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# Oscillatory Behavior of Third-order Difference Equations with Asynchronous Nonlinearities

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**Abstract**

This paper studied the oscillatory and asymptotic properties of the third order difference equation with asynchronous nonlinearities. Some sufficient conditions which ensure that all solutions are either oscillatory or converges to zero were established, and one example was presented to illustrate the main results.

**AMS Subject Classifications:** 39A21.
**Keywords:** Oscillation, third-order, asynchronous.

## 1 Introduction

In this paper, we are concerned with the following third-order difference equation of the form

$$\Delta(a_n\Delta^2(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2})) + q_n x^{\alpha}_{n+1-\sigma_1} + p_n x^{\beta}_{n+1+\sigma_2} = 0 \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{p_n\}$ and $\{q_n\}$ are real positive sequences, $\alpha \geq \beta$ are ratios of odd positive integers, $\tau_1$, $\tau_2$, $\sigma_1$, $\sigma_2$ are positive integers, and $n \in N = \{n_0, n_0 + 1, \cdots\}$, $n_0$ is a nonnegative integer.

Let $\theta = \max\{\tau_1, \sigma_1\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ and satisfying equation (1.1) for all $n \in \mathbb{N}$. A nontrivial

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solution $\{x_n\}$ is said to be nonoscillatory if it is either eventually positive or eventually negative and it is oscillatory otherwise.

Since the neutral type difference equations have various applications in problems dealing with vibrating masses attached to an elastic bar and in some variational problems, there was increasing interest in studying the oscillatory behavior of this kind of equations, the references therein. Recently, many authors discussed the oscillation problems of neutral difference equations [1–10]. Among of these, [8] considered the $m$-th order mixed linear neutral difference equation

$$\Delta^m(x_n + ax_{n-k} - bx_{n+\sigma}) + qx_{n-g} + px_{n+h} = 0 \tag{1.2}$$

and obtained some oscillation theorems for the oscillation of all solutions of equation (1.2). E.Thandapani et. al. [10] considered the third-order, mixed neutral type difference equation

$$\Delta(a_n\Delta^2(x_n + b_nx_{n-\tau_1} + c_nx_{n+\tau_2})) + q_nx^{\beta}_{n+1-\sigma_1} + p_nx^{\beta}_{n+1+\sigma_2} = 0, \tag{1.3}$$

and established some sufficient conditions for the oscillation of all solutions of equation (1.3).

It is easy to see that if $\alpha = \beta$ in (1.1), then it becomes (1.3), we call it is a synchronous case, if $\alpha \neq \beta$, then we call (1.1) is an equation with asynchronous nonlinearities. To the best of our knowledge, there is no paper dealing with the asynchronous case, which inspires us to discuss the oscillatory and asymptotic behavior of solutions of equation (1.1).

## 2 Oscillation Results

In this section, we present some oscillation criteria for equation (1.1). Without loss the generality, we admit that a functional inequality holds for all sufficiently large $n$ and the following conditions hold throughout of this paper.

(H$_1$) $\{a_n\}$ is a positive nondecreasing sequence such that $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$;

(H$_2$) $\{b_n\}$ and $\{c_n\}$ are real sequences such that $0 < b_n < b$ and $0 < c_n < c$ with $b + c < 1$.

**Lemma 2.1** (See [4]). *Assume $A \geq 0$ $B \geq 0$, $\alpha \geq 1$. Then*

$$(A + B)^{\alpha} \leq 2^{\alpha-1}(A^{\alpha} + B^{\alpha}).$$

Similar to [4], we have the following lemmas. Let $y_n = x_n + b_nx_{n-\tau_1} + c_nx_{n+\tau_2}$.

**Lemma 2.2.** *Let $\{x_n\}$ be a positive solution of equation* (1.1)*. Then there are only two cases hold for $n \geq n_1 \in \mathbb{N}$ sufficiently large:*
*(1) $y_n > 0, \Delta y_n > 0, \Delta^2 y_n > 0, \Delta(a_n \Delta^2 y_n) \leq 0$;*
*(2) $y_n > 0, \Delta y_n < 0, \Delta^2 y_n > 0, \Delta(a_n \Delta^2 y_n) \leq 0$.*

**Lemma 2.3.** *Let $y_n > 0, \Delta y_n > 0, \Delta^2 y_n > 0, \Delta^3 y_n \leq 0$ for all $n \geq N \in \mathbb{N}$. Then for any $\xi \in (0,1)$, and some integer $N_1$, the following inequalities*

$$\frac{y_{n+1}}{\Delta y_n} \geq \frac{n-N}{2} \geq \frac{\xi n}{2} \quad \text{for } n \geq N_1 \geq N \tag{2.1}$$

*hold.*

**Lemma 2.4.** *Suppose that $\{x_n\}$ be a bounded positive solution of equation* (1.1) *with the upper bound $M$, $y_n$ satisfies (2) of Lemma 2.2. If*

$$\sum_{n=n_0}^{\infty} \sum_{l=n}^{\infty} \left( \frac{1}{a_l} \sum_{r=l}^{\infty} (q_r + M^{\beta-\alpha} p_r) \right) = \infty \tag{2.2}$$

*holds, then $\lim_{n\to\infty} x_n = 0$.*

*Proof.* Let $\{x_n\}$ be a positive solution of equation (1.1) satisfying $x_n \leq M$. Since $y_n > 0$ and $\Delta y_n < 0$, then $\lim_{n\to\infty} y_n = \mu \geq 0$ exists. It can be proved that $\mu = 0$. If not, then $\mu > 0$, and for any $\epsilon > 0$, we have $\mu + \epsilon > y_n$ eventually. Choose

$$0 < \epsilon < \frac{\mu(1-b-c)}{b+c}.$$

Then we have

$$x_n = y_n - b_n x_{n-\tau_1} - c_n x_{n+\tau_2} > \mu - (b+c) y_{n-\tau_1} > \mu - (b+c)(\mu+\epsilon) = d(\mu+\epsilon) > d y_n,$$

where $d = \dfrac{\mu - (b+c)(\mu+\epsilon)}{\mu+\epsilon} > 0$. Further,

$$\Delta(a_n \Delta y_n) \leq -q_n d^\alpha y_{n+1-\sigma_1}^\alpha - p_n d^\beta y_{n+1-\sigma_2}^\beta \leq -d^\alpha (q_n + M^{\beta-\alpha} p_n) y_{n+1-\tau_1}^\alpha.$$

Summing the above inequality from $n$ to $\infty$, and using the relation $y_n \geq \mu$, we obtain

$$\Delta^2 y_n \geq (d\mu)^\alpha \left( \frac{1}{a_n} \sum_{r=n}^{\infty} (q_r + M^{\beta-\alpha} p_r) \right). \tag{2.3}$$

Summing the two sides of (2.3) from $n$ to $\infty$, we have

$$-\Delta y_n \geq (d\mu)^\alpha \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{r=s}^{\infty} (q_r + M^{\beta-\alpha} p_r).$$

Summing from $n_1$ to $\infty$ leads to

$$y_{n_1} \geq (d\mu)^\alpha \sum_{n=n_1}^{\infty} \sum_{s=n}^{\infty} \left( \frac{1}{a_s} \sum_{r=s}^{\infty} (q_r + M^{\beta-\alpha} p_r) \right),$$

which is a contradiction to (2.2). Then $\mu = 0$, which together with the inequality $0 < x_n < y_n$ implies that $\lim_{n\to\infty} x_n = 0$. The proof is complete. □

Let $Q_n = \min\{q_n, q_{n-\tau_1}, q_{n+\tau_2}\}$, $P_n = \min\{p_n, p_{n-\tau_1}, p_{n+\tau_2}\}$.

**Theorem 2.5.** *Suppose that* $\{x_n\}$ *be a bounded positive solution of equation* (1.1) *with the upper bound* $M$*, and that condition* (2.2) *holds,* $\sigma_1 \geq \tau_1$ *and* $\alpha, \beta \geq 1$*. If there exists a positive real sequence* $\{\gamma_n\}$ *and an integer* $N_1 \in \mathbb{N}$ *such that for some* $\xi \in (0,1)$ *and* $\delta > 0$

$$\limsup_{n\to\infty} \sum_{s=N_1}^{n-1} \left( \gamma_s \left( \left(\frac{\delta}{4}\right)^{\alpha-1} \frac{\xi(n-\sigma_1)^\alpha}{2^\alpha} W_s \right) - \frac{(1 + b^\alpha + \frac{c^\beta}{2^{\beta-1}})}{4} \frac{a_{s-\sigma_1}(\Delta\gamma_s)^2}{\gamma_s} \right) = \infty \tag{2.4}$$

*holds, then every such solution* $\{x_n\}$ *of equation* (1.1) *oscillates or* $\lim_{n\to\infty} x_n = 0$*, where* $W_s = Q_s + M^{\beta-\alpha} P_s$*.*

*Proof.* Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1), and $x_n \leq M$. Without loss of the generality, assume that there exists an integer $N \geq n_0$ such that $x_n$, $x_{n-\sigma_1}$, $x_{n+\sigma_2}, x_{n-\tau_1}, x_{n+\tau_2} \in (0, M]$ for all $n > N$, we have

$$\begin{aligned} &\Delta(a_n \Delta^2 y_n) + q_n x_{n+1-\sigma_1}^\alpha + p_n x_{n+1+\sigma_2}^\beta + b^\alpha \Delta(a_{n-\tau_1} \Delta^2 y_{n-\tau_1}) \\ &+ b^\alpha q_{n-\tau_1} x_{n+1-\tau_1-\sigma_1} + b^\alpha p_{n-\tau_1} x_{n+1-\tau_1+\sigma_2}^\beta + \frac{c^\beta}{2^{\beta-1}} \Delta(a_{n+\tau_2} \Delta^2 y_{n+\tau_2}) \\ &+ \frac{c^\beta}{2^{\beta-1}} q_{n+\tau_2} x_{n+1+\tau_2-\sigma_1}^\alpha + \frac{c^\beta}{2^{\beta-1}} p_{n+\tau_2} x_{n+1+\tau_2-\sigma_2}^\beta = 0. \end{aligned} \tag{2.5}$$

By Lemma 2.1 and $\beta \leq \alpha$ in (2.5), we have

$$\begin{aligned} &\Delta(a_n \Delta^2 y_n) + b^\alpha \Delta(a_{n-\tau_1} \Delta^2 y_{n-\tau_1}) + \frac{c^\beta}{2^{\beta-1}} \Delta(a_{n+\tau_2} \Delta^2 y_{n+\tau_2}) \\ &+ \frac{Q_n}{4^{\alpha-1}} y_{n+1-\sigma_1}^\alpha + \frac{M^{\beta-\alpha} P_n}{4^{\alpha-1}} y_{n+1+\sigma_2}^\alpha \leq 0. \end{aligned} \tag{2.6}$$

By Lemma 2.2, there are two cases for $y_n$ to be considered. Assume that case (1) holds for all $n \geq N_1 \geq N$. It follows from $\Delta y_n > 0$ that $y_{n+\sigma_2} > y_{n-\sigma_1}$. (2.6) tells us

$$\Delta a_n \Delta^2 y_n + b^\alpha \Delta(a_{n-\tau_1} \Delta^2 y_{n-\tau_1}) + \frac{c^\beta}{2^{\beta-1}} \Delta(a_{n+\tau_2} \Delta^2 y_{n+\tau_2}) + \frac{R_n}{4^{\alpha-1}} y_{n+1-\sigma_1}^\alpha \leq 0. \tag{2.7}$$

Define

$$\nu_1(n) = \gamma_n \frac{a_n \Delta^2 y_n}{\Delta y_{n-\sigma_1}}, \quad n \geq N_1. \tag{2.8}$$

Then $\nu_1(n) > 0$ for $n \geq N_1$. From (2.8), we see that

$$\Delta \nu_1(n) = \frac{\Delta \gamma_n}{\gamma_{n+1}} \nu_1(n+1) + \gamma_n \frac{\Delta(a_n \Delta^2 y_n)}{\Delta y_{n-\sigma_1}} - \nu_1(n+1) \left( \frac{\Delta^2 y_{n-\sigma_1}}{\Delta y_{n-\sigma_1}} \right).$$

By (1.1), we know that

$$\Delta(a_n \Delta^2 z_n) = -q_n x^{\alpha}_{n+1-\sigma_1} - p_n x^{\beta}_{n+1+\sigma_2} < 0, \tag{2.9}$$

and $a_{n-\sigma_1} \Delta^2 y_{n-\sigma_1} \geq a_{n+1} \Delta^2 y_{n+1}$. Thus from (2.8), we have

$$\Delta \nu_1(n) \leq \frac{\Delta \gamma_n}{\gamma_{n+1}} \nu_1(n+1) + \gamma_n \frac{\Delta(a_n \Delta^2 y_n)}{\Delta y_{n-\sigma_1}} - \gamma_n \frac{{\nu_1}^2(n+1)}{\gamma^2_{n+1} a_{n-\sigma_1}}. \tag{2.10}$$

Define

$$\nu_2(n) = \gamma_n \frac{a_{n-\tau_1} \Delta^2 y_{n-\tau_1}}{\Delta y_{n-\sigma_1}}, \quad n \geq N_1. \tag{2.11}$$

Then $\nu_2(n) > 0$ for $n > N_1$, which together with (2.11) yields

$$\Delta \nu_2(n) = \frac{\Delta \gamma_n}{\gamma_{n+1}} \nu_2(n+1) + \gamma_n \frac{\Delta(a_{n-\tau_1} \Delta^2 y_{n-\tau_1})}{\Delta y_{n-\sigma_1}} - \nu_2(n+1) \left( \frac{\Delta^2 y_{n-\sigma_1}}{\Delta y_{n-\sigma_1}} \right).$$

Note that $\sigma_1 > \tau_1$. By (2.9), we find $a_{n-\sigma_1} \Delta^2 y_{n-\sigma_1} \geq a_{n+1-\tau_1} \Delta^2 y_{n+1-\tau_1}$. Hence by (2.11), we obtain

$$\Delta \nu_2(n) \leq \frac{\Delta \gamma_n}{\gamma_{n+1}} \nu_2(n+1) + \gamma_n \frac{\Delta(a_{n-\tau_1} \Delta^2 y_{n-\tau_1})}{\Delta y_{n-\sigma_1}} - \gamma_n \frac{w^2_2(n+1)}{\gamma^2_{n+1} a_{n-\sigma_1}}. \tag{2.12}$$

Similarly, define

$$\nu_3(n) = \gamma_n \frac{a_{n+\tau_2} \Delta^2 y_{n+\tau_2}}{\Delta y_{n-\sigma_1}}, \quad n \geq N_1. \tag{2.13}$$

Then we have

$$\Delta \nu_3(n) \leq \frac{\Delta \gamma_n}{\gamma_{n+1}} \nu_3(n+1) + \gamma_n \frac{\Delta(a_{n+\tau_2} \Delta^2 y_{n+\tau_2})}{\Delta y_{n-\sigma_1}} - \gamma_n \frac{\nu^2_3(n+1)}{\gamma^2_{n+1} a_{n-\sigma_1}}. \tag{2.14}$$

Therefore (2.10), (2.12) and (2.14) show that

$$\begin{aligned}&\Delta\nu_1(n)+b^\alpha\Delta\nu_2(n)+\frac{c^\beta}{2^{\beta-1}}\Delta\nu_3(n)\\ &\le -\gamma_n\frac{R_n}{4^{\alpha-1}}\frac{y^\alpha_{n+1-\sigma_1}}{\Delta y_{n-\sigma_1}}+\frac{\Delta\gamma_n}{\gamma_{n+1}}-\gamma_n\frac{\nu_1^2(n+1)}{\gamma^2_{n+1}a_{n-\sigma_1}}\\ &+b^\alpha\left(\frac{\Delta\gamma_n\nu_2(n+1)}{\gamma_{n+1}}-\gamma_n\frac{\nu_2^2(n+1)}{\gamma^2_{n+1}a_{n-\sigma_1}}\right)\\ &+\frac{c^\beta}{2^{\beta-1}}\left(\frac{\Delta\gamma_n\nu_3(n+1)}{\gamma_{n+1}}-\gamma_n\frac{\nu_3^2(n+1)}{\gamma^2_{n+1}a_{n-\sigma_1}}\right).\end{aligned}\tag{2.15}$$

On the other hand, since $\{a_n\}$ is nondecreasing and $\Delta^2 y_n > 0$ for $n \ge N_1$. Then we have $\Delta^3 y_n < 0$ for $n \ge N_1$. By Lemma 2.3 for any $\xi \in (0,1)$, and $n$ sufficiently large

$$\frac{y_{n+1-\sigma_1}}{\Delta y_{n-\sigma_1}} \ge \frac{\xi(n-\sigma_1)}{2}.\tag{2.16}$$

Note that $y_n > 0, \Delta y_n > 0$ and $\Delta^2 y_n > 0$ for $n \ge N_1$. Thus

$$y_n = y_{N_1}+\sum_{s=N_1}^{n-1}\Delta y_s \ge (n-N_1)\Delta y_{N_1} \ge \frac{\delta n}{2}\tag{2.17}$$

for some $\delta > 0$ and $n$ sufficiently large. From (2.16), (2.17) and $\beta \ge 1$ we have $\dfrac{y^\alpha_{n+1-\sigma_1}}{\Delta y_{n-\sigma_1}} \ge \dfrac{\lambda^{\alpha-1}\xi(n-\sigma_1)^\alpha}{2^\alpha}$. Combining the last inequality with (2.15), and completing the square on the right-hand side of the resulting inequality, we obtain

$$\begin{aligned}&\Delta\nu_1(n)+b^\alpha\Delta\nu_2(n)+\frac{c^\beta}{2^{\beta-1}}\Delta\nu_3(n)\\ &\le -\gamma_n\left((\frac{\delta}{4})^{\alpha-1}\frac{\xi(n-\sigma_1)^\alpha}{2^\alpha}[Q_s+M^{\beta-\alpha}P_s]\right)+\frac{(1+b^\alpha+\frac{c^\beta}{2^{\beta-1}})a_{n-\sigma_1}(\Delta\gamma_n)^2}{4\gamma_n}.\end{aligned}$$

Summing the last inequality from $N_2 \ge N_1$ to $n-1$ implies that

$$\begin{aligned}&\sum_{s=N_2}^{n-1}\left(\gamma_s\left((\frac{\delta}{4})^{\alpha-1}\frac{\xi(n-\sigma_1)^\alpha}{2^\alpha}[Q_s+M^{\beta-\alpha}P_s]\right)+\frac{(1+b^\alpha+\frac{c^\beta}{2^{\beta-1}})a_{s-\sigma_1}(\Delta\gamma_s)^2}{4\gamma_s}\right)\\ &\le \nu(N_2)+b^\alpha\nu_2(N_2)+\frac{c^\beta}{2^{\beta-1}}\nu_3(N_2).\end{aligned}\tag{2.18}$$

Taking $\limsup$ in the last inequality yields a contradiction to (2.4).

The case (2) of Lemma 2.2 can be proved similarly, here it is omitted. The proof is complete. □

Let $\gamma_n = n$ and $\alpha = \beta = 1$. Then the following corollary is easily obtained.

**Corollary 2.6.** *Suppose that* $\{x_n\}$ *be a bounded positive solution of equation* (1.1) *with the upper bound* $M$*, and* $\alpha = 1$*. Assume that condition* (2.4) *holds and* $\sigma_1 > \tau_1$*. If there is an integer* $N_1 \in \mathbb{N}$ *such that for some* $\xi \in (0,1)$ *and* $\delta > 0$

$$\limsup_{n\to\infty}\sum_{s=N_1}^{n-1}\left(s\left(\left(\frac{\delta}{4}\right)^{\alpha-1}\frac{\xi(n-\sigma_1)^\alpha}{2^\alpha}W_s\right)-\frac{1+b+c}{4s}a_{s-\sigma_1}\right)=\infty$$

*holds, then every such solution* $\{x_n\}$ *of equation* (1.1) *oscillates or* $\lim_{n\to\infty} x_n = 0$*, where* $W_s = Q_s + M^{\beta-\alpha}P_s$*.*

**Theorem 2.7.** *Suppose that* $\{x_n\}$ *be a bounded positive solution of equation* (1.1) *with the upper bound* $M$*. Assume that condition* (2.4) *holds,* $\sigma_1 \le \tau_1$*, and* $\alpha \ge \beta \ge 1$*. If there exists a positive real sequence* $\gamma_n$*, and an integer* $N_1 \in \mathbb{N}$ *such that for some* $\xi \in (0,1)$ *and* $\delta > 0$

$$\limsup_{n\to\infty}\sum_{s=N_1}^{n-1}\left(\gamma_s\left((\frac{\delta}{4})^{\alpha-1}\frac{\xi(n-\sigma_1)^\alpha}{2^\alpha}W_s\right)-\frac{(1+b^\alpha+\frac{c^\beta}{2^{\beta-1}})a_{s-\tau_1}(\Delta\gamma_s)^2}{4\gamma_s}\right)=\infty \tag{2.19}$$

*holds, then every such solution* $\{x_n\}$ *of equation* (1.1) *oscillates or* $\lim_{n\to\infty} x_n = 0$*, where* $W_s = Q_s + M^{\beta-\alpha}P_s$*.*

*Proof.* Similar to Theorem 2.5, (2.6) is still true. By Lemma 2.2, there are two cases for $y_n$. Assume that case (1) holds for all $n \ge N_1 \ge N$. Then (2.7) holds. In like manner, define

$$\nu_1(n) = \gamma_n\frac{a_n\Delta^2 y_n}{\Delta y_{n-\tau_1}},\ \ n \ge N_1,$$

$$\nu_2(n) = \gamma_n\frac{a_{n-\tau_1}\Delta^2 y_{n-\tau_1}}{\Delta y_{n-\tau_1}},\ \ n \ge N_1,$$

and

$$\nu_3(n) = \gamma_n\frac{a_{n+\tau_2}\Delta^2 y_{n+\tau_2}}{\Delta y_{n-\tau_1}},\ \ n \ge N_1.$$

We have

$$\begin{aligned}
&\Delta\nu_1(n) + b^\alpha\Delta\nu_2(n) + \frac{c^\beta}{2^{\beta-1}}\Delta\nu_3(n)\\
&\le -\gamma_n\frac{R_n}{4^{\alpha-1}}\frac{z^\alpha_{n+1-\tau_1}}{\Delta y_{n-\tau_1}} + \frac{\Delta\gamma_n}{\gamma_{n+1}} - \gamma_n\frac{w_1^2(n+1)}{\gamma^2_{n+1}a_{n-\tau_1}}\\
&+b^\alpha\left(\frac{\Delta\gamma_n\nu_2(n+1)}{\gamma_{n+1}} - \gamma_n\frac{w_2^2(n+1)}{\gamma^2_{n+1}a_{n-\tau_1}}\right)\\
&+\frac{c^\beta}{2^{\beta-1}}\left(\frac{\Delta\gamma_n\nu_3(n+1)}{\gamma_{n+1}} - \gamma_n\frac{w_3^2(n+1)}{\gamma^2_{n+1}a_{n-\tau_1}}\right).
\end{aligned} \tag{2.20}$$

On the other hand, we have for any $\xi \in (0,1)$

$$\frac{y_{n+1-\sigma_1}}{\Delta y_{n-\tau_1}} = \frac{y_{n+1-\sigma_1}}{\Delta y_{n-\sigma_1}} \cdot \frac{\Delta y_{n-\sigma_1}}{\Delta y_{n-\tau_1}} \geq \frac{\xi(n-\sigma_1)}{2},$$

for all $n \geq N_2$. As in the proof of Theorem 2.5, we obtain

$$\Delta\nu_1(n) + b^\alpha \Delta\nu_2(n) + \frac{c^\beta}{2^{\beta-1}}\Delta\nu_3(n) \leq -\gamma_n\omega_n + \frac{(1+b^\alpha+\frac{c^\beta}{2^{\beta-1}})a_{n-\tau_1}(\Delta\gamma_n)^2}{4\gamma_n}.$$

Summing the last inequality from $N_2$ to $n-1$, we obtain

$$\sum_{s=N_2}^{n-1}\left(\gamma_s\left((\frac{\delta}{4})^{\alpha-1}\frac{\xi(n-\sigma_1)^\alpha}{2^\alpha}[Q_s+M^{\beta-\alpha}P_s]\right)+\frac{(1+b^\alpha+\frac{c^\beta}{2^{\beta-1}})a_{s-\tau_1}(\Delta\gamma_s)^2}{4\gamma_s}\right)$$
$$\leq \nu_1(N_2)+b^\alpha\nu_2(N_2)+\frac{c^\beta}{2^{\beta-1}}\nu_3(N_2). \tag{2.21}$$

Taking $\limsup$ on both sides of the last inequality yields a contradiction to (2.19).

The case (2) can be proved similarly. The proof is completed. □

Let $\gamma_n = n$ and $\alpha = \beta = 1$. Then, we can obtain the following result.

**Corollary 2.8.** *Suppose that $\{x_n\}$ be a bounded positive solution of equation* (1.1) *with the upper bound $M$. Assume that condition* (2.2) *holds and $\tau_1 \geq \sigma_1$. If*

$$\limsup_{n\to\infty}\sum_{s=N_1}^{n-1}\left(s\left((\frac{\delta}{4})^{\alpha-1}\frac{\xi(n-\sigma_1)^\alpha}{2^\alpha}W_s\right)-\frac{1+b+c}{4s}a_{s-\tau_1}\right)=\infty$$

*holds for all sufficiently large $N$, then every such solution $\{x_n\}$ of equation* (1.1) *oscillates or $\lim_{n\to\infty} x_n = 0$, where $W_s = Q_s + M^{\beta-\alpha}P_s$.*

# 3 Example

Next we present an example to illustrate the main results.

**Example 3.1.** Consider the third order difference equation

$$\Delta^3\left(x_n+\frac{1}{3}x_{n-1}+\frac{1}{3}x_{n+1}\right)+\frac{16^n}{10240}x_{n-2}^5+\frac{31}{30}4^n x_{n+1}^3=0. \tag{3.1}$$

Let $a_n = 1$, $b_n = c_n = \frac{1}{3}$, $q_n = \frac{16^n}{10240}$, $p_n = \frac{31}{30}4^n$, $\alpha=\beta=1$, $\tau_1 = 1$, $\tau_2 = 1$, $\sigma_1 = 3$, $\sigma_2 = 0$. Take $\gamma = 1$. Then condition (2.2) holds. On the other hand, condition (2.4) also holds. Therefore by Theorem 2.5, every solution $\{x_n\}$ with $x_n \leq 1$ of the equation (3.1) oscillates or $\lim_{n\to\infty} x_n = 0$. It is easy to see that $\{x_n\} = \{2^{-n}\}$ is a solution of the equation (3.1).

We conclude this paper with the following remark.

*Remark* 3.2. The established results presented in this paper are essentially new and include some of the existing results as special cases.

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