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Positive Solutions of Boundary Value Problems for Second-order p-Laplacian Difference Equations

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Abstract

This paper is devoted to the existence of at least three positive solutions for a second order p-Laplacian difference equation. The proofs of the main results are based on Bai and Ge's fixed point theorem. As an application, an example is given to illustrate the obtained results.

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1 Introduction

In recent years, there has been a large number of papers interested in proving the existence of positive solutions of the boundary value problems (BVPs) of the difference

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equations since these BVPs have extensive applications, see the papers [1, 2, 4, 5, 7–13] and the references therein. We now discuss briefly several of the appropriate papers on the topic.

Liu, [12] studied the following second order p- Laplacian difference equation with multi-point boundary conditions

$$\begin{cases} \Delta\left[\phi\left(\Delta x(n)\right)\right] + f\left(n, x(n+1), \Delta x(n), \Delta x(n+1)\right) = 0, \ n \in [0, N], \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\eta_i) = A, \\ x(N+2) - \beta_i x(\eta_i) = B. \end{cases}$$

Sufficient conditions to guarantee the existence of at least three positive solutions of the above multi-point boundary value problem were established by using a new fixed point theorem obtained in [6].

Liu, [13] studied the following boundary value problem

$$\begin{cases} \Delta\left[\phi\left(\Delta x(n)\right)\right] + f\left(n, x(n+1), \Delta x(n), \Delta x(n+1)\right) = 0, \ n \in [0, N], \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\eta_i) = 0, \\ x(N+2) - \beta_i x(\eta_i) = B. \end{cases}$$

By using five functionals fixed point theorem [3], Liu obtained the existence criteria of at least three positive solutions.

Motivated by the results above, in this study, we consider the following more generalized BVPs for the second order p-Laplacian difference equation with multi-point boundary conditions

$$\begin{cases} \Delta\left[\phi_p\left(\Delta x(n)\right)\right] + f\left(n, x(n+1), \Delta x(n), \Delta x(n+1)\right) = 0, \ n \in [0, N], \\ ax(0) - b\Delta x(0) = \sum_{i=1}^{m} \alpha_i x(\eta_i), \\ cx(N+2) + d\Delta x(N+1) = \sum_{i=1}^{m} \beta_i x(\eta_i), \end{cases} \tag{1.1}$$

where

- $N > 1$ an integer, $0 < n_1 < \cdots < n_m < N+2$;
- $\alpha_i, \beta_i \geq 0$ for all $i = 1, \ldots, m$;
- f is continuous and positive;
- ϕ_p is called p-Laplacian, $\phi_p(x) = |x|^{p-2}x$ with $p > 1$, its inverse function is denoted by $\phi_q(x)$ with $\phi_q(x) = |x|^{q-2}x$ with $1/p + 1/q = 1$;

- $\sum_{i=r}^{s} x(i) = 0$ if $r,\ s \in \mathbb{Z}$ and $s < r$, where $\mathbb{Z}$ is the integer set, denote $[r, s] = \{r, r+1, \cdots, s\}$ for $r,\ s \in \mathbb{Z}$ with $r \le s$.

Throughout this paper we assume that following conditions hold:

$(C1)$ $f : [0, N] \times [0, +\infty) \times \mathbb{R}^2 \to (0, +\infty)$ is continuous,

$(C2)$ $\alpha_i \ge 0$, $\beta_i \ge 0$ satisfy $\sum_{i=1}^{m} \alpha_i < a$ and $\sum_{i=1}^{m} \beta_i \le c$.

By using the Bai and Ge's fixed point theorem [6], we get the existence of at least three positive solution for the BVP (1.1).

This paper is organized as follows. In Section 2, we give some preliminary lemmas which are key tools for our proof. The main result is given in Section 3. Finally, in Section 4, we give an example to demonstrate our result.

2 Preliminaries

In this section we present some lemmas, which will be needed in the proof of the main result.

Let α and β be nonnegative continuous convex functionals on a cone $\mathcal{P}$, ψ be a nonnegative continuous concave functional on $\mathcal{P}$, and r, m, l be positive numbers with $r > m$, we define the following convex sets

$$\begin{aligned}
\mathcal{P}(\alpha, r; \beta, l) &= \{x \in \mathcal{P} :\ \alpha(x) < r,\ \beta(x) < l\}, \\
\bar{\mathcal{P}}(\alpha, r; \beta, l) &= \{x \in \mathcal{P} :\ \alpha(x) \le r,\ \beta(x) \le l\}, \\
\mathcal{P}(\alpha, r; \beta, l; \psi, m) &= \{x \in \mathcal{P} :\ \alpha(x) < r,\ \beta(x) < l,\ \psi(x) > m\}, \\
\bar{\mathcal{P}}(\alpha, r; \beta, l; \psi, m) &= \{x \in \mathcal{P} :\ \alpha(x) \le r,\ \beta(x) \le l,\ \psi(x) \ge m\}.
\end{aligned}$$

The following assumptions as regards the nonnegative continuous convex functions α, β are used:

$(B1)$ there exists $M > 0$ such that $\|x\| \le M \max\{\alpha(x),\ \beta(x)\}$ for all $x \in \mathcal{P}$;

$(B2)$ $\mathcal{P}(\alpha, r; \beta, l) \ne \emptyset$ for any $r > 0$ and $l > 0$.

To prove our main result, we need the following fixed point theorem due to Bai and Ge in [6].

Lemma 2.1 (See [6]). *Let $\mathcal{P}$ be a cone in a real Banach space $\mathbb{B}$ and let $r_2 \ge \tau > \nu > r_1 > 0$, $l_2 \ge l_1 > 0$. Assume that α and β are nonnegative continuous convex functions satisfying $(B1)$ and $(B2)$, ψ is a nonnegative continuous concave function on $\mathcal{P}$ such that $\psi(x) \le \alpha(x)$ for all $x \in \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2)$ and $T : \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2) \to \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2)$ is a completely continuous operator. Suppose that*

$(B3)$ $\left\{x \in \bar{\mathcal{P}}(\alpha,\tau;\beta,l_2;\psi,\nu):\ \psi(x) > \nu\right\} \neq \emptyset$, $\psi(Tx) > \nu$, *for* $x \in \bar{\mathcal{P}}(\alpha,\tau;\beta,l_2;\psi,\nu)$,

$(B4)$ $\alpha(Tx) < r_1$, $\beta(Tx) < l_1$ *for all* $x \in \bar{\mathcal{P}}(\alpha,r_1;\beta,l_1)$,

$(B5)$ $\psi(Tx) > \nu$ *for all* $x \in \bar{\mathcal{P}}(\alpha,r_2;\beta,l_2;\psi,\nu)$ *with* $\alpha(Tx) > \tau$.

Then T *has at least three fixed points* x_1, x_2 *and* $x_3 \in \bar{\mathcal{P}}(\alpha,r_2;\beta,l_2)$ *with*

$$x_1 \in \mathcal{P}(\alpha,r_1;\beta,l_1),\ x_2 \in \left\{\bar{\mathcal{P}}(\alpha,r_2;\beta,l_2;\psi,\nu):\ \psi(x) > \nu\right\},$$

and

$$x_3 \in \bar{\mathcal{P}}(\alpha,r_2;\beta,l_2) \setminus \left(\bar{\mathcal{P}}(\alpha,r_2;\beta,l_2;\psi,\nu) \cup \bar{\mathcal{P}}(\alpha,r_1;\beta,l_1)\right).$$

Let $h(n)(n \in [0,N])$ be a positive sequence. Consider the following BVP

$$\begin{cases} \Delta\left[\phi_p\left(\Delta x(n)\right)\right] + h(n) = 0,\ n \in [0,N], \\ ax(0) - b\Delta x(0) = \displaystyle\sum_{i=1}^{m} \alpha_i x(\eta_i), \\ cx(N+2) + d\Delta x(N+1) = \displaystyle\sum_{i=1}^{m} \beta_i x(\eta_i). \end{cases} \tag{2.1}$$

Lemma 2.2. *Suppose that* $(C2)$ *holds. If* x *is a solution of BVP* (2.1)*, then there exists unique* $n_0 \in [0,N]$ *such that* $\Delta x(n_0) > 0$ *and* $\Delta x(n_0+1) \leq 0$.

Proof. Suppose x satisfies (2.1). It follows that

$$x(n) = x(0) + \sum_{i=0}^{n-1} \phi_q\left(\phi_p\left(\Delta x(0)\right) - \sum_{s=0}^{i-1} h(s)\right),\ n \in [0,N+2]. \tag{2.2}$$

The BCs in (2.1) imply that

$$ax(0) = b\Delta x(0) + \sum_{i=1}^{m} \alpha_i \sum_{j=0}^{\eta_i-1} \phi_q\left(\phi_p\left(\Delta x(0)\right) - \sum_{s=0}^{j-1} h(s)\right) + x(0)\sum_{i=1}^{m}\alpha_i,$$

and

$$\begin{aligned} & cx(0) + c\sum_{i=0}^{N+1} \phi_q\left(\phi_p\left(\Delta x(0)\right) - \sum_{s=0}^{i-1} h(s)\right) \\ = & \sum_{i=1}^{m} \beta_i \sum_{j=0}^{\eta_i-1} \phi_q\left(\phi_p\left(\Delta x(0)\right) - \sum_{s=0}^{j-1} h(s)\right) - d\phi_q\left(\phi_p\left(\Delta x(0)\right) - \sum_{s=0}^{N} h(s)\right) \end{aligned}$$

$$+x(0)\sum_{i=1}^{m}\beta_i.$$

It follows that

$$x(0)=\frac{1}{a-\sum_{i=1}^{m}\alpha_i}\left(b\Delta x(0)+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(\phi_p\left(\Delta x(0)\right)-\sum_{s=0}^{j-1}h(s)\right)\right),$$

and

$$\begin{aligned}&\frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\left(b\Delta x(0)+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(\phi_p\left(\Delta x(0)\right)-\sum_{s=0}^{j-1}h(s)\right)\right)\\&+c\sum_{i=0}^{N+1}\phi_q\left(\phi_p\left(\Delta x(0)\right)-\sum_{s=0}^{i-1}h(s)\right)\\=&\sum_{i=1}^{m}\beta_i\sum_{j=0}^{\eta_i-1}\phi_q\left(\phi_p\left(\Delta x(0)\right)-\sum_{s=0}^{j-1}h(s)\right)-d\phi_q\left(\phi_p\left(\Delta x(0)\right)-\sum_{s=0}^{N}h(s)\right).\end{aligned}$$

Similarly, we can get that

$$x(n)=x(N+2)-\sum_{i=n}^{N+1}\phi_q\left(\phi_p\left(\Delta x(N+1)\right)+\sum_{s=i}^{N}h(s)\right),\ n\in[0,N+2]. \quad (2.3)$$

The BCs in (2.1) imply that

$$\begin{aligned}x(N+2)\ =\ &\frac{1}{a-\sum_{i=1}^{m}\alpha_i}\Bigg(b\phi_q\left(\phi_p\left(\Delta x(N+1)\right)+\sum_{s=0}^{N}h(s)\right)\\&+a\sum_{i=0}^{N+1}\phi_q\left(\phi_p\left(\Delta x(N+1)\right)+\sum_{s=i}^{N}h(s)\right)\\&-\sum_{i=1}^{m}\alpha_i\sum_{j=\eta_i}^{N+1}\phi_q\left(\phi_p\left(\Delta x(N+1)\right)+\sum_{s=j}^{N}h(s)\right)\Bigg),\end{aligned}$$

and

$$\begin{aligned}&\frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\Bigg[b\phi_q\left(\phi_p\left(\Delta x(N+1)\right)+\sum_{s=0}^{N}h(s)\right)\\&+a\sum_{i=0}^{N+1}\phi_q\left(\phi_p\left(\Delta x(N+1)\right)+\sum_{s=i}^{N}h(s)\right)\end{aligned}$$

$$
\begin{aligned}
& -\sum_{i=1}^{m}\alpha_i\sum_{j=\eta_i}^{N+1}\phi_q\left(\phi_p\left(\Delta x(N+1)\right)+\sum_{s=j}^{N}h(s)\right)\Bigg] \\
= & -\sum_{i=1}^{m}\beta_i\sum_{j=\eta_i}^{N+1}\phi_q\left(\phi_p\left(\Delta x(N+1)\right)+\sum_{s=j}^{N}h(s)\right)-d\Delta x(N+1)).
\end{aligned}
$$

Step 1. Prove that $\phi_p\left(\Delta x(0)\right)\geq 0$. In fact, let

$$
\begin{aligned}
F(e) = & \frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\left(b\phi_q(e)+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(e-\sum_{s=0}^{j-1}h(s)\right)\right) \\
& +c\sum_{i=0}^{N+1}\phi_q\left(e-\sum_{s=0}^{i-1}h(s)\right)-\sum_{i=1}^{m}\beta_i\sum_{j=0}^{\eta_i-1}\phi_q\left(e-\sum_{s=0}^{j-1}h(s)\right) \\
& +d\phi_q\left(e-\sum_{s=0}^{N}h(s)\right).
\end{aligned}
$$

Then $F\left(\phi_p\left(\Delta x(0)\right)\right)=0$. One can change $F(e)$ to

$$
\begin{aligned}
F(e) = & \frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\left(b\phi_q(e)+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(e-\sum_{s=0}^{j-1}h(s)\right)\right) \\
& +\left(c-\sum_{i=1}^{m}\beta_i\right)\sum_{i=0}^{N+1}\phi_q\left(e-\sum_{s=0}^{i-1}h(s)\right)+\sum_{i=1}^{m}\beta_i\sum_{j=\eta_i}^{N+1}\phi_q\left(e-\sum_{s=0}^{j-1}h(s)\right) \\
& +d\phi_q\left(e-\sum_{s=0}^{N}h(s)\right).
\end{aligned}
$$

One sees from $(C1)$ and $(C2)$ that $F(e)$ is increasing on $(-\infty,+\infty)$. $(C1)$ implies that

$$
\begin{aligned}
F(0) = & -\frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\left(\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(\sum_{s=0}^{j-1}h(s)\right)\right) \\
& -\left(c-\sum_{i=1}^{m}\beta_i\right)\sum_{i=0}^{N+1}\phi_q\left(\sum_{s=0}^{i-1}h(s)\right)-\sum_{i=1}^{m}\beta_i\sum_{j=\eta_i}^{N+1}\phi_q\left(\sum_{s=0}^{j-1}h(s)\right) \\
& -d\phi_q\left(\sum_{s=0}^{N}h(s)\right) \\
< & \ 0.
\end{aligned}
$$

It follows that

$$
\frac{F(e)}{\phi_q(e)} = \frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\left(b+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(1-\frac{\sum_{s=0}^{j-1}h(s)}{e}\right)\right)
$$

$$+\left(c-\sum_{i=1}^{m}\beta_i\right)\sum_{i=0}^{N+1}\phi_q\left(1-\frac{\sum_{s=0}^{i-1}h(s)}{e}\right)$$
$$+\sum_{i=1}^{m}\beta_i\sum_{j=\eta_i}^{N+1}\phi_q\left(1-\frac{\sum_{s=0}^{j-1}h(s)}{e}\right)+d\phi_q\left(1-\frac{\sum_{s=0}^{N}h(s)}{e}\right).$$

One sees

$$\lim_{e\to+\infty}\left|\frac{\sum_{s=0}^{i-1}h(s)}{e}\right|=0,\ i\in[0,N+1].$$

Hence

$$\lim_{e\to+\infty}\frac{F(e)}{\phi_q(e)} = \frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\left(b+\sum_{i=1}^{m}\alpha_i\eta_i\right)+\left(c-\sum_{i=1}^{m}\beta_i\right)(N+2)$$
$$+\sum_{i=1}^{m}\beta_i(N+2-\eta_i)+d>0.$$

Then, together with $F(0)<0$ and F is increasing on $(-\infty,+\infty)$, it follows that there is a unique $A_h\in(0,+\infty)$ such that $F(A_h)=0$. Since $F\left(\phi_p\left(\Delta x(0)\right)\right)=0$, we get that $\phi_p\left(\Delta x(0)\right)=A_h\in(0,+\infty)$. It follows that $\Delta x(0)>0$.

Step 2. Prove that $\phi_p\left(\Delta x(N+1)\right)\le 0$. In fact, let

$$G(e) = \frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\Bigg[b\phi_q\left(e+\sum_{s=0}^{N}h(s)\right)+a\sum_{i=0}^{N+1}\phi_q\left(e+\sum_{s=i}^{N}h(s)\right)$$
$$-\sum_{i=1}^{m}\alpha_i\sum_{j=\eta_i}^{N+1}\phi_q\left(e+\sum_{s=j}^{N}h(s)\right)\Bigg]$$
$$+\sum_{i=1}^{m}\beta_i\sum_{j=\eta_i}^{N+1}\phi_q\left(e+\sum_{s=j}^{N}h(s)\right)+d\phi_q(e)$$
$$= \frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\Bigg[b\phi_q\left(e+\sum_{s=0}^{N}h(s)\right)$$
$$+\left(a-\sum_{i=1}^{m}\alpha_i\right)\sum_{i=0}^{N+1}\phi_q\left(e+\sum_{s=i}^{N}h(s)\right)$$
$$+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(e+\sum_{s=j}^{N}h(s)\right)\Bigg]$$
$$+\sum_{i=1}^{m}\beta_i\sum_{j=\eta_i}^{N+1}\phi_q\left(e+\sum_{s=j}^{N}h(s)\right)+d\phi_q(e).$$

Similar to Step 1, we have that

$$\begin{aligned} G(0) &= \frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\left[b\phi_q\left(\sum_{s=0}^{N}h(s)\right)+\left(a-\sum_{i=1}^{m}\alpha_i\right)\sum_{i=0}^{N+1}\phi_q\left(\sum_{s=i}^{N}h(s)\right)\right. \\ &\quad \left.+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(\sum_{s=j}^{N}h(s)\right)\right]+\sum_{i=1}^{m}\beta_i\sum_{j=\eta_i}^{N+1}\phi_q\left(\sum_{s=j}^{N}h(s)\right) \\ &\geq 0, \end{aligned}$$

and

$$\begin{aligned} \lim_{e\to-\infty}\frac{G(e)}{\phi_q(e)} &= \frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\left[b+\left(a-\sum_{i=1}^{m}\alpha_i\right)(N+2)+\sum_{i=1}^{m}\alpha_i\eta_i\right] \\ &\quad +\sum_{i=1}^{m}\beta_i(N+2-\eta_i)+d>0. \end{aligned}$$

It is easy to see that $G(e)$ is increasing on $(-\infty,+\infty)$. Then there exists unique $D_h \leq 0$ such that $G(D_h)=0$. Since $G\left(\phi_p\left(\Delta x(N+1)\right)\right)=0$, we get that $D_h=\phi_p\left(\Delta x(N+1)\right)$. Then $\Delta x(N+1)\leq 0$.

It follows from Step 1 and Step 2 and the fact that $\Delta x(n)$ is decreasing on $[0,N+1]$ that there exists unique $n_0\in[0,N]$ such that $\Delta x(n_0)>0$ and $\Delta x(n_0+1)\leq 0$. The proof is complete. □

Lemma 2.3. *Suppose that* $(C2)$ *holds. If* x *is a solution of BVP* (2.1)*, then* $x(0)\geq 0,\ x(N+2)\geq 0$ *and* $x(n)>0$ *for all* $n\in[1,N+1]$.

Proof. Since $(C2)$ holds, we get from Lemma 2.2 that there exists unique $n_0\in[0,N]$ such that $\Delta x(n_0)>0$ and $\Delta x(n_0+1)\leq 0$. It follows from the equation (2.1) that

$$\phi_p\left(\Delta x(n)\right)=\begin{cases}\phi_p\left(\Delta x(n_0+1)\right)-\displaystyle\sum_{s=n_0+1}^{n-1}h(s),\ n\in[n_0+1,N+1],\\ \phi_p\left(\Delta x(n_0)\right)+\displaystyle\sum_{s=n}^{n_0-1}h(s),\ n\in[0,n_0].\end{cases}$$

Then

$$x(n)=\begin{cases}x(N+2)+\displaystyle\sum_{i=n}^{N+1}\phi_q\left(-\phi_p\left(\Delta x(n_0+1)\right)+\sum_{s=n_0+1}^{i-1}h(s)\right),\\ n\in[n_0+1,N+2],\\ x(0)-\displaystyle\sum_{i=0}^{n-1}\phi_q\left(\phi_p\left(\Delta x(n_0)\right)+\sum_{s=i}^{n_0-1}h(s)\right),\ n\in[0,n_0+1],\end{cases}$$

with

$$\begin{aligned} x(n_0+1) &= x(0)+\sum_{i=0}^{n_0}\phi_q\left(\phi_p\left(\Delta x(n_0)\right)+\sum_{s=i}^{n_0-1}h(s)\right) \\ &= x(N+2)+\sum_{i=n_0+1}^{N+1}\phi_q\left(-\phi_p\left(\Delta x(n_0+1)\right)+\sum_{s=n_0+1}^{i-1}h(s)\right). \end{aligned}$$

It follows from $h(n)$ positive, $\Delta x(n_0) > 0$ and $\Delta x(n_0+1) \le 0$ that

$$x(n) > \begin{cases} x(N+2), \ n \in [n_0+1, N+1], \\ x(0), \ n \in [1, n_0+1]. \end{cases}$$

So $x(n) \ge \min\{x(0), x(N+2)\}$ for all $n \in [0, N+2]$. From BCs in BVP (2.1), we get that

$$\begin{aligned} ax(0) &= \sum_{i=1}^{m}\alpha_i x(\eta_i)+b\Delta x(0) \\ &\ge \sum_{i=1}^{m}\alpha_i x(\eta_i) \\ &\ge \sum_{i=1}^{m}\alpha_i \min\{x(0), x(N+2)\}, \end{aligned}$$

$$\begin{aligned} cx(N+2) &= \sum_{i=1}^{m}\beta_i x(\eta_i)-d\Delta x(N+1) \\ &\ge \sum_{i=1}^{m}\beta_i x(\eta_i) \\ &\ge \sum_{i=1}^{m}\beta_i \min\{x(0), x(N+2)\}. \end{aligned}$$

Then

$$\min\{x(0), x(N+2)\} \ge \min\left\{\frac{1}{a}\sum_{i=1}^{m}\alpha_i, \frac{1}{c}\sum_{i=1}^{m}\beta_i\right\}\min\{x(0), x(N+2)\}.$$

Hence

$$\left(1-\min\left\{\frac{1}{a}\sum_{i=1}^{m}\alpha_i, \frac{1}{c}\sum_{i=1}^{m}\beta_i\right\}\right)\min\{x(0), x(N+2)\} \ge 0.$$

It follow from $(C2)$ that $\min\{x(0), x(N+2)\} \ge 0$. Hence $x(n) > 0$ for all $n \in [1, N+1]$. The proof is complete. □

Lemma 2.4. *Assume that $(C2)$ holds. If x is a solution of BVP* (2.1)*, then*

$$x(n) \geq \sigma_n \max_{n \in [0,N+2]} x(n) \text{ for all } n \in [0, N+2], \tag{2.4}$$

where $\sigma_n = \min\left\{\dfrac{n}{N+2}, \dfrac{N+2-n}{N+2}\right\}$.

Proof. It follows from Lemma 2.2 and Lemma 2.3 that $x(n) \geq 0$ for $n \in [0, N+2]$. Suppose that $x(n_0) = \max\{x(n) : n \in [0, N+2]\}$. Since $\Delta x(0) > 0$ and $\Delta x(N+1) \leq 0$, we get that $n_0 \in [1, N+1]$. For $n \in [1, n_0]$, it is easy to see that

$$\begin{aligned} \frac{x(n_0) - x(0)}{n_0} n + x(0) - x(n) &= \frac{n \sum_{s=0}^{n_0-1} \Delta x(s) - n_0 \sum_{s=0}^{n-1} \Delta x(s)}{n_0} \\ &= \frac{-(n_0 - n) \sum_{s=0}^{n-1} \Delta x(s) + n \sum_{s=n}^{n_0-1} \Delta x(s)}{n_0}. \end{aligned}$$

Since $\Delta\left[\phi_p\left(\Delta x(n)\right)\right] = -h(n) < 0$ for all $n \in [0, N]$, we get that $\Delta x(s) \leq \Delta x(j)$ for all $s \geq j$. Then $-(n_0 - n) \sum\limits_{s=0}^{n-1} \Delta x(s) + n \sum\limits_{s=n}^{n_0-1} \Delta x(s) \leq 0$. It follows that

$$\frac{x(n_0) - x(0)}{n_0} n + x(0) - x(n) \leq 0.$$

Then

$$x(n) \geq \frac{n}{n_0} x(n_0) + \left(1 - \frac{n}{n_0}\right) x(0) \geq \frac{n}{N+2} \max_{n \in [0,N+2]} x(n) \text{ for all } n \in [1, n_0].$$

Similarly, if $n \in [n_0, N+1]$, we get

$$x(n) \geq \frac{N+2-n}{N+2} \max_{n \in [0,N+2]} x(n) \text{ for all } n \in [n_0, N+1].$$

Then

$$x(n) \geq \min\left\{\frac{n}{N+2}, \frac{N+2-n}{N+2}\right\} \max_{n \in [0,N+2]} x(n) \text{ for all } n \in [0, N+2].$$

This completes the proof. □

Lemma 2.5. *Suppose that $(C2)$ holds. If x is a solution of BVP* (2.1)*, then*

$$x(n) = B_h + \sum_{i=0}^{n-1} \phi_q\left(A_h - \sum_{j=0}^{i-1} h(j)\right), \tag{2.5}$$

where A_h satisfies the equality

$$\begin{aligned}&\frac{c-\sum_{i=1}^m\beta_i}{a-\sum_{i=1}^m\alpha_i}\left(b\phi_q(A_h)+\sum_{i=1}^m\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(A_h-\sum_{s=0}^{j-1}h(s)\right)\right)\\&+c\sum_{i=0}^{N+1}\phi_q\left(A_h-\sum_{s=0}^{i-1}h(s)\right)\\=\;&\sum_{i=1}^m\beta_i\sum_{j=0}^{\eta_i-1}\phi_q\left(A_h-\sum_{s=0}^{j-1}h(s)\right)-d\phi_q\left(A_h-\sum_{s=0}^{N}h(s)\right),\end{aligned}$$

and

$$B_h=\frac{1}{a-\sum_{i=1}^m\alpha_i}\left(b\phi_q(A_h)+\sum_{i=1}^m\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(A_h-\sum_{s=0}^{j-1}h(s)\right)\right).$$

Proof. The proof follows from Lemma 2.2 and is omitted. □

Lemma 2.6. *Suppose that $(C2)$ holds. If x is a solution of BVP* (2.1)*, then there exists an $n_0\in[0,N]$ such that*

$$\begin{aligned}\max_{n\in[0,N+2]}x(n)&=x(n_0+1)\\&\geq\max\left\{\sum_{i=0}^{n_0}\phi_q\left(\sum_{j=i}^{n_0-1}h(j)\right),\sum_{i=n_0+1}^{N+1}\phi_q\left(\sum_{j=n_0+1}^{i-1}h(j)\right)\right\}.\end{aligned}$$

Proof. It follows from Lemma 2.2 that there is $n_0\in[0,N]$ such that $\Delta x(n_0)>0$ and $\Delta x(n_0{+}1)\leq 0$, $\Delta x(n)>0$ for all $n\in[0,n_0]$ and $\Delta x(n)\leq 0$ for all $n\in[n_0{+}1,N{+}1]$. Then

$$\max_{n\in[0,N+2]}x(n)=x(n_0+1),$$

there exists $\xi\in(n_0,n_0+1]$ such that

$$\frac{\Delta x(n_0+1)-\Delta x(n_0)}{n_0+1-n_0}=\frac{0-\Delta x(n_0)}{\xi-n_0}.$$

Then

$$\Delta x(n_0+1)=-\frac{n_0+1-\xi}{\xi-n_0}\Delta x(n_0).\tag{2.6}$$

It is easy to see from (2.1) that

$$0<\Delta x(n_0)=\phi_q\left(A_h-\sum_{i=0}^{n_0-1}h(i)\right),\tag{2.7}$$

$$0 \geq \Delta x(n_0+1) = \phi_q \left(A_h - \sum_{i=0}^{n_0} h(i) \right), \tag{2.8}$$

where $A_h = \phi_p(\Delta x(0))$. So (2.6)–(2.8) imply that

$$\phi_q \left(A_h - \sum_{i=0}^{n_0} h(i) \right) = -\frac{n_0+1-\xi}{\xi - n_0} \phi_q \left(A_h - \sum_{i=0}^{n_0-1} h(i) \right).$$

Then

$$A_h = \frac{1}{1+\phi_p\left(\frac{n_0+1-\xi}{\xi-n_0}\right)} \left[\sum_{i=0}^{n_0} h(i) + \phi_p \left(\frac{n_0+1-\xi}{\xi-n_0} \right) \sum_{i=0}^{n_0-1} h(i) \right].$$

We get that

$$\sum_{i=0}^{n_0-1} h(i) \leq A_h \leq \sum_{i=0}^{n_0} h(i). \tag{2.9}$$

Lemma 2.3 implies that $B_h = x(0) \geq 0$. Furthermore, one has from (2.6) that

$$\begin{aligned} x(n_0+1) &= B_h + \sum_{i=0}^{n_0} \phi_q \left(A_h - \sum_{j=0}^{i-1} h(j) \right) \\ &\geq \sum_{i=0}^{n_0} \phi_q \left(\sum_{j=0}^{n_0-1} h(j) - \sum_{j=0}^{i-1} h(j) \right) \\ &= \sum_{i=0}^{n_0} \phi_q \left(\sum_{j=i}^{n_0-1} h(j) \right). \end{aligned}$$

On the other hand, by a discussion similar to Lemma 2.2 and Lemma 2.3, we have $\overline{A_h} = \phi_p(\Delta x(N+1))$, $\overline{B_h} = x(N+2)$ with

$$\Delta x(n) = \phi_q \left(\overline{A_h} + \sum_{j=n}^{N} h(j) \right),$$

and

$$x(n) = \overline{B_h} - \sum_{i=n}^{N+1} \phi_q \left(\overline{A_h} + \sum_{j=i}^{N} h(j) \right).$$

It follows that,

$$\Delta x(n_0) = \phi_q \left(\overline{A_h} + \sum_{j=n_0}^{N} h(j) \right) > 0,$$

$$\Delta x(n_0+1) = \phi_q\left(\overline{A_h} + \sum_{j=n_0+1}^{N} h(j)\right) \le 0.$$

So

$$\phi_q\left(\overline{A_h} + \sum_{j=n_0+1}^{N} h(j)\right) = -\frac{n_0+1-\xi}{\xi - n_0}\phi_q\left(\overline{A_h} + \sum_{j=n_0}^{N} h(j)\right).$$

Then

$$\overline{A_h} = -\frac{1}{1+\phi_p\left(\frac{n_0+1-\xi}{\xi-n_0}\right)}\left(\sum_{j=n_0+1}^{N} h(j) + \phi_p\left(\frac{n_0+1-\xi}{\xi-n_0}\right)\sum_{j=n_0}^{N} h(j)\right).$$

We get that

$$-\sum_{j=n_0+1}^{N} h(j) \ge \overline{A_h} \ge -\sum_{j=n_0}^{N} h(j).$$

One has from Lemma 2.3 that $\overline{B_h} = x(N+2) \ge 0$. Therefore

$$\begin{aligned} x(n_0+1) &= \overline{B_h} - \sum_{i=n_0+1}^{N+1} \phi_q\left(\overline{A_h} + \sum_{j=i}^{N} h(j)\right) \\ &\ge -\sum_{i=n_0+1}^{N+1} \phi_q\left(\overline{A_h} + \sum_{j=i}^{N} h(j)\right) \\ &= \sum_{i=n_0+1}^{N+1} \phi_q\left(-\overline{A_h} - \sum_{j=i}^{N} h(j)\right) \\ &\ge \sum_{i=n_0+1}^{N+1} \phi_q\left(\sum_{j=n_0+1}^{N} h(j) - \sum_{j=i}^{N} h(j)\right) \\ &= \sum_{i=n_0+1}^{N+1} \phi_q\left(\sum_{j=n_0+1}^{i-1} h(j)\right). \end{aligned}$$

Hence

$$\begin{aligned} \max_{n\in[0,N+2]} x(n) &= x(n_0+1) \\ &\ge \max\left\{\sum_{i=0}^{n_0} \phi_q\left(\sum_{j=i}^{n_0-1} h(j)\right), \sum_{i=n_0+1}^{N+1} \phi_q\left(\sum_{j=n_0+1}^{i-1} h(j)\right)\right\}. \end{aligned}$$

The proof is complete. □

Let $h(n) = f\left(n, x(n+1), \Delta x(n), \Delta x(n+1)\right)$. Then A_x satisfies the equality:

$$\begin{aligned} & \frac{c - \sum_{i=1}^{m} \beta_i}{a - \sum_{i=1}^{m} \alpha_i} \Bigg(b\phi_q(A_x) + \sum_{i=1}^{m} \alpha_i \sum_{j=0}^{\eta_i - 1} \phi_q \Bigg(A_x \\ & - \sum_{s=0}^{j-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right) \Bigg) \Bigg) \\ & + c \sum_{i=0}^{N+1} \phi_q \left(A_x - \sum_{s=0}^{i-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right) \right) \\ = & \sum_{i=1}^{m} \beta_i \sum_{j=0}^{\eta_i - 1} \phi_q \left(A_x - \sum_{s=0}^{j-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right) \right) \\ & - d\phi_q \left(A_x - \sum_{s=0}^{N} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right) \right), \end{aligned}$$

and

$$\begin{aligned} B_x = & \frac{1}{a - \sum_{i=1}^{m} \alpha_i} \Bigg(b\phi_q(A_x) + \sum_{i=1}^{m} \alpha_i \sum_{j=0}^{\eta_i - 1} \phi_q \Bigg(A_x \\ & - \sum_{s=0}^{j-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right) \Bigg) \Bigg). \end{aligned}$$

Let $\mathbb{B} = \mathbb{R}^{N+3}$. We call $x \le y$ for $x, y \in \mathbb{B}$ if $x(n) \le y(n)$ for all $n \in [0, N+2]$, define the norm

$$\|x\| = \max \left\{ \max_{n \in [0,N+2]} x(n), \max_{n \in [0,N+1]} |\Delta x(n)| \right\}.$$

It is easy to see that $\mathbb{B}$ is a semi-ordered real Banach space. Choose

$$\mathcal{P} = \left\{ x \in \mathbb{B} : \begin{array}{c} x(n) \ge \sigma_n \max_{n \in [0,N+2]} x(n) \text{ for all } n \in [0, N+2], \\ \Delta^2 x(n) \le 0 \text{ for } n \in [0, N], \\ ax(0) - b\Delta x(0) = \sum_{i=1}^{m} \alpha_i x(\eta_i) \end{array} \right\}, \tag{2.10}$$

where $\sigma_n = \min \left\{ \dfrac{n}{N+2}, \dfrac{N+2-n}{N+2} \right\}$. Then $\mathcal{P}$ is a cone in $\mathbb{B}$. Define the operator $T : \mathcal{P} \to \mathbb{B}$ by

$$(Tx)(n) = B_x + \sum_{i=0}^{n-1} \phi_q \left(A_x - \sum_{j=0}^{i-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right) \right),$$

for $x \in \mathcal{P},\ n \in [0, N+2]$. Then

$$(Tx)(n) = \frac{\sum_{i=1}^{m} \alpha_i \sum_{j=0}^{\eta_i - 1} \phi_q \left(A_x - \sum_{s=0}^{j-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right) \right)}{a - \sum_{i=1}^{m} \alpha_i} + \frac{b\phi_q(A_x)}{a - \sum_{i=1}^{m} \alpha_i} + \sum_{i=0}^{n-1} \phi_q \left(A_x - \sum_{j=0}^{i-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right) \right).$$

Lemma 2.7. *Suppose that* $(C1)$ *and* $(C2)$ *hold. Then*

(i) Tx *satisfies the following:*

$$\begin{cases} \Delta \left[\phi_p \left(\Delta (Tx)(n)\right)\right] + f\left(n, x(n+1), \Delta x(n), \Delta x(n+1)\right) = 0, \\ n \in [0, N], \\ a(Tx)(0) - b\Delta(Tx)(0) = \sum_{i=1}^{m} \alpha_i (Tx)(\eta_i), \\ c(Tx)(N+2) + d\Delta(Tx)(N+1) = \sum_{i=1}^{m} \beta_i (Tx)(\eta_i). \end{cases} \tag{2.11}$$

(ii) $Tx \in \mathcal{P}$ *for each* $x \in \mathcal{P}$.

(iii) x *is a solution of BVP* (1.1) *if and only if* x *is a solution of the operator equation* $x = Tx$.

(iv) $T : \mathcal{P} \to \mathcal{P}$ *is completely continuous.*

Proof. (i) By the definition of Tx, we get (2.11).

(ii) Note the definition of $\mathcal{P}$. Since $(C1)$ and $(C2)$ hold, for $x \in \mathcal{P}$, (2.11), Lemma 2.2, Lemma 2.3 and Lemma 2.4 imply that $\Delta(Tx)(n)$ is decreasing on $[0, N+1]$ and $(Tx)(n) \geq \sigma_n \max_{n \in [0, N+2]} (Tx)(n)$ for all $n \in [0, N+2]$. Together with (2.11), it follows that $Tx \in \mathcal{P}$.

(iii) It is easy to see from (2.11) that x is a solution of BVP (1.1) if and only if x is a solution of the operator equation $x = Tx$.

(iv) It suffices to prove that T is continuous on $\mathcal{P}$ and T is relative compact.
We divide the proof into three steps:

Step 1. For each bounded subset $D \subset \mathcal{P}$, prove that $\left\{(A_x, B_x) : x \in \overline{D}\right\}$ is bounded in $\mathbb{R}^2$. Denote

$$L_1 = \max\left\{\max_{n\in[0,N+2]} x(n), \max_{n\in[0,N+1]} |\Delta x(n)| : x \in \overline{D}\right\},$$

and

$$L_2 = \max_{j\in[0,N]} f_{L_1}(j) = \max_{j\in[0,N]} \max_{|x|,|y|\leq L_1, |\omega|\leq L_1} |f(j,x,y,w)|.$$

It follows from (2.9) in the proof of Lemma 2.6 that

$$0 \leq A_x \leq \sum_{j=0}^{N} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right) \leq (N+1)L_2,$$

and B_x satisfies that

$$\begin{aligned}
|B_x| &= \frac{\left|b\phi_q(A_x) + \sum_{i=1}^{m}\alpha_i \sum_{j=0}^{\eta_i-1}\phi_q\left(A_x - \sum_{s=0}^{j-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right)\right)\right|}{a - \sum_{i=1}^{m}\alpha_i} \\
&\leq \frac{1}{a - \sum_{i=1}^{m}\alpha_i}\left(b\phi_q((N+1)L_2) + \sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left((N+1)L_2 + jL_2\right)\right).
\end{aligned}$$

Hence $\left\{(A_x, B_x) : x \in \overline{D}\right\}$ is bounded in $\mathbb{R}^2$.

Step 2. For each bounded subset $D \subset \mathcal{P}$, and each $x_0 \in D$, prove that T is continuous at x_0. For $x_0 \in D$ and $x_n \in D$ with $x_n \to x_0$ $(n \to +\infty)$ in D. Denote $u_n(k) = (Tx_n)(k)$, $u_0(k) = (Tx_0)(k)$ for all $k \in [0, N+2]$. We prove that T is continuous at x_0, i.e., $u_n \to u_0$ $(n \to +\infty)$. Let Ax_0, Bx_0 be defined by

$$\begin{aligned}
&\frac{c - \sum_{i=1}^{m}\beta_i}{a - \sum_{i=1}^{m}\alpha_i}\left(b\phi_q(A_{x_0}) + \sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(A_{x_0} - \sum_{s=0}^{j-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right)\right)\right) \\
&\qquad + c\sum_{i=0}^{N+1}\phi_q\left(A_{x_0} - \sum_{s=0}^{i-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right)\right) \\
&= \sum_{i=1}^{m}\beta_i\sum_{j=0}^{\eta_i-1}\phi_q\left(A_{x_0} - \sum_{s=0}^{j-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right)\right)
\end{aligned}$$

$$-d\phi_q\left(A_{x_0}-\sum_{s=0}^{N} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right)\right).$$

and

$$B_x=\frac{b\phi_q(A_{x_0})+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(A_{x_0}-\sum_{s=0}^{j-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right)\right)}{a-\sum_{i=1}^{m}\alpha_i}.$$

First, we prove that A_x, B_x are continuous in x, i.e.,

$$(A_{x_n}, B_{x_n}) \to (A_{x_0}, B_{x_0}),\ n\to+\infty.$$

It follows from Step 1 that (A_{x_n}, B_{x_n}) is bounded. Without loss of generality, suppose that $(A_{x_n}, B_{x_n}) \to (\overline{A}, \overline{B}) \neq (A_{x_0}, B_{x_0})$. It is easy to see that

$$\begin{aligned}
&\lim_{n\to+\infty} u_n(k)\\
= &\lim_{n\to+\infty}\left[B_{x_n}+\sum_{i=0}^{k-1}\phi_q\left(A_{x_n}-\sum_{j=0}^{i-1} f\left(j, x_n(j+1), \Delta x_n(j), \Delta x_n(j+1)\right)\right)\right]\\
= &\overline{B}+\sum_{i=0}^{k-1}\phi_q\left(\overline{A}-\lim_{n\to+\infty}\sum_{j=0}^{i-1} f\left(j, x_n(j+1), \Delta x_n(j), \Delta x_n(j+1)\right)\right)\\
= &\overline{B}+\sum_{i=0}^{k-1}\phi_q\left(\overline{A}-\lim_{n\to+\infty}\sum_{j=0}^{i-1} f\left(j, x_0(j+1), \Delta x_0(j), \Delta x_0(j+1)\right)\right)\\
= &\overline{u}(k).
\end{aligned}$$

One can easily see that $\overline{u}$ satisfies

$$a\overline{u}(0)-b\Delta\overline{u}(0)=\sum_{i=1}^{m}\alpha_i\overline{u}(\eta_i),$$

$$c\overline{u}(N+2)+d\Delta\overline{u}(N+1)=\sum_{i=1}^{m}\beta_i\overline{u}(\eta_i).$$

So

$$\frac{c-\sum_{i=1}^{m}\beta_i}{a-\sum_{i=1}^{m}\alpha_i}\left(b\phi_q(\overline{A})+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(\overline{A}-\sum_{s=0}^{j-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right)\right)\right)$$

$$
\begin{aligned}
&+c\sum_{i=0}^{N+1}\phi_q\left(\overline{A}-\sum_{s=0}^{i-1}f\left(s,x(s+1),\Delta x(s),\Delta x(s+1)\right)\right)\\
&-\sum_{i=1}^{m}\beta_i\sum_{j=0}^{\eta_i-1}\phi_q\left(\overline{A}-\sum_{s=0}^{j-1}f\left(s,x(s+1),\Delta x(s),\Delta x(s+1)\right)\right)\\
&+d\phi_q\left(\overline{A}-\sum_{s=0}^{N}f\left(s,x(s+1),\Delta x(s),\Delta x(s+1)\right)\right)=0,
\end{aligned}
$$

and

$$
\overline{B}=\frac{b\phi_q(\overline{A})+\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{\eta_i-1}\phi_q\left(\overline{A}-\sum_{s=0}^{j-1}f\left(s,x(s+1),\Delta x(s),\Delta x(s+1)\right)\right)}{a-\sum_{i=1}^{m}\alpha_i}.
$$

It follows from the proof of Step 1 in Lemma 2.2 that $\overline{A}=A_{x_0}$, then $\overline{B}=B_{x_0}$. Hence

$$
(A_{x_n},B_{x_n})\to(\overline{A},\overline{B})=(A_{x_0},B_{x_0}),\ n\to+\infty.
$$

It follows that

$$
\lim_{n\to+\infty}Tx_n=Tx_0.
$$

This implies that T is continuous at x_0.

Step 3. For each bounded subset $D\subset\mathcal{P}$, prove that T is relative compact on D. In fact, for each bounded subset $\Omega\subseteq D$ and $x\in\Omega$. Suppose

$$
\|x\|=\max\left\{\max_{n\in[0,N+2]}|x(n)|,\ \max_{n\in[0,N+1]}|\Delta x(n)|\right\}<M_1,
$$

and Step 1 implies that there exists a constant $M_2>0$ such that $|A_x|$, $|B_x|<M_2$. Then

$$
\begin{aligned}
|(Tx)(n)| &= \left|B_x+\sum_{i=0}^{n-1}\phi_q\left(A_x-\sum_{j=0}^{i-1}f\left(j,x(j+1),\Delta x(j),\Delta x(j+1)\right)\right)\right|\\
&\le M_2+\sum_{i=0}^{N+1}\phi_q\left(M_2+\sum_{j=i}^{N}\left|f\left(j,x(j+1),\Delta x(j),\Delta x(j+1)\right)\right|\right)\\
&\le M_2+\sum_{i=0}^{N+1}\phi_q\left(M_2+\sum_{j=i}^{N}f_{M_1}(j)\right)\\
&:= M_3,
\end{aligned}
$$

where $f_{M_1}(j) = \max\limits_{|x|\le M_1,|y|\le M_1,|\omega|\le M_1} |f(j,x,y,\omega)|$. Similarly, one has that

$$\begin{aligned} |\Delta(Tx)(n)| &= \left|\phi_q\left(A_x - \sum_{j=0}^{n-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right)\right)\right| \\ &\le \phi_q\left(M_2 + \sum_{j=0}^{n-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right)\right) \\ &\le \phi_q\left(M_2 + \sum_{j=0}^{N} f_{M_1}(j)\right) \\ &:= M_4. \end{aligned}$$

It follows that $T\Omega$ is bounded. Since $\mathbb{B} = \mathbb{R}^{N+3}$, one knows that $T\Omega$ is relative compact. Steps $1,\ 2$ and 3 imply that T is completely continuous. □

3 Main Result

In this section, our objective is to establish the existence of at least three positive solutions for the BVP (1.1) by using Bai and Ge's fixed point theorem [6].

Choose $\left[\dfrac{N+2}{2}\right] > k > 0$, where $[x]$ denotes the integer not greater than x, and denote $\sigma_k = \min\left\{\dfrac{k}{N+2}, \dfrac{N+2-k}{N+2}\right\}$. Define the functionals on $\mathcal{P} : \mathcal{P} \to [0,+\infty)$ by

$$\begin{aligned} \alpha(x) &= \max_{n\in[0,N+2]} x(n), \\ \beta(x) &= \max_{n\in[0,N+1]} |\Delta x(n)|, \\ \psi(x) &= \min_{n\in[k,N+2-k]} x(n). \end{aligned}$$

For $x \in \mathcal{P}$, since

$$\begin{aligned} x(n) &= x(n) - x(0) + x(0) \\ &\le \left|\sum_{i=0}^{n-1} \Delta x(i)\right| + x(0) \\ &= \left|\sum_{i=0}^{n-1} \Delta x(i)\right| + x(0) + \frac{\sum_{i=1}^{m} \alpha_i x(\eta_i) + b\Delta x(0) - ax(0)}{a - \sum_{i=1}^{m} \alpha_i} \end{aligned}$$

$$
\begin{aligned}
&= \left|\sum_{i=0}^{n-1} \Delta x(i)\right| + \frac{\sum_{i=1}^{m} \alpha_i \left(x(\eta_i) - x(0)\right) + b\Delta x(0)}{a - \sum_{i=1}^{m} \alpha_i} \\
&\le \left|\sum_{i=0}^{N+1} \Delta x(i)\right| + \frac{\sum_{i=1}^{m} \alpha_i \sum_{s=0}^{\eta_i - 1} \Delta x(s) + b\Delta x(0)}{a - \sum_{i=1}^{m} \alpha_i} \\
&\le \left(N + 2 + \frac{\sum_{i=1}^{m} \alpha_i \eta_i + b}{a - \sum_{i=1}^{m} \alpha_i}\right) \max_{n\in[0,N+1]} \left|\Delta x(n)\right|,
\end{aligned}
$$

we get that

$$
\alpha(x) \le \left(N + 2 + \frac{\sum_{i=1}^{m} \alpha_i \eta_i + b}{a - \sum_{i=1}^{m} \alpha_i}\right) \beta(x),\ x \in \mathcal{P}.
$$

Let

$$
\begin{aligned}
\Lambda &= \frac{1}{a - \sum_{i=1}^{m} \alpha_i} \sum_{j=1}^{m} \alpha_j \sum_{i=0}^{\eta_j - 1} \phi_q\left(N - i + 1\right) + \frac{b}{a - \sum_{i=1}^{m} \alpha_i} \phi_q(N+1) \\
&\quad + \sum_{i=0}^{N+1} \phi_q\left(N - i + 1\right), \\
H &= \sigma_k \min\left\{ \sum_{i=k}^{\left[\frac{N+2}{2}\right]} \phi_q\left(\left[\frac{N+2}{2}\right] - i\right), \sum_{i=\left[\frac{N+2}{2}\right]}^{N+2-k} \phi_q\left(i - \left[\frac{N+2}{2}\right]\right)\right\}.
\end{aligned}
$$

Theorem 3.1. *Suppose that* $(C1)$ *and* $(C2)$ *hold. If there are positive numbers* $r_2 \ge \tau = \dfrac{\nu}{\sigma_k} > \nu > r_1,\ l_2 \ge l_1$, *with* $\phi_p\left(\dfrac{\nu}{H}\right) \le \min\left\{\phi_p\left(\dfrac{r_2}{\Lambda}\right), \dfrac{1}{2(N+1)}\phi_p\left(l_2\right)\right\}$ *such that the following conditions are satisfied:*

$(C3)$ $f(n,u,v,w) < \min\left\{\phi_p\left(\dfrac{r_1}{\Lambda}\right), \dfrac{1}{2(N+1)}\phi_p\left(l_1\right)\right\}$ *for all* $(n,u,v,w) \in [0,N] \times [0,r_1] \times [-l_1,l_1] \times [-l_1,l_1]$*;*

$(C4)$ $f(n,u,v,w) > \phi_p\left(\frac{\nu}{H}\right)$ *for all* $(n,u,v,w) \in [k, N+2-k] \times \left[\nu, \frac{\nu}{\sigma_k}\right] \times [-l_2, l_2] \times [-l_2, l_2]$*;*

$(C5)$ $f(n,u,v,w) \leq \min\left\{\phi_p\left(\frac{r_2}{\Lambda}\right), \frac{1}{2(N+1)}\phi_p(l_2)\right\}$ *for all* $(n,u,v,w) \in [0,N] \times [0, r_2] \times [-l_2, l_2] \times [-l_2, l_2]$.

Then the BVP (1.1) *has at least three positive solutions* x_1, x_2, x_3 *such that*

$$\max_{n\in[0,N+2]} x_1(n) < r_1 \text{ and } \max_{n\in[0,N+1]} |\Delta x_1(n)| < l_1;$$

$$\nu < \min_{t\in[k,N+2-k]} x_2(n) \leq \max_{n\in[0,N+2]} x_2(n) \leq r_2 \text{ and } \max_{n\in[0,N+1]} |\Delta x_2(n)| \leq l_2;$$

$$r_1 < \max_{n\in[0,N+2]} x_3(n) < r_2, \quad \min_{n\in[k,N+2-k]} x_3(n) < \nu, \text{ and } \max_{n\in[0,N+2]} |\Delta x_3(n)| \leq l_2.$$

Proof. To apply Lemma 2.1, we prove that all conditions in Lemma 2.1 are satisfied. By the definitions, it is easy to see that α and β are two nonnegative continuous convex functionals on the cone $\mathcal{P}$ satisfying $(B1)$ and $(B2)$, ψ is a nonnegative continuous concave functional on the cone $\mathcal{P}$ with $\psi(x) \leq \alpha(x)$ for all $x \in \mathcal{P}$. Lemma 2.7 implies that $x = x(n)$ is a solution of BVP (1.1) if and only if x is a solution of the operator equation $x = Tx$ and $T : \mathcal{P} \to \mathcal{P}$ is completely continuous. Now, we prove all conditions of Lemma 2.1 hold.

Step 1. Prove that $T\bar{\mathcal{P}}(\alpha, r_2; \beta, l_2) \subseteq \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2)$. For $x \in \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2)$, we get $x \in \mathcal{P}$, $\alpha(x) \leq r_2$ and $\beta(x) \leq l_2$. Then

$$0 \leq x(n) \leq r_2, \ n \in [0, N+2], \ |\Delta x(n)| \leq l_2 \text{ for all } n \in [0, N+1].$$

$(C5)$ implies that

$$f(n, x(n+1), \Delta x(n), \Delta x(n+1)) \leq \min\left\{\phi_p\left(\frac{r_2}{\Lambda}\right), \frac{1}{2(N+1)}\phi_p(l_2)\right\}.$$

Since $A_x \in \left[0, \sum_{i=0}^{N} f(i, x(i+1), \Delta x(i), \Delta x(i+1))\right]$, given by (2.9) in the proof of Lemma 2.6 and $(C3)$ holds, we get

$$\begin{aligned}
\alpha(Tx) &= \max_{n\in[0,N+2]} |Tx(n)| \\
&= \max_{n\in[0,N+2]} \left(B_x + \sum_{i=0}^{n-1} \phi_q\left(A_x - \sum_{j=0}^{i-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right)\right)\right) \\
&= \max_{n\in[0,N+2]} \left\{ \frac{\sum_{j=1}^{m} \alpha_j \sum_{i=0}^{\eta_j - 1} \phi_q\left(A_x - \sum_{s=0}^{i-1} f\left(s, x(s+1), \Delta x(s), \Delta x(s+1)\right)\right)}{a - \sum_{i=1}^{m} \alpha_i} \right.
\end{aligned}$$

$$+\frac{b}{a-\sum_{i=1}^{m}\alpha_i}\phi_q(A_x)+\sum_{i=0}^{n-1}\phi_q\left(A_x-\sum_{j=0}^{i-1}f\left(j,x(j+1),\Delta x(j),\Delta x(j+1)\right)\right)\Bigg\}$$

$$\begin{aligned}
\leq\ & \max_{n\in[0,N+2]}\Bigg[\frac{1}{a-\sum_{i=1}^{m}\alpha_i}\sum_{j=1}^{m}\alpha_j\sum_{i=0}^{\eta_j-1}\phi_q\Bigg(\sum_{s=0}^{N}f(s,x(s+1),\Delta x(s+1),\Delta x(s+1))\\
&-\sum_{s=0}^{i-1}f\left(s,x(s+1),\Delta x(s),\Delta x(s+1)\right)\Bigg)\\
&+\frac{b}{a-\sum_{i=1}^{m}\alpha_i}\phi_q\left(\sum_{s=0}^{N}f(s,x(s+1),\Delta x(s+1),\Delta x(s+1))\right)\\
&+\sum_{i=0}^{n-1}\phi_q\Bigg(\sum_{j=0}^{N}f(j,x(j+1),\Delta x(j+1),\Delta x(j+1))\\
&-\sum_{j=0}^{i-1}f\left(j,x(j+1),\Delta x(j),\Delta x(j+1)\right)\Bigg)\Bigg]\\
=\ & \max_{n\in[0,N+2]}\Bigg[\frac{1}{a-\sum_{i=1}^{m}\alpha_i}\sum_{j=1}^{m}\alpha_j\sum_{i=0}^{\eta_j-1}\phi_q\left(\sum_{s=i}^{N}f\left(s,x(s+1),\Delta x(s),\Delta x(s+1)\right)\right)\\
&+\frac{b}{a-\sum_{i=1}^{m}\alpha_i}\phi_q\left(\sum_{s=0}^{N}f\left(s,x(s+1),\Delta x(s),\Delta x(s+1)\right)\right)\\
&+\sum_{i=0}^{n-1}\phi_q\left(\sum_{j=i}^{N}f\left(j,x(j+1),\Delta x(j),\Delta x(j+1)\right)\right)\Bigg]\\
\leq\ & \max_{n\in[0,N+2]}\Bigg(\frac{1}{a-\sum_{i=1}^{m}\alpha_i}\sum_{j=1}^{m}\alpha_j\sum_{i=0}^{\eta_j-1}\phi_q\left(N-i+1\right)\\
&+\frac{b}{a-\sum_{i=1}^{m}\alpha_i}\phi_q(N+1)+\sum_{i=0}^{n-1}\phi_q\left(N-i+1\right)\Bigg)\frac{r_2}{\Lambda}
\end{aligned}$$

$$\begin{aligned}
&= \Bigg(\frac{1}{a-\sum_{i=1}^{m}\alpha_i} \sum_{j=1}^{m} \alpha_j \sum_{i=0}^{\eta_j -1} \phi_q\,(N-i+1) \\
&\quad + \frac{b}{a-\sum_{i=1}^{m}\alpha_i} \phi_q(N+1) + \sum_{i=0}^{N+1} \phi_q\,(N-i+1) \Bigg) \frac{r_2}{\Lambda} \\
&= r_2,
\end{aligned}$$

and

$$\begin{aligned}
\beta(Tx) &= \max_{n\in[0,N+1]} |\Delta(Tx)(n)| \\
&= \max_{n\in[0,N+1]} \phi_q \left| \left(A_x - \sum_{j=0}^{n-1} f\,(j, x(j+1), \Delta x(j), \Delta x(j+1)) \right) \right| \\
&\le \max_{n\in[0,N+1]} \phi_q \Bigg(\sum_{j=0}^{N} f\,(j, x(j+1), \Delta x(j), \Delta x(j+1)) \\
&\quad + \sum_{j=0}^{n-1} f\,(j, x(j+1), \Delta x(j), \Delta x(j+1)) \Bigg) \\
&= \max_{n\in[0,N+1]} \phi_q \left(2\sum_{j=0}^{N} f\,(j, x(j+1), \Delta x(j), \Delta x(j+1)) \right) \\
&\le \phi_q \left(2(N+1) \frac{\phi_p(l_2)}{2(N+1)} \right) = l_2.
\end{aligned}$$

So, $Tx \in \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2)$. It follows that $T\bar{\mathcal{P}}(\alpha, r_2; \beta, l_2) \subseteq \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2)$. These completes Step 1.

Step 2. Prove that $(B4)$ in Lemma 2.1 holds. For each $x \in \bar{\mathcal{P}}(\alpha, r_1; \beta, l_1)$, we prove that $\alpha(Tx) < r_1$ and $\beta(Tx) < l_1$ by using $(C3)$. The proof is similar to above discussion and is omitted.

Step 3. Prove that $(B3)$ in Lemma 2.1 holds. We prove that

$$\{x \in \bar{\mathcal{P}}(\alpha, \tau; \beta, l_2; \psi, \nu) | \psi(x) > \nu\} \neq \emptyset \text{ and } \psi(Tx) > \nu,$$

for every $x \in \bar{\mathcal{P}}(\alpha, \tau; \beta, l_2; \psi, \nu)$. Choose $x(n) = \nu/(2\sigma_k) = \tau/2$ for all $n \in [0, N+2]$. It is easy to see that

$$\alpha(x) = \frac{\nu}{2\sigma_k} \le \tau,\ \beta(x) = 0 \le l_2,\ \psi(x) = \frac{\nu}{2\sigma_k} > \nu,$$

since $\sigma_k = \min\left\{ \frac{k}{N+2}, \frac{N+2-k}{N+2} \right\} < \frac{1}{2}$. Hence

$$\{x \in \bar{\mathcal{P}}(\alpha, \tau; \beta, l_2; \psi, \nu) | \psi(x) > \nu\} \neq \emptyset.$$

For $x \in \bar{\mathcal{P}}(\alpha, \tau; \beta, l_2; \psi, \nu)$, one sees that

$$|\Delta x(n)| \leq l_2,\ n \in [0, N+1],\ \nu \leq x(n) \leq \frac{\nu}{\sigma_k},\ n \in [k, N+2-k].$$

It follows from $(C4)$ that

$$f(n,u,v,w) > \phi_p\left(\frac{\nu}{H}\right),\ (n,u,v,w) \in [k, N+2-k] \times \left[\nu, \frac{\nu}{\sigma_k}\right] \times [-l_2, l_2]$$
$$\times [-l_2, l_2].$$

Similarly to Lemma 2.6 that there exists $n_0 \in [k, N+2-k]$ such that $\Delta x(n_0) > 0$ and $\Delta x(n_0+1) \leq 0$ and

$$\begin{aligned}\max_{n\in[0,N+2]} Tx(n) &= Tx(n_0+1)\\ &\geq \max\left\{(Tx)(n_0), (Tx)(n_0+2)\right\},\end{aligned}$$

we get from (2.4), $(C1)$ and $(C2)$, Lemma 2.6 that

$$\begin{aligned}\psi(Tx) &= \min_{n\in[k,N+2-k]}(Tx)(n) \geq \sigma_k \max_{n\in[0,N+2]}(Tx)(n) = \sigma_k (Tx)(n_0)\\
&\geq \sigma_k \max\left\{\sum_{i=0}^{n_0}\phi_q\left(\sum_{j=i}^{n_0-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right)\right),\right.\\
&\left.\sum_{i=n_0+1}^{N+1}\phi_q\left(\sum_{j=n_0+1}^{i-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right)\right)\right\}\\
&\geq \sigma_k \max\left\{\sum_{i=0}^{\left[\frac{N+2}{2}\right]}\phi_q\left(\sum_{j=i}^{\left[\frac{N+2}{2}\right]-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right)\right),\right.\\
&\left.\sum_{i=\left[\frac{N+2}{2}\right]}^{N+1}\phi_q\left(\sum_{j=\left[\frac{N+2}{2}\right]}^{i-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right)\right)\right\}\\
&\geq \sigma_k \min\left\{\sum_{i=k}^{\left[\frac{N+2}{2}\right]}\phi_q\left(\sum_{j=i}^{\left[\frac{N+2}{2}\right]-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right)\right),\right.\\
&\left.\sum_{i=\left[\frac{N+2}{2}\right]}^{N+2-k}\phi_q\left(\sum_{j=\left[\frac{N+2}{2}\right]}^{i-1} f\left(j, x(j+1), \Delta x(j), \Delta x(j+1)\right)\right)\right\}\\
&\geq \sigma_k \min\left\{\sum_{i=k}^{\left[\frac{N+2}{2}\right]}\phi_q\left(\left[\frac{N+2}{2}\right]-i\right), \sum_{i=\left[\frac{N+2}{2}\right]}^{N+2-k}\phi_q\left(i-\left[\frac{N+2}{2}\right]\right)\right\}\frac{\nu}{H}\end{aligned}$$

$$= \nu.$$

We get $\psi(Tx) > \nu$. Step 3 is complete.

Step 4. Prove that $(B5)$ in Lemma 2.1 holds. We prove that $\psi(Tx) > \nu$ for $x \in \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2; \psi, \nu)$ with $\alpha(Tx) > \dfrac{\nu}{\sigma_k}$. Suppose $x \in \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2; \psi, \nu)$ with $\alpha(Tx) > \dfrac{\nu}{\sigma_k}$. We get

$$\psi(Tx) = \min_{n\in[k,N+2-k]}(Tx)(n) \geq \sigma_k \max_{n\in[0,N+2]}(Tx)(n) = \sigma_k\alpha(Tx) > \sigma_k\frac{\nu}{\sigma_k} = \nu.$$

This completes Step 4.

Consequently, from Lemma 2.1 BVP (1.1) has at least three positive solutions x_1, x_2, $x_3 \in \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2)$ with

$$x_1 \in \mathcal{P}(\alpha, r_1; \beta, l_1),$$
$$x_2 \in \left\{x \in \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2; \psi, \nu) : \psi(x) > \nu\right\},$$
$$x_3 \in \bar{\mathcal{P}}(\alpha, r_2; \beta, l_2) \setminus \left[\bar{\mathcal{P}}(\alpha, r_2; \beta, l_2; \psi, \nu) \cup \bar{\mathcal{P}}(\alpha, r_1; \beta, l_1)\right].$$

The proof is complete. □

4 An Example

In this section, we present an example to illustrate the main result.

Example 4.1. Consider the following BVP

$$\begin{cases} \Delta^2 x(n) + f\left(n, x(n+1), \Delta x(n), \Delta x(n+1)\right) = 0,\ n \in [0, 200], \\ x(0) - \Delta x(0) = \dfrac{1}{2}x(100), \\ 3x(202) + \Delta x(201) = 2x(100), \end{cases} \tag{4.1}$$

where $f(n, u, v, w)$ is continuous and positive for all $(n, u, v, w) \in [0, N] \times [0, +\infty) \times \mathbb{R}^2$. Corresponding to BVP (1.1) we have $N = 200$, $m = 1$, $p = 2$, $\eta_1 = 20$, $\alpha_1 = \dfrac{1}{2}$, $\beta_1 = 2$, $a = b = d = 1$, $c = 3$, $\phi_2(x) = x$. It is easy to see that $(C1)$ and $(C2)$ hold. Choose the constant $k = 10$, then $\sigma_{10} = \min\left\{\dfrac{10}{202}, \dfrac{192}{202}\right\} = \dfrac{5}{101}$, $r_1 = 100$, $r_2 = 20000$, $l_1 = 300$, $l_2 = 1000$ and $\nu = 150$, then one sees that $\Lambda = 24533$, $H = \dfrac{20930}{101}$,

$$20000 = r_2 \geq \tau = 3030 > \nu = 150 > r_1 = 100,\ 1000 = l_2 \geq l_1 = 300,$$
$$\frac{1515}{2093} \leq \min\left\{\frac{20000}{24533}, \frac{500}{201}\right\}.$$

If

$f(n,u,v,w) < \min\left\{\frac{100}{24533}, \frac{150}{201}\right\}$ for all $(n,u,v,w) \in [0,200] \times [0,100] \times [-300,300] \times [-300,300]$;

$f(n,u,v,w) > \frac{1515}{2093}$ for all $(n,u,v,w) \in [10,192] \times \left[150, \frac{150}{3030}\right] \times [-1000,1000] \times [-1000,1000]$;

$f(n,u,v,w) \leq \min\left\{\frac{20000}{24533}, \frac{500}{201}\right\}$ for all $(n,u,v,w) \in [0,200] \times [0,20000] \times [-1000,1000] \times [-1000,1000]$.

then Theorem 3.1 implies that BVP (4.1) has at least three positive solutions such that

$$\max_{n\in[0,202]} x_1(n) < 100, \quad \max_{n\in[0,201]} |\Delta x_1(n)| < 300,$$

$$150 < \min_{n\in[10,192]} x_2(n) \leq \max_{n\in[0,202]} x_2(n) \leq 20000, \quad \max_{n\in[0,201]} |\Delta x_2(n)| \leq 1000,$$

$$100 < \max_{n\in[0,202]} x_3(n) < 20000, \quad \min_{n\in[10,192]} x_3(n) < 150, \text{ and } \max_{n\in[0,202]} |\Delta x_3(n)| \leq 1000.$$

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