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Global Attractivity in a Nonlinear Difference Equation and Applications to a Biological Model

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Abstract

Consider the following difference equation of order $k + 1$

$$x_{n+1} = f(x_n) + g(x_{n-k}),\ n = 0, 1, \ldots,$$

where $f, g : [0, \infty) \to [0, \infty)$ are continuous functions, and k is a nonnegative integer. We establish a sufficient condition for the global attractivity of positive solutions of this equation. Our result can be applied to the following biological model

$$x_{n+1} = \frac{ax_n^2}{b + x_n} + c\frac{e^{p-qx_n}}{1 + e^{p-qx_n}},\ n = 0, 1, \ldots,$$

where $0 < a < 1, b, c, p$ and q are positive constants, whose dynamics of solutions has been studied in [8] recently. A new global stability result for the positive solutions of this model is obtained, which is a significant improvement of the corresponding result obtained in [8].

AMS Subject Classifications: 39A10, 92D25.
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1 Introduction

The dynamics of positive solutions of the difference equation

$$x_{n+1} = \frac{ax_n^2}{b + x_n} + c\frac{e^{p-qx_n}}{1 + e^{p-qx_n}},\ n = 0, 1, \ldots, \tag{1.1}$$

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where $0 < a < 1, b, c, p$ and q are positive constants, is studied in [8] recently. Eq. (1.1) is a biological model derived from the evaluation of a perennial grass [13]. The boundedness and the persistence of positive solutions, the existence, the attractivity and the global asymptotic stability of the unique positive equilibrium and the existence of periodic solutions have been discussed in [8]. Motivated by the work in [8], we study here the asymptotic behavior of positive solutions of the following general higher order difference equation

$$x_{n+1} = f(x_n) + g(x_{n-k}),\ n = 0, 1, \ldots, \tag{1.2}$$

where $f, g : [0, \infty) \to [0, \infty)$ are continuous functions with f nondecreasing and k is a nonnegative integer.

Clearly, Eq. (1.1) is a special case of Eq. (1.2) with $f(x) = \dfrac{ax^2}{b + x}, g(x) = c\dfrac{e^{p-qx}}{1 + e^{p-qx}}$ and $k = 0$. In this paper, we will establish a sufficient condition for the global attractivity of positive solutions of Eq. (1.2). In particular, by applying our result to Eq. (1.1), we will obtain a new result on the globally asymptotic stability of positive solutions of Eq. (1.1), which is a significant improvement of the corresponding result obtained in [8].

Because of its theoretical interest and many applications, the study of difference equations has become an active area of research, see, for example, [1–13] and references cited therein. Recently, the asymptotic behavior of positive solutions of the higher order difference equation

$$x_{n+1} - x_n = p_n[f(x_{n-k}) - g(x_{n+1})],\ n = 0, 1, \ldots, \tag{1.3}$$

where $k \in \{0, 1, \ldots\}, f, g \in [[0, \infty), [0, \infty)]$ with g nondecreasing, and $\{p_n\}$ is a nonnegative sequence, is studied and some global attractivity results are obtained in [7]. These results may be applied to several difference equations derived from mathematical biology. However, Eq. (1.1) can not be written in the form (1.3) and so the results obtained in [7] cannot be applied to this case.

2 Main Results

In this section, we establish a sufficient condition for the global attractivity of positive solutions of Eq. (1.2). In the following discussion, we assume that Eq. (1.2) has a unique positive equilibrium $\bar{x}$, that is, $\bar{x}$ is the only positive solution of the equation $f(x) + g(x) = x$. In addition, we adopt the notation $\sum_{i=m}^{n} a_n = 0$ whenever $m > n$.

Theorem 2.1. *Assume that $f(x)$ is nondecreasing and there is a positive number $a < 1$ such that $f(x) \le ax$ and $f(x) - ax$ is nonincreasing for $x \ge 0$. Suppose also that*

$$(x - \bar{x})(f(x) + g(x) - x) < 0\ for\ x > 0\ and\ x \neq \bar{x}, \tag{2.1}$$

and that g is L-Lipschitz with

$$\left[\frac{1-a^{k+1}}{1-a}\right] L < 1.$$

Then every positive solution $\{x_n\}$ of Eq. (1.2) *converges to $\bar{x}$ as $n \to \infty$.*

Proof. First, we show that $\{x_n\}$ is bounded. Otherwise, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} > \max\{x_n : -k \leq n < n_i\},\ i = 1, 2, \ldots,\ \text{and}\ \lim_{i\to\infty} x_{n_i} = \infty.$$

From Eq. (1.2) we see that

$$g(x_{n_i-1-k}) + f(x_{n_i}) - x_{n_i} = f(x_{n_i}) - f(x_{n_i-1}) \geq 0$$

and so it follows that

$$g(x_{n_i-1-k}) \geq x_{n_i} - f(x_{n_i}). \tag{2.2}$$

Since $f(x) - ax$ is nondecreasing, $x - f(x)$ is increasing. Then (2.2) yields

$$g(x_{n_i-1-k}) > x_{n_i-k-1} - f(x_{n_i-k-1}).$$

Hence,

$$f(x_{n_i-1-k}) + g(x_{n_i-k-1}) - x_{n_i-k-1} > 0$$

which, in view of (2.1), implies that $x_{n_i-k-1} < \bar{x}$. Then it follows that $\{g(x_{n_i-k-1})\}$ is bounded which clearly contradicts (2.2) since

$$x_{n_i} - f(x_{n_i}) \geq x_{n_i} - ax_{n_i} \to \infty \ as\ i \to \infty.$$

Hence, $\{x_n\}$ must be bounded.

Now, we are ready to show that $x_n \to \bar{x}$ as $n \to \infty$. First, we assume that $\{x_n\}$ is a nonoscillatory (about $\bar{x}$) solution. Suppose that $x_n - \bar{x}$ is eventually nonnegative. The proof for the case that $x_n - \bar{x}$ is eventually nonpositive is similar and will be omitted. Let $\lim sup_{n\to\infty} x_n = A$. Then $\bar{x} \leq A < \infty$. Clearly, it suffices to show that $A = \bar{x}$. First, we assume that $\{x_n\}$ is nonincreasing eventually. Then $\lim\limits_{n\to\infty} x_n = A$. If $A > \bar{x}$, then by noting (2.1), it follows from Eq. (1.2) that

$$x_{n+1} - x_0 = \sum_{i=0}^{n} (f(x_i) + g(x_{i-k}) - x_i) \to -\infty \ as\ n \to \infty,$$

which is a contradiction. Hence, $A = \bar{x}$.

Next, assume that $\{x_n\}$ is not eventually nonincreasing. Then, there is a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that

$$\lim_{m\to\infty} x_{n_m} = A \text{ and } x_{n_m} > x_{n_m-1},\ m = 0, 1, \ldots.$$

Hence, it follows from Eq. (1.2) that

$$g(x_{n_m-1-k}) + f(x_{n_m}) - x_{n_m} = f(x_{n_m}) - f(x_{n_m-1}) \geq 0,\ m = 0, 1, \ldots,$$

and so

$$g(x_{n_m-1-k}) \geq x_{n_m} - f(x_{n_m}),\ m = 0, 1, \ldots. \tag{2.3}$$

Since $x_{n_m-1-k} \geq \bar{x}$,

$$g(x_{n_m-1-k}) + f(x_{n_m-1-k}) - x_{n_m-1-k} \leq 0$$

and it follows that

$$g(x_{n_m-1-k}) \leq x_{n_m-1-k} - f(x_{n_m-1-k}). \tag{2.4}$$

Hence, in view of (2.3) and (2.4), we see that

$$x_{n_m-1-k} - f(x_{n_m-1-k}) \geq x_{n_m} - f(x_{n_m}).$$

By noting that $x - f(x)$ is increasing, we see that $x_{n_m-1-k} \geq x_{n_m}$ which yields $\lim\limits_{m\to\infty} x_{n_m-1-k} = A$. Then by taking limit on both sides of (2.3), we find that

$$g(A) + f(A) - A \geq 0$$

which implies that $A \leq \bar{x}$. Hence, $A = \bar{x}$.

Finally, assume that $\{x_n\}$ is a solution of Eq. (1.2) and oscillates about $\bar{x}$. Let $y_n = x_n - \bar{x}$. Then $\{y_n\}$ satisfies

$$y_{n+1} = f(y_n + \bar{x}) - f(\bar{x}) + g(y_{n-k} + \bar{x}) - g(\bar{x}), n = 0, 1, \ldots, \tag{2.5}$$

and $\{y_n\}$ oscillates about zero. Since $\{x_n\}$ is bounded, there is a positive constant M such that $|y_n| = |x_n - \bar{x}| \leq M,\ n = 0, 1, \ldots$. Then by noting the Lipschitz property of g, we see that

$$|g(y_{n-k} + \bar{x}) - g(\bar{x})| \leq L|y_{n-k}| \leq LM, n \geq k.$$

Let y_l and y_s be two consecutive members of the solution $\{y_n\}$ with $N_0 < l < s$ such that

$$y_l \leq 0, y_{s+1} \leq 0 \text{ and } y_n > 0 \text{ for } l + 1 \leq n \leq s. \tag{2.6}$$

Let

$$y_r = \max\{y_{l+1}, y_{l+2}, \ldots, y_s\} \tag{2.7}$$

where y_r is chosen as the first one to reach the maximum among $y_{l+1}, y_{l+2} \dots y_s$. We claim that

$$r - (l+1) \le k. \tag{2.8}$$

Suppose, for the sake of contradiction, that $r - (l+1) > k$. Then, $y_r > y_{r-1-k} > 0$. By noting $y_{r-1-k} + \bar{x} > \bar{x}$ and (2.1), we see that

$$g(y_{r-1-k} + \bar{x}) + f(y_{r-1-k} + \bar{x}) - (y_{r-1-k} + \bar{x}) < 0. \tag{2.9}$$

Since $f(x) - x$ is decreasing,

$$f(y_{r-1-k} + \bar{x}) - (y_{r-1-k} + \bar{x}) > f(y_r + \bar{x}) - (y_r + \bar{x})$$

and then it follows from (2.9) that

$$g(y_{r-1-k} + \bar{x}) + f(y_r + \bar{x}) - (y_r + \bar{x}) < 0.$$

However, on the other hand, (2.5) yields

$$g(y_{r-1-k} + \bar{x}) + f(y_r + \bar{x}) - (y_r + \bar{x}) = f(y_r + \bar{x}) - f(y_{r-1} + \bar{x}) \ge 0$$

which contradicts (2.9). Hence, (2.8) holds.

Now, observe that

$$y_{n+1} - a y_n = f(y_n + \bar{x}) - f(\bar{x}) - a y_n + g(y_{n-k} + \bar{x}) - g(\bar{x})$$

and so it follows that

$$\frac{y_{n+1}}{a^{n+1}} - \frac{y_n}{a^n} = \frac{1}{a^{n+1}}[f(y_n + \bar{x}) - f(\bar{x}) - a y_n] + \frac{1}{a^{n+1}}[g(y_{n-k} + \bar{x}) - g(\bar{x})].$$

Summing up from l to $r-1$, we see that

$$\frac{y_r}{a^r} - \frac{y_l}{a^l} = \sum_{j=l}^{r-1} \frac{1}{a^{j+1}}\left[f(y_j + \bar{x}) - f(\bar{x}) - a y_j\right] + \sum_{j=l}^{r-1} \frac{1}{a^{j+1}}[g(y_{j-k} + \bar{x}) - g(\bar{x})]$$

and so

$$\begin{aligned}
y_r &= a^r \left(\frac{y_l}{a^l} + \sum_{j=l}^{r-1} \frac{1}{a^{j+1}}\left[f(y_j + \bar{x}) - f(\bar{x}) - a y_j\right] + \sum_{j=l}^{r-1} \frac{1}{a^{j+1}}[g(y_{j-k} + \bar{x}) - g(\bar{x})] \right) \\
&= a^r \left(\frac{y_l}{a^l} + \frac{1}{a^{l+1}}[f(y_l + \bar{x}) - f(\bar{x}) - a y_l] + \sum_{j=l}^{r-2} \frac{1}{a^{j+1}}[f(y_j + \bar{x}) - f(\bar{x}) - a y_j] \right. \\
&\quad \left. + \sum_{j=l}^{r-1} \frac{1}{a^{j+1}}[g(y_{j-k} + \bar{x}) - g(\bar{x})] \right) \\
&= a^r \left(\frac{1}{a^{l+1}}[f(y_l + \bar{x}) - f(\bar{x})] + \sum_{j=l}^{r-2} \frac{1}{a^{j+1}}[f(y_j + \bar{x}) - f(\bar{x}) - a y_j] \right. \\
&\quad \left. + \sum_{j=l}^{r-1} \frac{1}{a^{j+1}}[g(y_{j-k} + \bar{x}) - g(\bar{x})] \right).
\end{aligned} \tag{2.10}$$

Then by noting (2.6), $f(x)$ is nondecreasing and $f(x)-ax$ is nonincreasing, we see that

$$f(y_l+\bar{x})-f(\bar{x})\le 0 \text{ and } f(y_j+\bar{x})-f(\bar{x})-ay_j\le 0,\ j=l+1,\dots,r-1.$$

Hence, it follow from (2.10) that

$$y_r\le a^r\sum_{j=l}^{r-1}\frac{1}{a^{j+1}}[g(y_{j-k}+\bar{x}))-g(\bar{x}))]\le\left[\frac{1-a^{k+1}}{1-a}\right]LM$$

and so

$$y_n\le\left[\frac{1-a^{k+1}}{1-a}\right]LM,\ l\le n\le s.$$

Since y_l and y_s are two arbitrary members of the solution with property (2.6), we see that there is a positive integer $N_1'\ge N_0$ such that $y_n\le[\frac{1-a^{k+1}}{1-a}]ML,\ n\ge N_1'$. Then, by a similar argument, it can be shown that there is a positive integer $N_1''\ge N_0$ such that $y_n\ge-[\frac{1-a^{k+1}}{1-a}]ML,\ n\ge N_1''$. Hence, there is a positive integer $N_1\ge N_0$ such that

$$|y_n|\le\left[\frac{1-a^{k+1}}{1-a}\right]LM,\ n\ge N_1. \tag{2.11}$$

Now, by noting the Lipschitz property of $f(x)$ and (2.11), we see that

$$|g(y_{n-k}+\bar{x})-g(\bar{x})|\le L|y_{n-k}|\le\left[\frac{1-a^{k+1}}{1-a}\right]L^2M, n\ge N_1+k.$$

Let y_l and y_s be two consecutive members of the solution $\{y_n\}$ with $t_0\le l<s$ such that (2.6) holds. Let y_r be defined by (2.7). By a similar argument, we may show that (2.8) holds and

$$y_r\le\left(\left[\frac{1-a^{k+1}}{1-a}\right]L\right)^2M.$$

Then it follows that $y_n(\le[\frac{1-a^{k+1}}{1-a}]L)^2M,\ l\le n\le s$ and so again by noting y_l and y_s are two arbitrary members of the solution with property (2.6), there is a positive integer $N_2'\ge N_1+k$ such tat $y_n\le([\frac{1-a^{k+1}}{1-a}]L)^2M, n\ge N_2'$. Similarly, it can be shown that there is a positive integer $N_2''\ge N_1+K$ such that $y_n\ge-([\frac{1-a^{k+1}}{1-a}]L)^2M, n\ge N_2''$. Hence, there is a positive integer $N_2\ge N_1+k$ such that $|y_n|\le([\frac{1-a^{k+1}}{1-a}]L)^2M, n\ge N_2$. Finally, by induction, we find that for any positive integer m, there is a positive integer N_m with $N_m\to\infty$ as $m\to\infty$ such that

$$|y_n|\le\left(\left[\frac{1-a^{k+1}}{1-a}\right]L\right)^mM, n\ge N_m.$$

Then, by noting the hypotheses $\left[\dfrac{1-a^{k+1}}{1-a}\right] L < 1$, we see that $y_n \to 0$ as $n \to \infty$, and so it follows that $x_n \to \bar{x}$ as $n \to \infty$. The proof is complete. □

3 Applications

Consider the difference equation

$$x_{n+1} = \frac{ax_n^2}{b+x_n} + c\frac{e^{p-qx_{n-k}}}{1+e^{p-qx_{n-k}}},\ n = 0, 1, \ldots, \tag{3.1}$$

where $0 < a < 1, b, c, p$ and q are positive constants, and k is a nonnegative integer. Eq. (3.1) is in the form of (1.2) with

$$f(x) = \frac{ax^2}{b+x} \text{ and } g(x) = c\frac{e^{p-qx}}{1+e^{p-qx}}.$$

Clearly, f is increasing and $f(x) \leq ax$. By noting

$$(f(x) - ax)' = -\frac{ab^2}{(b+x)^2} < 0$$

we see that $f(x) - ax$ is decreasing. Let

$$F(x) = \frac{ax^2}{b+x} + c\frac{e^{p-qx}}{1+e^{p-qx}} - x.$$

It is easy to check that $F(0) > 0,\ \lim_{x\to\infty} F(x) = -\infty$ and $F'(x) < 0$. Hence, there is a unique positive number $\bar{x}$ such that $F(\bar{x}) = 0$, that is, Eq. (3.1) has a unique positive equilibrium $\bar{x}$, and

$$(x - \bar{x})(f(x) + g(x) - x) < 0,\ x > 0 \text{ and } x \neq \bar{x}.$$

Now, observe that

$$g'(x) = -cq\frac{e^{p-qx}}{(1+e^{p-qx})^2} \text{ and } g''(x) = cq^2\frac{e^{p-qx}(1-e^{p-qx})}{(1+e^{p-qx})^3}.$$

By noting that $x = \dfrac{p}{q}$ is the only point such that $g''(x) = 0$, we see that $|g'(x)|$ takes maximum when $x = \dfrac{p}{q}$ and $|g'(\dfrac{p}{q})| = \dfrac{cq}{4}$. Hence, g is L-Lipschitz with $L = \dfrac{cq}{4}$. Then by Theorem 2.1, we have that following conclusion immediately.

Theorem 3.1. *Assume that*

$$\left[\frac{1-a^{k+1}}{1-a}\right]\frac{cq}{4} < 1. \tag{3.2}$$

Then every positive solution $\{x_n\}$ *of Eq.* (3.1) *tends to its positive equilibrium* $\bar{x}$ *as* $n \to \infty$.

We claim that (3.2) is also a sufficient condition for the positive equilibrium $\bar{x}$ to be globally asymptotically stable when $k = 0$ or $k = 1$. In fact, when $k = 0$, (3.2) reduces to $cq < 4$ and the linearized equation of (3.1) about the equilibrium $\bar{x}$ is

$$x_{n+1} = \left(\frac{a\bar{x}^2 + 2ab\bar{x}}{(\bar{x}+b)^2} - cq\frac{e^{p-q\bar{x}}}{(1+e^{p-q\bar{x}})^2} \right) x_n. \tag{3.3}$$

It is obvious that

$$\frac{a\bar{x}^2 + 2ab\bar{x}}{(\bar{x}+b)^2} - cq\frac{e^{p-q\bar{x}}}{(1+e^{p-q\bar{x}})^2} < 1$$

since

$$\frac{a\bar{x}^2 + 2ab\bar{x}}{(\bar{x}+b)^2} < \frac{\bar{x}^2 + 2b\bar{x}}{(\bar{x}+b)^2} < 1.$$

By noting $cq < 4$, we see that

$$cq\frac{e^{p-q\bar{x}}}{(1+e^{p-q\bar{x}})^2} < 4\frac{e^{p-q\bar{x}}}{(1+e^{p-q\bar{x}})^2} \le 1 < 1 + \frac{a\bar{x}^2 + 2ab\bar{x}}{(\bar{x}+b)^2}$$

and so

$$-1 < \frac{a\bar{x}^2 + 2ab\bar{x}}{(\bar{x}+b)^2} - cq\frac{e^{p-q\bar{x}}}{(1+e^{p-q\bar{x}})^2}.$$

Hence, it follows that

$$\left| \frac{a\bar{x}^2 + 2ab\bar{x}}{(\bar{x}+b)^2} - cq\frac{e^{p-q\bar{x}}}{(1+e^{p-q\bar{x}})^2} \right| < 1$$

and so the zero solution of Eq. (3.3) is asymptotically stable. By the linearization stability theory, the equilibrium $\bar{x}$ of Eq. (3.1) is locally asymptoticly stable. Then by combining this fact and Theorem 3.1, we see that the equilibrium $\bar{x}$ of Eq. (3.1) is globally asymptoticly stable.

When $k = 1$, (3.2) reduces to $cq < \dfrac{4}{1+a}$ and the linearized equation of (3.1) about the equilibrium $\bar{x}$ is

$$x_{n+1} = \frac{a\bar{x}^2 + 2ab\bar{x}}{(\bar{x}+b)^2} x_n - cq\frac{e^{p-q\bar{x}}}{(1+e^{p-q\bar{x}})^2} x_{n-1}. \tag{3.4}$$

It is well known (see, for example [6]) for the linear equation

$$y_{n+1} + \alpha y_n + \beta y_{n-1} = 0,$$

where α and β are constants, a necessary and sufficient condition for the asymptotic stability is

$$|\alpha| < 1 + \beta < 2.$$

Hence, by noting that under the condition $cq < \dfrac{4}{1+a}$,

$$\frac{a\bar{x}^2 + 2ab\bar{x}}{(\bar{x}+b)^2} < 1 + cq\frac{e^{p-q\bar{x}}}{(1+e^{p-q\bar{x}})^2} < 1 + 4\frac{e^{p-q\bar{x}}}{(1+e^{p-q\bar{x}})^2} \le 2,$$

we see that the zero solution of Eq. (3.4) is asymptoticly stable. Then it follows that the equilibrium $\bar{x}$ of Eq. (3.1) is locally asymptotically stable, which together with Theorem 3.1 implies that $\bar{x}$ is globally asymptotically stable.

Remark 3.2. In particular, when $k = 0$, Eq. (3.1) reduces to the first order difference equation

$$x_{n+1} = \frac{ax_n^2}{b+x_n} + c\frac{e^{p-qx_n}}{1+e^{p-qx_n}}, \; n = 0, 1, \dots \tag{3.5}$$

which is a biological model mentioned in Section 1. It has been shown in [8] that if

$$cq < 2(1-a), \tag{3.6}$$

then the positive equilibrium $\bar{x}$ of Eq. (3.5) is globally asymptotically stable. Clearly, our condition $cq < 4$ is a significant improvement of (3.6).

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