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Basins of Attraction of Certain Linear Fractional Systems of Difference Equations in the Plane

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Abstract

We investigate global dynamics of the following system of difference equations

$$\begin{cases} x_{n+1} &= \dfrac{\beta_1 x_n}{A_1 + y_n} \\ y_{n+1} &= \dfrac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \end{cases}, \quad n = 0, 1, 2, \ldots$$

where the parameters $\gamma_2, A_1, \beta_1, \beta_2$ are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers such that $x_0 + y_0 > 0$. Our results give an affirmative answers to Conjecture 5.1 and complete answer to Open problem 5.1 from [4].

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1 Introduction and Preliminaries

In this paper we study the global dynamics of the following rational system of difference equations

$$\left\{\begin{array}{lcl} x_{n+1} & = & \dfrac{\beta_1 x_n}{A_1 + y_n} \\ y_{n+1} & = & \dfrac{\beta_2 x_n + \gamma_2 y_n}{x_n + y_n} \end{array}\right., \quad n = 0, 1, 2, \ldots \tag{1.1}$$

where the parameters $\gamma_2, A_1, \beta_1, \beta_2$ are positive numbers and initial conditions x_0 and y_0 are arbitrary nonnegative numbers such that $x_0 + y_0 > 0$.

System (1.1) was mentioned in [3] as a part of Open Problem 4 which asked for description of global dynamics of 55 systems. According to the labeling in [3], System (1.1) is called $(14, 36)$.

In this paper, we provide the precise descriptions of global dynamics and the basins of attraction of all attractors of System (1.1) including the point at infinity (∞, β_2). Our results give an affirmative answers to Conjecture 5.1 and complete answer to Open problem 5.1 from [4]. The technique we use seems to be interesting as we use reduction of System (1.1) to a single second order difference equation for y_n, when $\gamma_2 < \beta_2$ and we use the theory of competitive systems when $\beta_2 < \gamma_2$. This technique is based on a fact that System (1.1) is competitive only for $\beta_2 < \gamma_2$. Indeed, while the description of the basins of attraction is simple for $\gamma_2 \leq \beta_2$ it is more complicated for $\beta_2 < \gamma_2$, where the separatrix between the basins of attraction of two attractors is the global stable manifold of a saddle point equilibrium solution.

The rest of this section contains some preliminary results as well as some basic facts about System (1.1). The second section of the paper presents local stability results for two equilibrium solutions of System (1.1) and the third section consists of the results on global dynamics of System (1.1). which includes complete analysis of the basins of attraction of all attractors of (1.1).

Consider a first order system of difference equations of the form

$$\left\{\begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array}\right., \quad n = 0, 1, 2, \ldots, \quad (x_{-1}, x_0) \in \mathcal{I} \times \mathcal{I} \tag{1.2}$$

where $f, g : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ are continuous functions on an interval $\mathcal{I} \subset \mathbb{R}$, $f(x, y)$ is nondecreasing in x and nonincreasing in y, and $g(x, y)$ is nonincreasing in x and nondecreasing in y. Such system is called *competitive*. One may associate a competitive map T to a competitive system (1.2) by setting $T = (f, g)$ and considering T on $\mathcal{B} = \mathcal{I} \times \mathcal{I}$. Theory of competitive systems and maps in the plane have been extensively developed and main results are given in [15–17, 19, 20] with emphasis on nonhyperbolic dynamics in [1]. Some results on global dynamics of anti-competitive systems are given in [12]. The general theory of monotone systems in ordered Banach spaces is given in [10, 11] with many applications to different types of dynamical systems such as elliptic and parabolic differential equations, ordinary differential equations and delay differential equations. The advantage of the results in [2, 15–17] is that they provide powerful tool for determining basins of attraction of equilibrium and periodic points which becomes the main objective in problems with several or infinite number of equilibrium or periodic points. See also [1,5–8,18,19] for different examples of planar competitive systems and their applications.

For additional definitions and results (e.g., repeller, hyperbolic fixed points, stability, asymptotic stability, stable and unstable manifolds) see [11] for competitive maps, and [13–16] for difference equations.

Let I be interval of real numbers and $f \in C[I \times I, [0, \infty)]$. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}) \tag{1.3}$$

The following result is from [9].

Theorem 1.1. *Let $I \subset [0, \infty)$ be some interval and assume that $f \in C[I \times I; [0, \infty)]$ satisfies the following conditions:*

(C_1) $f(x, y)$ is nondecreasing in each of its arguments;

(C_2) Eq. (1.3) has a unique positive equilibrium point $\bar{x} \in I$ and the function $f(x, x)$ satisfies the negative feedback condition

$$(x - \bar{x})(f(x, x) - x) < 0 \quad \textit{for every } x \in I \setminus \{\bar{x}\}$$

Then every positive solution of Eq. (1.3) with initial conditions in I converges to $\bar{x}$.

The following result is from [2].

Theorem 1.2. *Consider Eq. (1.3) subject to (C_1) and the following conditions:*

(C_3) There exist two equilibrium points $0 \leq x_1 < x_2$ of Eq. (1.3)

(C_4) Either the negative feedback condition (NFC) with respect to x_1 holds

$$(x - x_1)(f(x, x) - x) < 0 \quad x \in (x_1, x_2) \quad \textit{or}$$

(C_5) the negative feedback condition (NFC) with respect to x_2 holds

$$(x - x_2)(f(x, x) - x) < 0 \quad x \in (x_1, x_2)$$

Then every bounded solution of Eq. (1.3) converges to an equilibrium. The box $(x_1, x_2)^2$ is a part of the basin of attraction of x_1 if (C_4) is satisfied or x_2 if (C_5) is satisfied.

Now we present some basic facts about System (1.1), which will be used in our main results.

The map $T(x,y)$ associated to System (1.1) is given by

$$T(x,y)=(T_1(x,y),T_2(x,y))=\left(\frac{\beta_1 x}{A_1+y},\frac{\beta_2 x+\gamma_2 y}{x+y}\right),\quad (x,y)\in[0,\infty)^2\setminus\{(0,0)\} \tag{1.4}$$

The Jacobian matrix of T at the point (x,y) is given by:

$$J_T(x,y)=\left(\begin{array}{cc}\frac{\beta_1}{y+A_1} & -\frac{x\beta_1}{(y+A_1)^2}\\ \frac{y(\beta_2-\gamma_2)}{(x+y)^2} & \frac{x(\gamma_2-\beta_2)}{(x+y)^2}\end{array}\right). \tag{1.5}$$

The determinant of the Jacobian matrix (1.5) is given by

$$\det J_T(x,y)=\frac{A_1 x\beta_1(\gamma_2-\beta_2)}{(A_1+y)^2(x+y)^2}. \tag{1.6}$$

The trace of the Jacobian matrix (1.5) is given by

$$\operatorname{tr} J_T(x,y)=\frac{\beta_1}{A_1+y}+\frac{x(\gamma_2-\beta_2)}{(x+y)^2}. \tag{1.7}$$

Lemma 1.3. *Let* $(x_n,y_n):=T^n(x_0,y_0)$*, where* T *is given by* (1.4)*. Then the following holds:*

(i) If $x_0=0$ *and* $y_0>0$*, then* $x_n=0$ *and* $y_n=\gamma_2$ *for all* $n>0$ *.*

(ii) For all $x_0,y_0\in[0,\infty)^2\setminus\{(0,0)\}$*, the sequence* y_n *is bounded and*

$$\min\{\gamma_2,\beta_2\}\le y_n\le\max\{\gamma_2,\beta_2\}.\quad n=1,2,\dots.$$

iii) If $\beta_2<\gamma_2$*, then System* (1.1) *is competitive.*

(iv) If $\beta_2\neq\gamma_2$ *then* T *is injective.*

(v) If $\beta_2=\gamma_2$*, then for all* $(x_0,y_0)\in[0,\infty)^2\setminus\{(0,0)\}$ *and* $n\ge 0$ *the following holds:*

(v.1) If $\beta_1<A_1+\gamma_2$ *then* $x_n\to 0$ *as* $n\to\infty$ *and* $y_n=\gamma_2$ *for* $n>0$*.*

(v.2) If $\beta_1>A_1+\gamma_2$ *then* $x_n\to\infty$ *as* $n\to\infty$ *and* $y_n=\gamma_2$ *for* $n>0$*.*

(v.3) If $\beta_1=A_1+\gamma_2$ *then* $x_n=\beta_1 x_0/(A_1+y_0)$ *and* $y_n=\gamma_2$ *for* $n>0$*.*

Proof. The proof of the statements (i) and (iii) is immediate, so we skip it.

(ii) The proof follows from

$$\min\{\beta_2,\gamma_2\} = \frac{\min\{\beta_2,\gamma_2\}(x_n+y_n)}{(x_n+y_n)} \leq y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n+y_n}n$$

and

$$y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{x_n+y_n}n \leq \frac{\max\{\beta_2,\gamma_2\}(x_n+y_n)}{(x_n+y_n)} = \max\{\beta_2,\gamma_2\}.$$

(iv) We show that $T(x_1,y_1) = T(x_2,y_2)$ implies $x_1 = x_2$ and $y_1 = y_2$. Indeed, we have that

$$\begin{aligned} & T(x_1,y_1) - T(x_2,y_2) \\ = & \left(\frac{\beta_1\left(A_1\left(x_1-x_2\right)-x_2y_1+x_1y_2\right)}{\left(A_1+y_1\right)\left(A_1+y_2\right)}, \frac{\left(x_1y_2-x_2y_1\right)\left(\beta_2-\gamma_2\right)}{\left(x_1+y_1\right)\left(x_2+y_2\right)}\right) \\ = & (0,0) \end{aligned}$$

is equivalent to

$$\beta_1\left(A_1\left(x_1-x_2\right)-x_2y_1+x_1y_2\right)=0, \quad \left(x_1y_2-x_2y_1\right)\left(\beta_2-\gamma_2\right)=0. \tag{1.8}$$

By using the second equation of (1.8) we have that $y_1 = (x_1y_2)/x_2$. Substituting this into the first equation of (1.8) we get $A_1\beta_1(x_1-y_1) = 0$, from which it follows that $x_1 = x_2$. From $y_1 = (x_1y_2)/x_2$, we have that $y_1 = y_2$, which completes the proof.

(v) The proof follows from

$$x_{n+1} = \frac{\beta_1}{A_1+\gamma_2}x_n \quad \text{and} \quad y_n = \gamma_2 \quad \text{for } n>0.$$

□

2 Linearized Stability

The equilibrium points $(\bar{x},\bar{y})$ of System (1.1) satisfy

$$\bar{x} = \frac{\beta_1\bar{x}}{A_1+\bar{y}}, \quad \bar{y} = \frac{\beta_2\bar{x}+\gamma_2\bar{y}}{\bar{x}+\bar{y}} \tag{2.1}$$

Solutions of System (2.1) are:

(i) $\bar{x} = 0, \quad \bar{y} = \gamma_2$ i.e. $E_1 = (0,\gamma_2)$. Hence, the equilibrium $E_1 = (0,\gamma_2)$ exists for all parameter values.

(ii) If $\bar{x} \neq 0$, then using System (2.1), we obtain

$$\begin{aligned} \bar{x}_2 &= \frac{(A_1-\beta_1)(A_1-\beta_1+\gamma_2)}{A_1-\beta_1+\beta_2} \\ \bar{y}_2 &= \beta_1 - A_1. \end{aligned} \tag{2.2}$$

Hence, System (2.1) may have the second equilibrium point $E_2 = (\bar{x}_2, \bar{y}_2)$ in some regions of parameters. The criteria for the existence of the two equilibrium points are summarized in Table 1.

Table 1: The criteria for the existence of the equilibrium points of System (2.1).

E_1	$A_1 > \beta_1$ or $A_1 < \beta_1$ and $(\beta_1 > \max\{\gamma_2, \beta_2\} + A_1$ or $\beta_1 < \min\{\gamma_2, \beta_2\} + A_1)$
$E_1 \neq E_2$	$A_1 < \beta_1$ and $(\beta_2 + A_1 > \beta_1 > \gamma_2 + A_1$ or $\beta_2 + A_1 < \beta_1 < \gamma_2 + A_1)$
E_1=E_2	$A_1 < \beta_1,\ \beta_2 \neq \beta_1 - A_1$ and $\gamma_2 = \beta_1 - A_1$

Theorem 2.1. *The following statements hold:*

(i) The equilibrium point $E_1(0, \gamma_2)$ satisfies:

(i.1) E_1 is locally asymptotically stable if $\beta_1 < A_1 + \gamma_2$,

(i.2) E_1 is a saddle point if $\beta_1 > A_1 + \gamma_2$

(i.3) E_1 is nonhyperbolic if $\beta_1 = A_1+\gamma_2$. The eigenvalues are $\lambda_1 = 0,$ and $\lambda_2 = 1$. The corresponding eigenvectors are $\{0, 1\}$ and $\{\frac{\gamma_2}{\beta_2-\gamma_2}, 1\}$, respectively.

(ii) The equilibrium point $E_2(\bar{x}_2, \bar{y}_2)$ satisfies:

(ii.1) E_2 is locally asymptotically stable if $A_1 < \beta_1$ and $A_1 + \gamma_2 < \beta_1 < A_1 + \beta_2$,

(ii.2) E_2 is a saddle point if $A_1 < \beta_1$ and $A_1 + \gamma_2 > \beta_1 > A_1 + \beta_2$.

(ii.3) E_2 is nonhyperbolic if $A_1 + \gamma_2 = \beta_1,\ A_1 < \beta_1$ and $\beta_2 \neq A_1 + \gamma_2$.

Proof. The proof of the statement (i) follows from the fact that the eigenvalues of the Jacobian matrix

$$J_T(E_1) = \begin{pmatrix} \frac{\beta_1}{A_1+\gamma_2} & 0 \\ \frac{\beta_2}{\gamma_2} - 1 & 0 \end{pmatrix}$$

of T at E_1 are given by $\lambda_1 = 0$ and $\lambda_2 = \beta_1/(A_1+\gamma_2)$, with the corresponding eigenvectors

$$v_1 = (0,1)^T \qquad v_2 = \left(\frac{\beta_1\gamma_2}{(A_1+\gamma_2)(\beta_2-\gamma_2)}, 1\right)^T,$$

respectively.

Now, we prove the statement (ii). It is easy to see that

$$\begin{aligned} J_T(E_2) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\dfrac{(A_1-\beta_1)(A_1-\beta_1+\gamma_2)}{\beta_1(A_1-\beta_1+\beta_2)} \\ \dfrac{(A_1-\beta_1+\beta_2)^2}{(A_1-\beta_1)(\gamma_2-\beta_2)} & -\dfrac{(A_1-\beta_1+\beta_2)(A_1-\beta_1+\gamma_2)}{(A_1-\beta_1)(\beta_2-\gamma_2)} \end{pmatrix}, \end{aligned}$$

from which we have that

$$\det J_T(E_2) = -\frac{A_1(A_1-\beta_1+\beta_2)(A_1-\beta_1+\gamma_2)}{\beta_1(A_1-\beta_1)(\beta_2-\gamma_2)}$$

and

$$\operatorname{tr} J_T(E_2) = -\frac{\gamma_2(2A_1-2\beta_1+\beta_2)+(A_1-\beta_1)^2}{(A_1-\beta_1)(\beta_2-\gamma_2)}.$$

(ii.1) The equilibrium E_2 is locally asymptotically stable if

$$|\operatorname{tr} J_T(E_2)| < 1 + \det J_T(E_2) < 2.$$

Since $A_1 < \beta_1, A_1+\gamma_2 < \beta_1 < A_1+\beta_2$, and using (1.6) we have that $1 - \det J_T(E_2) > 0$. Under the given assumptions, we show that $|\operatorname{tr} J_T(E_2)| < 1 + \det J_T(E_2) \Leftrightarrow -1 - \det J_T(E_2) < \operatorname{tr} J_T(E_2) < 1 + \det J_T(E_2)$. After a straightforward calculation, we get

$$\det J_T(E_2) - \operatorname{tr} J_T(E_2) + 1 = -\frac{(A_1-\beta_1+\beta_2)(A_1-\beta_1+\gamma_2)}{\beta_1(\beta_2-\gamma_2)}, \tag{2.3}$$

$$\begin{aligned} &\det J_T(E_2) + \operatorname{tr} J_T(E_2) + 1 = \\ &-\frac{\gamma_2(A_1(2\beta_1+\beta_2)+A_1^2+\beta_1(\beta_2-3\beta_1))+(A_1+\beta_1+\beta_2)(A_1-\beta_1)^2}{\beta_1(A_1-\beta_1)(\beta_2-\gamma_2)}. \end{aligned} \tag{2.4}$$

Since $A_1 < \beta_1, A_1+\gamma_2 < \beta_1 < A_1+\beta_2$ and (2.3) we have that $\det J_T(E_2) - \operatorname{tr} J_T(E_2) + 1 > 0$. Further, the following holds:

$$\det J_T(E_2) + \operatorname{tr} J_T(E_2) + 1 > 0 \tag{2.5}$$

$$\Leftrightarrow \gamma_2 \left(A_1 \left(2\beta_1+\beta_2\right)+A_1^2+\beta_1 \left(\beta_2-3\beta_1\right)\right)+\left(A_1+\beta_1+\beta_2\right)\left(A_1-\beta_1\right)^{2}>0 \tag{2.6}$$

$$\Leftrightarrow \beta_2>-\frac{\left(A_1-\beta_1\right)\left(A_1\gamma_2+A_1^2-\beta_1\left(\beta_1-3\gamma_2\right)\right)}{\gamma_2\left(A_1+\beta_1\right)+\left(A_1-\beta_1\right)^{2}}. \tag{2.7}$$

Since, $\beta_2>\beta_1-A_1$ and

$$-\frac{\left(A_1-\beta_1\right)\left(A_1\gamma_2+A_1^2-\beta_1\left(\beta_1-3\gamma_2\right)\right)}{\gamma_2\left(A_1+\beta_1\right)+\left(A_1-\beta_1\right)^{2}}+A_1-\beta_1 = -\frac{2\beta_1\left(A_1-\beta_1\right)\left(A_1-\beta_1+\gamma_2\right)}{\gamma_2\left(A_1+\beta_1\right)+\left(A_1-\beta_1\right)^{2}}<0$$

we obtain that the inequality (2.7) is true. Hence the equilibrium point E_2 is locally asymptotically stable.

(ii.2) Since $A_1<\beta_1$ and $A_1+\gamma_2>\beta_1>A_1+\beta_2$, we have that $\beta_2<\gamma_2$, which in view of (1.6) and (1.7) implies $\det J_T(E_2)>0$ and $\operatorname{tr} J_T(E_2)>0$. The equilibrium point E_2 is a saddle point if $|\operatorname{tr} J_T(E_2)|>|1+\det J_T(E_2) \Leftrightarrow \det J_T(E_2)-\operatorname{tr} J_T(E_2)+1<0$. It follows from

$$\det J_T(E_2)-\operatorname{tr} J_T(E_2)+1=-\frac{\left(A_1-\beta_1+\beta_2\right)\left(A_1-\beta_1+\gamma_2\right)}{\beta_1\left(\beta_2-\gamma_2\right)},$$

and the assumption of the theorem, that $\det J_T(E_2)-\operatorname{tr} J_T(E_2)+1<0$. Therefore, the equilibrium point E_2 is a saddle point.

(ii.3) The proof follows from the fact that the eigenvalues of Jacobian matrix

$$J_T(E_2)=\begin{pmatrix} 1 & 0 \\ -\dfrac{\beta_2}{A_1-\beta_1}-1 & 0 \end{pmatrix}$$

of T at E_2 are given by $\lambda_1=0$ and $\lambda_2=1$ with the corresponding eigenvectors

$$v_1=(0,1)^T, \qquad v_2=\left(\frac{\beta_1-A_1}{A_1-\beta_1+\beta_2},1\right)^T,$$

respectively.

This completes the proof. □

3 Global Behavior and Basins of Attraction

In this section we describe global behavior of solutions of System (1.1).

Lemma 3.1. *The following statements hold:*

(i) Let $x_0, y_0 \in [0,\infty)^2 \setminus \{(0,0)\}$, *and let* $\{(x_n, y_n)\}$ *be a solution of System* (1.1). *Then,* $\{y_n\}_{n=1}^{\infty}$ *satisfies the equations* $y_1 = \dfrac{x_0\beta_2 + y_0\gamma_2}{y_0 + x_0}$ *and*

$$y_{n+2} = \frac{-\beta_2 y_{n+1}\left(A_1\gamma_2 + y_n\left(\beta_1+\gamma_2\right)\right) + \gamma_2 y_{n+1}^2\left(A_1 + y_n\right) + \beta_1\beta_2\gamma_2 y_n}{-y_{n+1}\left(A_1\beta_2 + \left(\beta_1+\beta_2\right)y_n\right) + y_{n+1}^2\left(A_1+y_n\right) + \beta_1\gamma_2 y_n}, \quad n \geq 0. \tag{3.1}$$

(ii) Equation (3.1) *has three equilibrium points* $\bar{u}_1 = \gamma_2$, $\bar{u}_2 = \beta_2$ *and* $\bar{u}_3 = \beta_1 - A_1$.

Proof. (i) The proof follows by elimination of the variable x_n from System (1.1).

(ii) Let $f(u,v)$ be the map associated to the equation (3.1), which is given by

$$f(u,v) = \frac{u^2\gamma_2\left(A_1+v\right) - u\beta_2\left(A_1\gamma_2 + v\left(\beta_1+\gamma_2\right)\right) + v\beta_1\beta_2\gamma_2}{u^2\left(A_1+v\right) - u\left(A_1\beta_2 + v\left(\beta_1+\beta_2\right)\right) + v\beta_1\gamma_2}. \tag{3.2}$$

The equilibrium points of the equation (3.1) are solutions of equation $f(u,u) = u$. After a straightforward calculation, we get

$$\begin{aligned} f(u,u) - u &= \frac{u^2\gamma_2\left(A_1+u\right) - u\beta_2\left(A_1\gamma_2 + u\left(\beta_1+\gamma_2\right)\right) + u\beta_1\beta_2\gamma_2}{u^2\left(A_1+u\right) - u\left(A_1\beta_2 + u\left(\beta_1+\beta_2\right)\right) + u\beta_1\gamma_2} - u \\ &= -\frac{\left(u-\beta_2\right)\left(u-\gamma_2\right)\left(A_1+u-\beta_1\right)}{A_1\left(u-\beta_2\right) + u^2 - u\left(\beta_1+\beta_2\right) + \beta_1\gamma_2} \end{aligned} \tag{3.3}$$

from which it follows that the equilibrium points of the equation (3.1) are given by $\bar{u}_1 = \gamma_2$, $\bar{u}_2 = \beta_2$ i $\bar{u}_3 = \beta_1 - A_1$.

The proof is complete. □

Lemma 3.2. *The following statements hold:*

(i) If $\gamma_2 < \beta_2$, *then*

(i.1) f *is continuous and* $f([\gamma_2,\beta_2]^2) \subseteq [\gamma_2,\beta_2]$.

(i.2) f *is nondecreasing in each of its arguments on* $[\gamma_2,\beta_2]^2$.

(ii) If $\gamma_2 > \beta_2$, *then*

(ii.1) f *is continuous function and* $f([\beta_2,\gamma_2]^2) \subseteq [\beta_2,\gamma_2]$.

(ii.2) f *is nondecreasing in each of its arguments on* $[\beta_2,\gamma_2]^2$.

Proof. The proof of the statement (ii) is similar to the proof of (i) and will be omitted. Now, we prove the statement (i).

(i.1) Assume that $(u,v) \in [\gamma_2, \beta_2]^2$. We show that $f(u,v) \in [\gamma_2, \beta_2]$. Since, $\gamma_2 \leq u, v \leq \beta_2$ and

$$\begin{aligned} f(u,v) - \gamma_2 &= \frac{u^2\gamma_2 (A_1+v) - u\beta_2 (A_1\gamma_2 + v(\beta_1+\gamma_2)) + v\beta_1\beta_2\gamma_2}{u^2 (A_1+v) - u (A_1\beta_2 + v(\beta_1+\beta_2)) + v\beta_1\gamma_2} - \gamma_2 \\ &= \frac{v\beta_1 (u-\gamma_2)(\gamma_2-\beta_2)}{A_1 u (v1 - \beta_2) + v(u^2 - u(\beta_1+\beta_2) + \beta_1\gamma_2)} \\ &= \frac{v\beta_1 (u-\gamma_2)(\gamma_2-\beta_2)}{A_1 u (u-\beta_2) + uv(u-\beta_2) + v\beta_1(\gamma_2 - u)} \geq 0 \end{aligned} \tag{3.4}$$

we have that $f(u,v) \geq \gamma_2$. On the other hand, in view of $\gamma_2 \leq u, v \leq \beta_2$ we have that

$$\begin{aligned} f(u,v) - \beta_2 &= \frac{u^2\gamma_2 (A_1+v) - u\beta_2 (A_1\gamma_2 + v(\beta_1+\gamma_2)) + v\beta_1\beta_2\gamma_2}{u^2 (A_1+v) - u (A_1\beta_2 + v(\beta_1+\beta_2)) + v\beta_1\gamma_2} - \beta_2 \\ &= \frac{(A_1+v)u\beta_1 (u-\beta_2)(\gamma_2-\beta_2)}{A_1 u (u - \beta_2) + v(u^2 - u(\beta_1+\beta_2) + \beta_1\gamma_2)} \\ &= \frac{(A_1+v)u\beta_1 (u-\beta_2)(\gamma_2-\beta_2)}{A_1 u (u-\beta_2) + uv(u-\beta_2) + v\beta_1(\gamma_2 - v)} \leq 0. \end{aligned} \tag{3.5}$$

Hence, we obtain $\gamma_2 \leq f(u,v) \leq \beta_2$ whenever $\gamma_2 \leq u, v \leq \beta_2$. One can show that $f\left((\gamma_2, \beta_2)^2\right) \subseteq (\gamma_2, \beta_2)$.

The denominator of the function $f(u,v)$ is given by

$$\begin{aligned} D(u,v) :=& u^2 (A_1+v) - u (A_1\beta_2 + v(\beta_1+\beta_2)) + v\beta_1\gamma_2 \\ =& u(A_1+v)(u-\beta_2) + v\beta_1(\gamma_2 - u). \end{aligned}$$

Since $\gamma_2 < \beta_2$ we have that $D(u,v) \neq 0$ for all $u, v \in [\gamma_2, \beta_2]$. From this it follows that $f(u,v)$ is continuous function on $[\gamma_2, \beta_2]^2$. This proves our statement.

(i.2) Since

$$\begin{aligned} \frac{\partial f}{\partial v}(u,v) &= -\frac{A_1 u\beta_1 (u-\beta_2)(u-\gamma_2)(\beta_2-\gamma_2)}{(A_1 u (u-\beta_2) + v(u^2 - u(\beta_1+\beta_2) + \beta_1\gamma_2))^2} \geq 0, \\ \frac{\partial f}{\partial u}(u,v) &= \frac{v\beta_1 (A_1+v)(\beta_2-\gamma_2)(u^2 - 2u\gamma_2 + \beta_2\gamma_2)}{(A_1 u (u-\beta_2) + v(u^2 - u(\beta_1+\beta_2) + \beta_1\gamma_2))^2} \\ &= \frac{v\beta_1 (A_1+v)(\beta_2-\gamma_2)(u(u-\gamma_2) + \gamma_2(\beta_2 - u))}{(A_1 u (u-\beta_2) + v(v^2 - u(\beta_1+\beta_2) + \beta_1\gamma_2))^2} \geq 0 \end{aligned} \tag{3.6}$$

it follows that $f(u,v)$ is nondecreasing in each of its arguments on $[\gamma_2, \beta_2]^2$.

The proof is complete. □

Theorem 3.3. *Assume that $\gamma_2 < \beta_2$. Then the following statements are true:*

(i) If $\gamma_2 > \beta_1 - A_1$ then System (1.1) *has the unique equilibrium point E_1 which is globally asymptotically stable. Its basin of attraction is $\mathcal{B}(E_1) = [0,\infty)^2 \setminus \{(0,0)\}$.*

(ii) If $\gamma_2 < \beta_1 - A_1 < \beta_2$ then System (1.1) *has two equilibrium points E_1 and E_2 : E_1 is a saddle point and E_2 is locally asymptotically stable. Furthermore, the basins of attraction of E_1 and E_2 are $\mathcal{B}(E_1) = \{(0,y)|y > 0\}$ and $\mathcal{B}(E_2) = (0,\infty)^2$, respectively.*

(iii) If $\beta_1 - A_1 > \beta_2$ then System (1.1) *has the unique equilibrium point E_1 which is saddle point. The basin of attraction of E_1 is given by $\mathcal{B}(E_1) = \{(0,y)|y > 0\}$. Furthermore, basin of attraction of (∞,β_2) is $\mathcal{B}(\infty,\beta_2) = (0,\infty)^2$.*

(iv) If $\gamma_2 = \beta_1 - A_1 > 0$ and $\beta_2 \neq \beta_1 - A_1 > 0$ then System (1.1) *has the unique nonhyperbolic equilibrium point $E = E_1 = E_2$ which is global attractor with the basin of attraction $\mathcal{B}(E) = [0,\infty)^2 \setminus \{(0,0)\}$.*

Proof. In view of Lemmas 1.3 and 3.2 the sequence $\{y_n\}$ is bounded and $\gamma_2 < y_n < \beta_2$ holds for all $n \geq 1$ and $(x_0,y_0) \in (0,\infty)^2$. Take $(x_0,y_0) \in (0,\infty)^2$. Then $\gamma_2 < y_n < \beta_2$ implies $y_1, y_2 \in (\gamma_2,\beta_2)$ where $(x_n,y_n) = T^n(x_0,y_0)$ for $n \geq 0$. By Lemma 3.1 the sequence $\{y_n\}$ is solution of Eq. (3.1) for $y_1 = (x_0\beta_2 + y_0\gamma_2)/(y_0 + x_0)$ and $y_2 = (x_1\beta_2 + y_1\gamma_2)/(y_1 + x_1)$ as initial conditions.

Now, we prove our statements.

(i) In view of Table 1 and statement i) of Theorem 2.1, the map T has the unique equilibrium point E_1 which is locally asymptotically stable. Since, $\beta_1 - A_1 < \gamma_2 < u < \beta_2$ we obtain

$$\begin{aligned}
&(u-\bar{u}_1)(f(u,u)-u)\\
&= (u-\gamma_2)\left(\frac{u^2\gamma_2\left(A_1+u\right) - u\beta_2\left(A_1\gamma_2 + u\left(\beta_1+\gamma_2\right)\right) + u\beta_1\beta_2\gamma_2}{u^2\left(A_1+u\right) - u\left(A_1\beta_2 + u\left(\beta_1+\beta_2\right)\right) + u\beta_1\gamma_2} - u\right)\\
&= -\frac{\left(u-\beta_2\right)\left(u-\gamma_2\right)^2\left(A_1+u-\beta_1\right)}{\left(A_1+u\right)\left(u-\beta_2\right) + \beta_1\left(\gamma_2-u\right)} < 0 \text{ for } u \in (\gamma_2,\beta_2).
\end{aligned} \tag{3.7}$$

In view of Lemma 3.2 and inequality (3.7) conditions (C_1), (C_3) and (C_4) of Theorem 1.2 for Eq. (3.1) and the equilibrium point $\bar{u}_1 = \gamma_2 < \bar{u}_2 = \beta_2$, are satisfied. Consequently, $(\gamma_2,\beta_2)^2$ is a subset of the basin of attraction of $\bar{u}_1 = \gamma_2$. Since $y_1, y_2 \in (\gamma_2,\beta_2)$, we have that $y_n \to \gamma_2$ as $n \to \infty$. From the second equation of System (1.1) we have

$$x_n = \frac{y_n y_{n+1} - \gamma_2 y_n}{\beta_2 - y_{n+1}}. \tag{3.8}$$

From (3.8) we obtain $x_n \to 0$ as $n \to \infty$. If $x_0 = 0$ and $y_0 > 0$ then from Lemma 1.3 we have $y_n = \gamma_2$. This proves our statement.

(ii) In view of Table 1 and statement ii) of Theorem 2.1 the map T has two equilibrium points E_1 and E_2, where E_1 is a saddle point and E_2 is locally asymptotically stable. Since $\gamma_2 < u < \beta_2$, we obtain

$$\begin{aligned}
&(u-\bar{u}_3)(f(u,u)-u)\\
&= (u-\beta_1+A_1)\left(\frac{u^2\gamma_2\left(A_1+u\right)-u\beta_2\left(A_1\gamma_2+u\left(\beta_1+\gamma_2\right)\right)+u\beta_1\beta_2\gamma_2}{u^2\left(A_1+u\right)-u\left(A_1\beta_2+u\left(\beta_1+\beta_2\right)\right)+u\beta_1\gamma_2}-u\right)\\
&= -\frac{\left(u-\beta_2\right)\left(u-\gamma_2\right)\left(A_1+u-\beta_1\right)^2}{\left(A_1+u\right)\left(u-\beta_2\right)+\beta_1\left(\gamma_2-u\right)} < 0 \text{ for } u \in (\gamma_2,\beta_2)\setminus\{\beta_1-A_1\}.
\end{aligned} \tag{3.9}$$

By Lemma 3.2 and (3.9) conditions (C_1) and (C_2) of Theorem 1.1 for Eq. (3.1) and $I = (\gamma_2, \beta_2)$ are satisfied. Consequently, $(\gamma_2, \beta_2)^2$ is a subset of the basin of attraction of $u_3 = \gamma_1 - A_1$. Hence, $y_n \to \gamma_1 - A_1$ as $n \to \infty$. From (3.8) we obtain

$$x_n \to \frac{\left(A_1-\beta_1\right)\left(A_1-\beta_1+\gamma_2\right)}{A_1-\beta_1+\beta_2} \quad \text{as} \quad n\to\infty,$$

from which it follows that $\mathcal{B}(E_2) = (0,\infty)^2$. By Lemma 1.3 it follows that $\mathcal{B}(E_1) = \{(0,y)|y>0\}$. This proves our statement. See Figure 3.1 case a).

(iii) In view of Table 1 and statement ii) of Theorem 2.1 the map T has the unique equilibrium point E_1 which is a saddle point. Similarly to the proof of the statement i) we obtain that $y_n \to \beta_2$ as $n \to \infty$. From (3.8) we obtain $x_n \to \infty$ as $n \to \infty$, which implies that $\mathcal{B}(\infty, \beta_2) = (0,\infty)^2$. By Lemma 1.3 it follows that $\mathcal{B}(E_1) = \{(0,y)|y>0\}$.

(iv) In view of Table 1 and statement ii) of Theorem 2.1 the map T has a unique equilibrium point $E = E_1 = E_2$ which is nonhyperbolic. Similarly to the proof of the statement i), by using

$$\begin{aligned}
&(u-\bar{u}_1)(f(u,u)-u)\\
&= (u-\gamma_2)\left(\frac{u^2\gamma_2\left(A_1+u\right)-u\beta_2\left(A_1\gamma_2+u\left(\beta_1+\gamma_2\right)\right)+u\beta_1\beta_2\gamma_2}{u^2\left(A_1+u\right)-u\left(A_1\beta_2+u\left(\beta_1+\beta_2\right)\right)+u\beta_1\gamma_2}-u\right)\\
&= -\frac{\left(u-\beta_2\right)\left(u-\gamma_2\right)^3}{A_1\left(u-\beta_1-\beta_2\right)+u^2-u\left(\beta_1+\beta_2\right)+\beta_1^2}
\end{aligned} \tag{3.10}$$

we obtain that $y_n \to \gamma_2 = \beta_1 - A_1$ as $n \to \infty$. From (3.8) we obtain $x_n \to 0$ as $n \to \infty$. This proves our statement.

The proof is complete. □

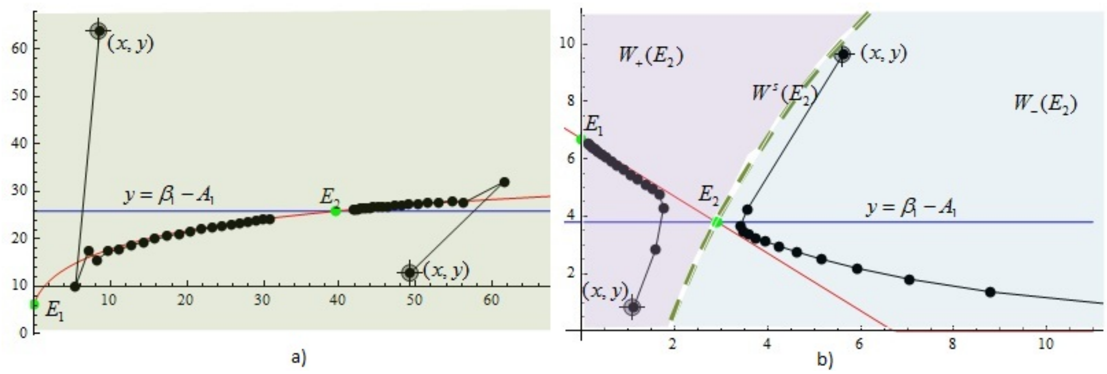

Figure 3.1: Visual illustration of statement (ii) of Theorem 3.3 and Theorem 3.5.

Theorem 3.4. *Assume that $\gamma_2 > \beta_2$. Then, the following statements are true:*

i) If $\beta_2 > \beta_1 - A_1$ then System (1.1) *has the unique equilibrium point E_1 which is globally asymptotically stable and $\mathcal{B}(E_1) = [0,\infty)^2 \setminus \{(0,0)\}$.*

ii) If $\beta_1 - A_1 > \gamma_2$ then System (1.1) *has the unique equilibrium point E_1 which is a saddle point. The global stable manifold is given by $\mathcal{W}^s(E_1) = \{(0,y)|y > 0\}$. Furthermore, $\mathcal{B}(\infty,\beta_2) = (0,\infty)^2$.*

iii) If $\gamma_2 = \beta_1 - A_1 > 0$ and $\beta_2 \neq \beta_1 - A_1 > 0$, then there exists the unique nonhyperbolic equilibrium point $E = E_1 = E_2$ which is global attractor with the basin of attraction $\mathcal{B}(E) = [0,\infty)^2 \setminus \{(0,0)\}$.

Proof. The proof is similar to the proof of Theorem 3.3 and will be omitted. □

Theorem 3.5. *Assume that $\gamma_2 > \beta_2$. If $\beta_2 < \beta_1 - A_1 < \gamma_2$ then System* (1.1) *has two equilibrium points E_1 and E_2, where E_1 is locally asymptotically stable and E_2 is a saddle point. Furthermore, there exists the global stable manifold $\mathcal{W}^s(E_2)$ which is the graph of a strictly increasing function E_2 that separates the positive quadrant so that all orbits above this manifold are asymptotic to the equilibrium point E_1, and all orbits below this manifold are asymptotic to (∞,β_2). All orbits that starts on $\mathcal{W}^s(E_2)$ are attracted to E_2.*

Proof. In view of (1.5) the map T is competitive. By Table 1 and Theorem 2.1 the map T has two equilibrium points E_1 and E_2, where E_1 is locally asymptotically stable and E_2 is a saddle point. Since $det J_T(x,y) > 0$ and by Lemma 1.3 all conditions of Theorems $1, 2$ and 4 in [17] are satisfied for T on $\mathcal{R} = \mathbb{R}^2_+$. This implies that there exists the global stable manifold $\mathcal{W}^s(E_2)$ which separates the first quadrant into two invariant regions, $\mathcal{W}_-(E_2)$ (below the stable manifold) and $\mathcal{W}_+(E_2)$ (above the stable manifold).

In view of [17, Theorem 4] for all $(x,y) \in \mathcal{W}_-(E_2)$, there exists $n_0 > 0$ such that for all $n > n_0$, $T^n(x,y) \in int(Q_4(E_2) \cap \mathcal{R})$ and for all $(x,y) \in \mathcal{W}_+(E_2)$, there exists

$n_1 > 0$ such that for all $n > n_1$, $T^n(x,y) \in int(Q_2(E_2) \cap \mathcal{R})$. New, we show that each orbit starting in the region $int(Q_4(E_2))$ is asymptotic to (∞, β_2) and each orbit starting in the region $int(Q_2(E_2))$ converges to E_1.

Let $U(t) = (t, \beta_1 - A_1)$ for $t \geq 0$. One can show that

$$E_1 = (0, \gamma_2) \preceq_{se} U(\bar{x}_2) = E_2 \quad \text{and}$$

$$T(U(t)) - U(t) = \left\{0, \frac{(A_1 - \beta_1)(A_1 - \beta_1 + \gamma_2) - t(A_1 - \beta_1 + \beta_2)}{A_1 - t - \beta_1}\right\},$$

where $(a_1, b_1) \preceq_{se} (a_2, b_2)$ means $a_1 \leq a_2, b_2 \leq b_1$. Since $\bar{x}_2$ is a solution of equation $(A_1 - \beta_1)(A_1 - \beta_1 + \gamma_2) - t(A_1 - \beta_1 + \beta_2) = 0$ and $\beta_1 - A_1 < 0$, we obtain $A_1 - t - \beta_1 < 0$ and $T(U(t)) \preceq_{se} U(t)$ for $0 \leq t \leq \bar{x}_2$ and $U(t) \preceq_{se} T(U(t))$ for $x_2 \leq t < \infty$.

By using monotonicity of T we have $E_1 \preceq_{se} T^{n+1}(U(t)) \preceq_{se} T^n(U(t)) \preceq_{se} E_2$, for $0 \leq t < \bar{x}_2$. This implies $T^n(U(t)) \to E_1$ as $n \to \infty$. For $x_2 \leq t$, we have that $E_1 \ll E_2 \preceq_{se} T^n(U(t)) \preceq_{se} T^{n+1}(U(t))$, from which it follows that the sequence $(x'_n, y'_n) = T^n(U(t))$ is monotonically increasing. Since $\{y'_n\}$ is decreasing and bounded, it has to converge. The sequence $\{x'_n\}$ is increasing, since T has no other equilibrium points except E_1 and E_2, which implies $x'_n \to \infty$ as $n \to \infty$. By using the second equation of System (1.1) we obtain

$$y'_{n+1} = \frac{\beta_2 + \frac{y'_n}{x'_n}\gamma_2}{1 + \frac{y'_n}{x'_n}}, \tag{3.11}$$

which implies $y'_n \to \beta_2$ as $n \to \infty$. This implies $T^n(U(t)) \to (\infty, \beta_2)$ as $n \to \infty$.

Take any point $(x,y) \in int(Q_2(E_2) \cap \mathcal{R})$. Then, there are $0 < t^* < \bar{x}_2$ and $u^* > 0$ such that $(0, u^*) \preceq_{se} (x,y) \preceq_{se} U(t^*)$. By using monotonicity of T we have that $E_1 = (0, \gamma_2) = T^n(0, u^*) \preceq_{se} T^n(x,y) \preceq_{se} T^n(U(t^*))$ for $n \geq 1$. This implies that $T^n(x,y) \to E_1$ as $n \to \infty$. In addition, for any $(x,y) \in int(Q_4(E_3) \cap \mathcal{R})$, there exists $t_1 > \bar{x}_2$ such that $U(t_1) \preceq_{se} (x,y) \preceq_{se} (\infty, 0)$. By using monotonicity of T we have that $E_1 \ll E_2 \preceq_{se} T^n(U(t_1)) \preceq_{se} T^n(x,y)$. Since $T^n(U(t_1)) \to (\infty, \beta_2)$ as $n \to \infty$ we obtain that $x_n \to \infty$ as $n \to \infty$, where $(x_n, y_n) = T^n(x,y)$ for $n \geq 1$. By (3.11) for the sequence (x_n, y_n) we obtain that $y_n \to \beta_2$ as $n \to \infty$. This implies that $T^n(x,y) \to (\infty, \beta_2)$ as $n \to \infty$, which complete the proof of theorem. See Figure 3.1 case b). □

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