

\[ (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} \] (distributive)

\[ \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \] (\( \parallel \)) unit vectors

\[ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0 \] (\( \perp \)) unit vectors

**Dot Product with Components**

The numerical value of the dot product of two vectors can be evaluated using the values of their components in any given axis system.

\[ \mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \]

\[ \mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k} \]

One can show that:

\[ \mathbf{A} \cdot \mathbf{B} = AB \cos \Theta_{AB} = A_x B_x + A_y B_y + A_z B_z \]
A PHYSICIST’S VIEW OF WORK

The dot product can be used to define a quantity, WORK, that accounts for the effect of forces that change kinetic energy, i.e., forces parallel to velocity.

Work done on an object by force \( \mathbf{F} \) as it moves on path \( \mathbf{r}(t) \) from initial position \( \mathbf{r}_i \) to final position \( \mathbf{r}_f \) is:

\[
(W_F)_{i-f} = \int_{r_i}^{r_f} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{\ell} = \int_{r_i}^{r_f} \mathbf{F}(\| \mathbf{v} \| \mathbf{v}) \cdot d\mathbf{\ell} = \int_{r_i}^{r_f} F_{\parallel} d\mathbf{\ell}
\]

where \( d\mathbf{\ell} \) is a small distance step and \( F_{\parallel} \) is the component of force in direction of the instantaneous velocity \( \mathbf{v} \):

→ We have picked out the component of force that causes a change in speed and thus kinetic energy.

WORK - KINETIC ENERGY THEOREM

\[
(W_F)_{i-f} = \int_{r_i}^{r_f} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{\ell} = \int_{r_i}^{r_f} F_{\parallel} d\mathbf{\ell}
\]

• \( F_{\parallel} \) changes the speed → changes Kinetic Energy

∴ Change in KE is related to physicist's Work!

\[
(W_{\text{net}})_{i-f} = \Delta K = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2
\]

Change in KE of object = How much work done on it!

FORCE CAN VARY OVER PATH

A force can change in magnitude and/or direction as the particle moves along a path.

We can break the path into segments \( \Delta \mathbf{\ell}_n \) (i.e., vector steps), over each of which the force is approximately constant

\[
W_F = \sum \mathbf{F}(\mathbf{r}_n) \cdot \Delta \mathbf{\ell}_n = \int_{r_i}^{r_f} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{\ell}
\]

W_{\text{NET}} Is Net Work (not Network)

The net work \( W_{\text{net}} \) done on an object in going from some initial state to some final state is the work done along the path by the net force \( \mathbf{F}_{\text{net}} \) on the object. It is also the sum of “works” done by the individual forces:

\[
(W_{\text{net}})_{i-f} = \sum W_n = \text{sum of individual works}
\]

This conjoined with \( (W_{\text{net}})_{i-f} = \Delta K \) (Work-KE theorem) gives a much simpler method of determining changes in speed, particularly if a non-constant force is involved!!

Video: car commercial “Isn’t it nice when things just work”
**SPECIAL DISPENSATIONS**

1. **Constant Force (both in magnitude & direction)**  
   
   If \( \mathbf{F} \) is constant over a path segment, i.e. \( \mathbf{F}(\mathbf{r}) = \mathbf{F} \) and your diagram shows it, then you are permitted to just write \( W_f = \mathbf{F} \cdot \Delta \mathbf{D} \) or make the corresponding substitution because
   
   \[
   \left( W_f \right)_{i \rightarrow f} = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} \]
   
   where \( \Delta \mathbf{D} \) is the vector displacement for the segment.

2. **Force is always perpendicular to path (whether constant in magnitude or not)**  
   
   If \( \mathbf{F} \) is always perpendicular to the path segment, i.e. \( \mathbf{F}(\mathbf{r}) \perp d\mathbf{r} \) and your diagram shows it, then you are permitted to just write \( W_f = 0 \) or make the corresponding substitution because
   
   \[
   \left( W_f \right)_{i \rightarrow f} = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} = 0
   \]

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**Example:** Determine the change in KE of a block of mass \( m \) pulled a distance \( D \) over a rough horizontal surface with coefficient of friction \( \mu \).

\[
\Delta K = W_{\text{net}} = \sum W_a = W_N + W_p + W_{\text{grav}} + W_f
\]

\[
K_f - K_i = N \cdot \Delta \mathbf{D} + \mathbf{F}_p \cdot \Delta \mathbf{D} + \mathbf{F}_g \cdot \Delta \mathbf{D} + \mathbf{F}_f \cdot \Delta \mathbf{D}
\]

\[
= 0 + PD \cos \theta + 0 + [-(\mu N)D]
\]

To get \( N \):  
\[
\Sigma F_x = N_x + P_x + F_{g_x} + F_f = ma_x
\]

\[
= (+N) + (+P \sin \theta) + (-mg) + 0 = m(0)
\]

**Caution:**  
\( N = (mg - P \sin \theta) \); never assume \( N = mg \)

**Example:** A ball of mass \( m \) is attached to a massless rigid rod of length \( L \). The rod is released from rest in a horizontal orientation. What is its speed at the bottom of its swing?

What is relationship between forces and speed? The Work-KE theorem!

\[
\Delta K = (W_{\text{net}})_{i \rightarrow f}
\]

But outward force \( \mathbf{T} \) produced by the rod varies. What to do?

Don’t lose heart; follow Litany!

2 forces → 2 W terms →  \( \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W_T + W_{\text{grav}} \)
½mv_f^2 - ½mv_i^2 = W_T + W_g

dT and T always perpendicular

\[ W_T = \int_{t_i}^{t_f} \mathbf{T} \cdot \mathbf{d} \mathbf{t} = W(0) = 0 \]

\( F_g \) constant \( \rightarrow \) \( W_{grav} = F_g \cdot D = F_{g \text{ direct grav}} = mg(+L) \)

\( ½mv_f^2 - ½m(0)^2 = W_T + W_g = mgL \)

\( ½mv_i^2 = mgL \) (same as direct drop because \( T \) does no work!!)

**Demo:** Speed of swinging mass through photo gate

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\[ W_{net} = K_f - K_i = ½mv_f^2 - ½mv_i^2 \]

\[ \mathbf{N} + W_T + W_{grav} = ½m(0)^2 - ½m(V)^2 \]

\[ 0 + (-fD) + mg D \cos(90°-0) \]

\[ = 0 - ½m(V)^2 \]

\[ -\mu N D + mg D \sin\theta = -½mV^2 \]

*To get N: \( \Sigma F_y = N_y + F_{y \parallel} + f_y = m a_y \)

\[ = (+N) + (-mg \cos\theta) + 0 = m(0) \]

*Caution: \( N = (mg \cos\theta) \); *never assume* \( N = mg \)

\[ -D \mu (mg \cos\theta) + D mg \sin\theta = -½mV^2 \]

\[ D = \frac{½V^2}{(\mu g \cos\theta - g \sin\theta)} \]

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**Example:** A truck is going at speed \( V \) down a hill that makes angle \( \theta \) with the horizontal. The driver sees a brick wall ahead in the middle of the road. If coefficient of friction between the tires and the road is \( \mu \), at what distance \( D \) ahead of the wall must the driver apply the brakes if she chooses not to hit the wall?

Power is the *rate of work* that is done on an object; i.e., how fast work (an energy) is done on something. This means it is a *scalar* quantity.

The instantaneous power \( P \) supplied by force \( \mathbf{F} \) to an object moving with velocity vector \( \mathbf{v} \) is a scalar that equals the dot product of the velocity \( \mathbf{v} \) with the force vector \( \mathbf{F} \):

\[ P_F = \mathbf{F} \cdot \mathbf{v} \]

Note that \( P \) can be positive or negative.
BIENIEK’S RULES OF POWER

1. If the power acting on object is positive, it speeds up because there is a component of force in the direction of velocity vector; and vice versa.

2. If the power on an object is negative, it slows down because there is a component of force opposite to the velocity.

3. If a force acts perpendicular to velocity, it delivers no power to the object and the speed doesn’t change.

Example: One day you are pressing against a wooden block that is sliding down a vertical wall and decelerating. The force $\vec{F}$ you’re applying to the block has magnitude $F = 50$ N, and is inclined upward with respect to the horizontal by angle $\theta = 20^\circ$. The coefficient of friction between the block and wall is $\mu_k = 0.10$. When the speed of the block is $V = 2$ m/s, what is the power delivered to the block by force $\vec{F}$?

$$P_F = \vec{F} \cdot \vec{V} = F V \cos \theta_{FV}$$

$$= F V \cos (\theta + 90^\circ) = F V \cos (20^\circ + 90^\circ)$$

$$= 50 \times 2 \times (-0.342) = -34.2 \text{ W}$$

Note: Coord system doesn’t come into it.