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Existence of Almost Periodic Solutions of Discrete Ricker Delay Models

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Abstract

The aim of this article is to investigate the sufficient conditions for the existence of almost periodic solutions of a generalized Ricker delay model,

$$N(n+1) = N(n)\exp\{f(n, N(n-r(n)))\},$$

when f is an almost periodic function in n, which appears as a model for dynamics with single species in changing almost periodic environments, by applying the technique of stability conditions which derives the uniformly asymptotically stable of solutions for above equation. Moreover, we final consider the existence of an almost periodic solution of the case where f has the Volterra term with an infinite delay.

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1 Introduction

For ordinary differential equations and functional differential equations, the existence of almost periodic solutions of systems has been studied by many authors. One of the most popular methods is to find the certain Liapunov functions/functionals [2, 3, 7, 10, 13] for stability conditions. For the periodic functional difference equation, the existence of uniform bounded and uniform ultimately bounded solutions imply the existence of a periodic solution, see [8], However, for an almost periodic equation, the boundedness

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of solutions does not necessarily imply the existence of an almost periodic solution even for scalar differential equations with no delay.

Recently, He, Zhang and Gopalsamy [9] have shown the existence of periodic and almost periodic solutions for a nonautonomous scalar delay differential equation of modeling single species dynamics in a temporally changing environment. Their results are to extend results in Gopalsamy [3] to a nonautonomous differential equation by using the 3/2 stability conditions [12] when its almost periodic case. To the best of our knowledge, there are no relevant results on the existence of almost periodic solutions for discrete Ricker models by means of our approach of discrete stability theorems [15]. We emphasize that our results extend [9] and [4] as a delay discrete almost periodic case. The equation (5.1) below does not contain a delay independent stabilising negative feedback term and hence (5.1) requires a different approach from that in the literature [10]. In this paper, we discuss the existence of almost periodic solutions for a generalized nonautonomous discrete Ricker type difference equations with finite delay and infinite delay in f, respectively.

In what follows, we denote by $\mathbb{R}$ the real Euclidean space, $\mathbb{Z}$ is the set of integers, $\mathbb{Z}^+ := [0, \infty)$ is the set of nonnegative integers and $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}$. For any discrete interval $I \subset \mathbb{Z} := (-\infty, \infty)$, we denote by $\mathrm{BS}(I)$ the set of all bounded functions mapping I into $\mathbb{R}$, and set $|\phi|_I = \sup\{|\phi(s)| : s \in I\}$.

We introduce an almost periodic function $f(n, \phi) : \mathbb{Z} \times \mathrm{BS} \to \mathbb{R}$. where BS is an open set in $\mathbb{R}$. After, this BS is defined by $I = [-h, 0]$ for some $h > 0$.

Definition 1.1. $f(n, \phi)$ is said to be almost periodic in n uniformly for $\phi \in \mathrm{BS}$, if for any $\epsilon > 0$ and any compact set K in BS, there exists a positive integer $L(\epsilon, K)$ such that any interval of length $L(\epsilon, K)$ contains an integer τ for which

$$|f(n + \tau, \phi) - f(n, \phi)| \le \epsilon$$

for all $n \in \mathbb{Z}$ and all $\phi \in K$. Such a number τ in above inequality is called an ϵ-translation number of $f(n, \phi)$.

In order to formulate a property of almost periodic functions, which is equivalent to the above definition, we discuss the concept of the normality of almost periodic functions. Namely, Let $f(n, \phi)$ be almost periodic in n uniformly for $\phi \in \mathrm{BS}$. Then, for any sequence $\{h_k^{'}\} \subset \mathbb{Z}$, there exists a subsequence $\{h_k\}$ of $\{h_k^{'}\}$ and function $g(n, \phi)$ such that

$$f(n + h_k, \phi) \to g(n, \phi) \tag{1.1}$$

uniformly on $\mathbb{Z} \times K$ as $k \to \infty$, where K is a compact set in BS. There are many properties of the discrete almost periodic functions [1, 10], which are corresponding properties of the continuous almost periodic functions $f(t, x) \in C(\mathbb{R} \times D, \mathbb{R})$, see, e.g., [13]. We shall denote by $T(f)$ the function space consisting of all translates of f, that is, $f_\tau \in T(f)$, where

$$f_\tau(n, \phi) = f(n + \tau, \phi), \qquad \tau \in \mathbb{Z}. \tag{1.2}$$

Let $H(f)$ denote the uniform closure of $T(f)$ in the sense of (1.2). $H(f)$ is called the hull of f. In particular, we denote by $\Omega(f)$ the set of all limit functions $g \in H(f)$ such that for some sequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ and $f(n+n_k,\phi) \to g(n,\phi)$ uniformly on $\mathbb{Z} \times S$ for any compact subset S in BS. By (1.1), if $f : \mathbb{Z} \times \mathrm{BS} \to \mathbb{R}$ is almost periodic in n uniformly for $\phi \in \mathrm{BS}$, so is a function in $\Omega(f)$.

The following concept of asymptotic almost periodicity was introduced by Fréchet in the case of continuous functions (see, e.g., [13]).

Definition 1.2. $u(n)$ is said to be asymptotically almost periodic if it is a sum of a almost periodic function $p(n)$ and a function $q(n)$ defined on $I^* = [a,\infty) \subset \mathbb{Z}^+$ which tends to zero as $n \to \infty$, that is,

$$u(n) = p(n) + q(n).$$

$u(n)$ is asymptotically almost periodic if and only if for any sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ there exists a subsequence $\{n_k\}$ for which $u(n+n_k)$ converges uniformly on $n; a \le n < \infty$.

2 Preliminary Auxiliary Result

In this paper, we shall consider a discrete nonautonomous Ricker delay difference equation of the form

$$N(n+1) = N(n)\exp\{f(n, N(n-r(n)))\} \qquad \text{for } n \ge n_0,\ \ n_0 \in \mathbb{Z}, \tag{2.1}$$

and in particular, we first obtain sufficient conditions for equation (2.1) to have an almost periodic solution when f is an almost periodic function. As example of the type (2.1), we provide

$$N(n+1) = N(n)\exp\{a + bN^p(n-r) - cN^q(n-r)\}.$$

We set the following assumptions for equation (2.1).

(H$_1$) f is almost periodic in n uniformly for ϕ and is continuous in the second variable defined on $[n_0,\infty) \times \mathbb{R}^+$ to $\mathbb{R}$. r is defined on $[n_0,\infty) \subset \mathbb{Z}$ to $\mathbb{Z}^+$ and r is almost periodic in n, $0 < r^l = \liminf_{n\to\infty} r(n) \le r(n) \le \limsup_{n\to\infty} r(n) = r^L < \infty$ for $n \ge n_0$, and some $n_0 \in \mathbb{Z}$.

(H$_2$) There exist continuous functions $F_i : \mathbb{R}^+$ to $\mathbb{R}$ ($i = 1,2$) such that $F_1(y) \le f(n,y) \le F_2(y)$ for $(n,y) \in [n_0,\infty) \times \mathbb{R}^+$.

(H$_3$) There exist $\xi_i > 0$ ($i = 1,2$) such that

$$\begin{cases} F_i(\xi_i) = 0, \quad (i = 1,2), \\ F_i(x) > 0 \quad \text{for} \quad x \in [0,\xi_i), \\ F_i(x) < 0 \quad \text{for} \quad x \in (\xi_i,\infty). \end{cases}$$

We assume that the initial conditions associated with (2.1) are as follows:

$$N(s)=\phi(s)\geq 0,\ s\in[n_0-r^L,n_0],\phi(n_0)>0 \text{ and } \phi \text{ defined on } [n_0-r^L,n_0]. \quad (2.2)$$

By (2.1) and (H_3), we can see that solutions of (2.1) satisfy

$$\phi(n_0)\exp\left[\sum_{j=n_0}^{n-1}F_1(N(j-r(j)))\right]\leq N(n)$$
$$\leq \phi(n_0)\exp\left[\sum_{j=n_0}^{n-1}F_2(N(j-r(j)))\right],\quad n>n_0. \quad (2.3)$$

To prove this, from (2.1), (2.2) and (H_2), we have

$$\frac{N(n+1)}{N(n)}=\exp[f(n,N(n-r(n)))]\leq \exp[F_2(N(n-r(n)))]. \quad (2.4)$$

By recursively using inequality (2.4), we have

$$\begin{aligned}\frac{N(n+1)}{N(n_0)} &= \frac{N(n+1)}{N(n)}\frac{N(n)}{N(n-1)}\cdots\frac{N(n_0+1)}{N(n_0)}\\ &\leq \prod_{j=n_0}^{n}\exp[F_2(N(j-r(j)))]\\ &= \exp\left[\sum_{j=n_0}^{n}F_2(N(j-r(j)))\right], \end{aligned} \quad (2.5)$$

where $N(n_0)=\phi(n_0)$. Thus, we obtain

$$N(n)\leq \phi(n_0)\exp\left[\sum_{j=n_0}^{n-1}F_2(N(j-r(j)))\right]. \quad (2.6)$$

Similarly, we get the first inequality in (2.3). Then solutions of (2.1) are defined for all $n\geq n_0$ and furthermore that $N(n)>0$ for $n\geq n_0$.

We show the following key lemma in which the solution of (2.1) is at permanence.

Lemma 2.1. *If the assumptions (H_1), (H_2) and (H_3) hold, then there exists an $n_1\geq n_0$ such that any solution $N(n)$ of* (2.1) *satisfies*

$$N(n)\leq M\equiv \xi_2 e^{\alpha r^L}\qquad \textit{for } n\geq n_1. \quad (2.7)$$

where $\alpha=\max_{0\leq x\leq \xi_2}F_2(x)$. Furthermore, if we assume (H_1), (H_2) and (H_3), and if

$$F_1(x)\geq F_1(M)\qquad \textit{for } 0\leq x\leq M, \quad (2.8)$$

then there exists an $n_2\geq n_1$ such that

$$N(n)\geq m\equiv \xi_1 e^{F_1(M)r^L}\qquad \textit{for } n\geq n_2. \quad (2.9)$$

Proof. Let $N(n)$ be any positive solution of (2.1) defined for all $n \geq n_0$. From (2.1) and (H_2),

$$N(n+1) \leq N(n)\exp[F_2(N(n-r(n)))]. \tag{2.10}$$

By (H_3), there exists a $\xi_2 > 0$ such that

$$F_2(\xi_2) = 0 \;\text{ and }\; F_2(x) > 0 \;\text{ for }\; 0 \leq x < \xi_2, \quad F_2(x) < 0 \;\text{ for }\; x > \xi_2. \tag{2.11}$$

Since F_2 is continuous on $[0, \xi_2]$, it must attain its maximum say α at some point say $\eta_2 \in [0, \xi_2)$, that is $\alpha = F_2(\eta_2)$. We suppose first that $N(n)$ is not oscillatory about ξ_2, then there exists an $n_3 \geq n_0$ such that

$$\text{either }\; N(n) < \xi_2 \qquad \text{or} \qquad N(n) > \xi_2 \;\text{ for }\; n \geq n_3.$$

If $N(n) < \xi_2$, then (2.7) follows, because $N(n) < \xi_2 \leq \xi_2 e^{\alpha r^L}$ by $\alpha r^L \geq 0$. If $N(n) > \xi_2$ for $n \geq n_3$, then it follows from (2.11) that

$$F_2(N(n-r(n))) < 0 \qquad \text{for }\; n \geq n_3 + r^L.$$

Now (2.10) leads to

$$N(n+1) \leq N(n)\exp[F_2(N(n-r(n)))] \leq N(n) \qquad \text{for }\; n \geq n_3 + r^L.$$

Thus $\lim_{n\to\infty} N(n) = N^* \geq \xi_2$ exists. We claim now that $N^* = \xi_2$, if otherwise $N^* > \xi_2$, by (2.11), there exist $\epsilon > 0$ and $n_4 \geq n_3 + 2r^L$ such that

$$F_2(N(n-r(n))) < -\epsilon \qquad \text{for }\; n \geq n_4. \tag{2.12}$$

We can derive from (2.10) and (2.12),

$$N(n+1) \leq N(n)\exp[F_2(N(n-r(n)))] \leq N(n)\exp[-\epsilon] \qquad \text{for }\; n \geq n_4. \tag{2.13}$$

By (2.13), on $[n_4, n], (n > n_4)$, we have

$$N(n+1) \leq N(n)e^{-\epsilon} < N(n).$$

Thus, $N(n) \to 0$ as $n \to \infty$, which contradicts $N(n) > \xi_2 > 0$ for $n \geq n_3$. Hence, we obtain $\lim_{n\to\infty} N(n) = \xi_2$ from which (2.7) follows. Next, we suppose that $N(n)$ is oscillates about ξ_2 that is, there exists a sequence $\{n_k\}$ satisfying $n_{k+1} - n_k > 2r^L,\; n_k \to \infty$ as $k \to \infty$ such that

$$N(n_k) = \xi_2 \qquad n = 1, 2, \cdots.$$

Let $N(\alpha_k)$ denote a local maximum of N on (n_k, n_{k+1}). Then, from (2.10),

$$0 \leq N(\alpha_k + 1) \leq N(\alpha_k)\exp[F_2(N(\alpha_k - r(\alpha_k)))]$$

which together with (2.11) implies

$$N(\alpha_k - r(\alpha_k))) \le \xi_2.$$

Similarly as in (2.5), on $[\alpha_k - r(\alpha_k), \alpha_k]$, we derive that

$$\frac{N(\alpha_k)}{N(\alpha_k - r(\alpha_k))} \le \exp\left[\sum_{j=\alpha_k - r(\alpha_k)}^{\alpha_k - 1} F_2(N(j - r(j)))\right].$$

Then,

$$\log \frac{N(\alpha_k)}{N(\alpha_k - r(\alpha_k))} \le \sum_{j=\alpha_k - r(\alpha_k)}^{\alpha_k - 1} F_2(N(j - r(j))) \le \alpha(r(\alpha_k)) \le \alpha r^L.$$

Thus, we arrive at $N(n) \le \xi_2 e^{\alpha r^L}$. Since $N(\alpha_k)$ is an arbitrary local maximum of N, we can conclude that there exists an $n_2 > n_1$ such that (2.7) holds. To prove (2.9), we note first that we have from (2.1) and (H_2),

$$N(n+1) \ge N(n) \exp[F_1(N(n - r(n)))]. \tag{2.14}$$

From (H_3), we can see that there exists a $\xi_1 > 0$ such that $F_1(\xi_1) = 0$. Suppose that $N(n)$ does not oscillate about ξ_1. Then similar to the above arguments, we can show that there exists an n_2 such that (2.9) holds. We have already show that there exists an n_1 such that (2.7) holds. Hence when $N(n)$ oscillates about ξ_1 we have from (2.8)

$$F_1(N(n - r(n))) \ge F_1(M) \qquad \text{for } n \ge n_1 + r^L. \tag{2.15}$$

Now, by using (2.14), (2.15) and arguments similar as above, we can show that there exists an $n_2 \ge n_1$ such that (2.9) holds. This completes the proof. □

3 Almost Periodic Models with Finite Delay

In order to construct discrete type existence theorem of almost periodic solutions, we first consider the scalar general functional difference equation

$$x(n+1) = g(n, x_n). \tag{3.1}$$

Here x_n is the segment of $x(s)$ on $[n-h, n]$ shifted to $[-h, 0]$, where $h > 0$ is a fixed constant integer. For $h > 0$ and $\phi \in \mathrm{BS} = \mathrm{BS}([-h, 0])$, $|\phi| = \sup_{-h \le s \le 0} |\phi(s)|$. In (3.1), $g(n, \phi)$ is continuous for ϕ (second term) defined on $\mathbb{Z} \times \mathrm{BS}$ to $\mathbb{R}$ and it takes bounded sets into bounded sets. Moreover, it satisfies a local Lipschitz condition in ϕ and the $g(n, \phi)$ is almost periodic in n uniformly for $\phi \in \mathrm{BS}$. A solution through (n_0, ϕ) is denoted by $x(n_0, \phi)$ with value at being $x(n, n_0, \phi)$ and with $x(n_0, n_0, \phi) = \phi$. The solution of (3.1) is unique for the initial function ϕ (see, e.g., [6, 8]).

In what follows, we need the following definitions of stability.

Definition 3.1. The bounded solution $x(n)$ of (3.1) is said to be

(i) uniformly stable (in short, US) if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $n_0 \geq 0$, $|x_{n_0} - u_{n_0}| < \delta(\epsilon)$, then $|x_n - u_n| < \epsilon$ for all $n \geq n_0$, where $u(n)$ is a solution of (3.1) through (n_0, ψ) such that $u_{n_0}(s) = \psi(s)$ for all $-h \leq s \leq 0$.

(ii) uniformly asymptotically stable (in short, UAS) if it is US and if for any $\epsilon > 0$ there exists a $\delta_0 > 0$ and a $T(\epsilon) > 0$ such that if $n_0 \geq 0$, $|x_{n_0} - u_{n_0}| < \delta_0$, then $|x_n - u_n| < \epsilon$ for all $n \geq n_0 + T(\epsilon)$, where $u(n)$ is a solution of (3.1) through (n_0, ψ) such that $u_{n_0}(s) = \psi(s)$ for all $-h \leq s \leq 0$. The δ_0 and the T above are independent of n_0.

(iii) globally uniformly asymptotically stable (in short, GUAS) if it is US and if for any $\epsilon > 0$ and $\alpha > 0$, there exists a $T(\epsilon, \alpha) > 0$ such that if $n_0 \geq 0$, $|x_{n_0} - u_{n_0}| < \alpha$, then $|x_n - u_n| < \epsilon$ for all $n \geq n_0 + T(\epsilon, \alpha)$, where $u(n)$ is a solution of (3.1) through (n_0, ψ) such that $u_{n_0}(s) = \psi(s)$ for all $-h \leq s \leq 0$. The T above is independent of n_0.

We consider the scalar delay difference equation as special case of (3.1):

$$\Delta x(n) = -a(n)x(n - r(n)), \tag{3.2}$$

where Δ is difference of $x(n)$, $a : \mathbb{Z}^+ \to \mathbb{R}^+$ and $r : \mathbb{Z}^+ \to [0, h]$. We assume that $a(n)$ and $r(n)$ are almost periodic functions in n, and $0 < r^l = \liminf_{n\to\infty} r(n) \leq r(n) \leq \limsup_{n\to\infty} r(n) = r^L$.

The following lemma is derived from the results of [15, Theorem 1.1 and 1.2], and so we omit the proof.

Lemma 3.2. *If $a(n)$ and $r(n)$ are in $\mathbb{R}$ and*

$$\sum_{s=0}^{\infty} a(s) = \infty, \qquad \lambda \equiv \limsup_{n\geq 0} \sum_{s=n-r(n)}^{n} a(s) \leq 1 + \frac{r^l + 2}{2(r^L + 1)},$$

then the zero solution $x(n) \equiv 0$ of (3.2) is uniformly stable. Moreover if $\lambda < 1 + \dfrac{r^l + 2}{2(r^L + 1)}$, then the zero solution of (3.2) is uniformly asymptotically stable.

We have the following theorem by Lemma 3.2.

Theorem 3.3. *Suppose that all assumptions of Lemma 3.2 are satisfied. Let $\tilde{g}(n, z) : [n_0, \infty) \times \mathbb{R} \to \mathbb{R}^+$ be defined by*

$$\tilde{g}(n, z) = -\frac{\partial f(n, e^z)}{\partial z}. \tag{3.3}$$

We assume that there exists a function $G : [n_0, \infty) \to \mathbb{R}^+$ *such that*

$$\sum_{s=n_0}^{\infty} \tilde{g}(s, x) = \infty, \qquad \textit{and } \tilde{g}(n, x) \le G(n) \textit{ for } m \le x \le M \tag{3.4}$$

where m *and* M *are defined by* (2.7) *and* (2.9). *If*

$$\limsup_{n\to\infty} \sum_{s=n-r(n)}^{n} G(s) \le 1 + \frac{r^l + 2}{2(r^L + 1)}, \tag{3.5}$$

then any positive solution of (2.1) *is uniformly asymptotically stable.*

Proof. Let $x(n)$ and $y(n)$ be any two positive solutions of (2.1) and define u, v and w by

$$e^{u(n)} = x(n), \qquad e^{v(n)} = y(n), \qquad \text{and} \qquad w(n) = u(n) - v(n).$$

Then it follows from (2.1) that $w(n)$ is governed by

$$\begin{aligned} \Delta w(n) &= w(n+1) - w(n) = \Delta u(n) - \Delta v(n) \\ &= f(n, x(n - r(n)) - f(n, y(n - r(n)) \\ &= f(n, e^{u(n-r(n))}) - f(n, e^{v(n-r(n))}). \end{aligned} \tag{3.6}$$

From (3.3) and (3.6), we can rewrite (2.1) as

$$\Delta w(n) = -[\tilde{g}(n, \xi(n - r(n)))]w(n - r(n)), \tag{3.7}$$

where $\xi(s)$ lies between $x(s)$ and $y(s)$. By replacing $\tilde{g}$ of equation (3.7) with the function $G(n)$, we can apply the results of Lemma 3.2 to equation (3.7) by the hypotheses of (3.4) and (3.5). Since (3.7) is equivalent to (2.1), the proof is complete. □

Remark 3.4. From [8, Theorem 2] and Theorem 3.3 above, we can see that, under the assumptions of Theorem 3.3, if f is a periodic function in n with period $T > 0$, then (2.1) has a periodic solution with period T which is uniformly asymptotically stable.

From Theorem 3.3, we obtain the following main theorem.

Theorem 3.5. *Under the assumptions* (H_1), (H_2) *and* (H_3), *if* $f(n, \phi)$ *in* (2.1) *is continuous in the second variable, satisfies a local Lipschitz condition in* ϕ *and is almost periodic in* $n \in [n_0, \infty)$ *uniformly for* $\phi \in \mathrm{BS}([-h, 0], \mathbb{R}^+)$, *then equation* (2.1) *has a unique positive almost periodic solution which is globally uniformly asymptotically stable.*

Proof. Let $N(n)$ denote a positive solution of (2.1) such that

$$0 < m \leq N(n) \leq M, \qquad \text{for } n \geq n_0,$$

where m and M are defined by (2.7) and (2.9). Let $\{n_j\}$ be a sequence of integers such that $n_j \to \infty$ as $j \to \infty$. Define $x_i(n)$ and $x_j(n)$ by

$$e^{x_i(n)} = N(n+n_i), \qquad e^{x_j(n)} = N(n+n_j).$$

It follows from (2.1) that

$$\begin{aligned} \Delta x_i = x_i(n+1) - x_i(n) &= f(n+n_i, e^{x_i(n-r(n))}), \\ \Delta x_j = x_j(n+1) - x_j(n) &= f(n+n_j, e^{x_j(n-r(n))}). \end{aligned}$$

We let $y_{i,j}(n) = x_i(n) - x_j(n)$ and note that

$$\begin{aligned} \Delta y_{i,j}(n) &= y_{i,j}(n+1) - y_{i,j}(n) \\ &= f(n+n_i, e^{x_i(n-r(n))}) - f(n+n_j, e^{x_j(n-r(n))}) \\ &= f(n+n_i, e^{x_i(n-r(n))}) - f(n+n_i, e^{x_j(n-r(n))}) + h(n; n_i, n_j), \quad (3.8) \end{aligned}$$

where $h(n; n_i, n_j) = f(n+n_i, e^{x_j(n-r(n))}) - f(n+n_j, e^{x_j(n-r(n))})$. By (3.3), we can rewrite (3.8) in the form

$$\Delta y_{i,j}(n) = -[\tilde{g}(n+n_i, \xi_{i,i}(n-r(n)))]y_{i,j}(n-r(n)) + h(n; n_i, n_j), \tag{3.9}$$

in which $\xi_{i,j}(n)$ lies between $N(n+n_i)$ and $N(n+n_j)$. By hypotheses and Lemma 3.2, the trivial solution of the linear equation

$$\Delta z(n) = -e(n)z(n-r(n)) \tag{3.10}$$

with $e(n) = \tilde{g}(n+n_i, \xi_{i,i}(n-r(n)))$ is uniformly asymptotically stable and hence the fundamental solution $W(n)$ associated with (3.10) satisfies an estimate of the type (see [2, 6])

$$|W(n)W^{-1}(s)| \leq C^* \eta^{\gamma(n-s)} \ \text{ for } \ \gamma \in (0, \infty), \ \ C^* \in [1, \infty), \eta \in (0,1). \tag{3.11}$$

By the variation of constants formula [2], we have from (3.9) that

$$y_{i,j}(n) = W(n)W^{-1}(n_0)y_{i,j}(n_0) + \sum_{s=n_0}^{n-1} W(n)W^{-1}(s)h(s; n_i, n_j) \tag{3.12}$$

Using (3.11) and (3.12), we obtain

$$|y_{i,j}(n)| = C^* \eta^{\gamma(n-n_0)} |y_{i,j}(n_0)| + \sum_{s=n_0}^{n-1} C^* \eta^{\gamma(n-n_0)} |h(s; n_i, n_j)|, \ \ n \geq n_0. \tag{3.13}$$

From the almost periodic property of $f(n,\phi)$ in n uniformly for $\phi \in \mathrm{BS}$, we can choose integers n_i^* and n_j^* large enough such that

$$\begin{aligned} |h(n;n_i,n_j)| &= |f(n+n_i, e^{x_j(n-r(n))}) - f(n+n_j, e^{x_j(n-r(n))})| \\ &< \frac{\epsilon\gamma}{2C^*} \qquad \text{for } n_i \ge n_i^*, n_j \ge n_j^* \end{aligned} \tag{3.14}$$

and for arbitrary $\epsilon > 0$. Also, we have from $|y_{i,j}(n_0)| = |x(n_0+n_i) - x(n_0+n_j)|$ that

$$|y_{i,j}(n_0)| < \frac{\epsilon}{2} \qquad \text{for } n_i \ge n_i^*, n_j \ge n_j^* \tag{3.15}$$

due to the uniform convergence of the sequence $\{x(n+n_i)\}$ for n in compact subsets of $[n_0,\infty)$. Thus, it follows from (3.13), (3.14) and (3.15) that

$$|y_{i,j}(n)| = |x(n+n_i) - x(n+n_j)| < \epsilon \qquad \text{for } n \in [n_0,\infty), n_i \ge n_i^*, n_j \ge n_j^*$$

and hence, $x(n)$ is asymptotically almost periodic in the sense that there exist x^* and $\bar{x}$ satisfying

$$x(n) = x^*(n) + \bar{x}(n) \tag{3.16}$$

where x^* is almost periodic on $[n_0,\infty)$ and $\bar{x}$ is on $[n_0,\infty)$ with $\bar{x}(n) \to 0$ as $n \to \infty$. We can now proceed as in [7] to show that the almost periodic part x^* in (3.16) is a solution of

$$\Delta x^*(n) = f(n, e^{x^*(n-r(n))}).$$

It follows that $N^*(n) = e^{x^*(n)}$ is an almost periodic solution of (2.1). The uniform asymptotic stability of $N^*(n)$ is an immediate consequence of Theorem 3.3, that is, the proof is complete. □

Conjecture 3.6. For the functional differential equation with finite delay, it is well known that the condition (3.5) and others is able to change the $3/2$ stability, see, e.g., [12]. Then, we have the following conjecture. It seems that if (3.5) is replaced by

$$\limsup_{n\to\infty} \sum_{s=n-r(n)}^{n} G(s) \le \frac{3}{2} + \frac{1}{2(r^L+1)},$$

then the result of Theorem 3.3 is also right for (2.1). However, we have no proof, yet.

4 Examples

Example 4.1. We first consider the almost periodic delay model of the form

$$N(n+1) = N(n)\exp\{a(n) + b(n)N^p(n-r(n)) - c(n)N^q(n-r(n))\}, \tag{4.1}$$

where a, b, c and r are defined on $[n_0, \infty)$ with $a(n) > 0,\ b(n) \in \mathbb{R},\ c(n) > 0$ and $r(n) > 0$ are almost periodic functions in n, and

$$0 < p < q,\ 0 < r^l = \liminf_{n\to\infty} \le r(n) \le \limsup_{n\to\infty} r(n) = r^L, \tag{4.2}$$

$$\begin{aligned} 0 < a^l &= \liminf_{n\to\infty} a(n) \le a(n) \le \limsup_{n\to\infty} a(n) = a^L, \\ b^l &= \liminf_{n\to\infty} b(n) \le b(n) \le \limsup_{n\to\infty} b(n) = b^L \text{ and} \\ 0 < c^l &= \liminf_{n\to\infty} c(n) \le c(n) \le \limsup_{n\to\infty} c(n) = c^L \end{aligned} \tag{4.3}$$

for $n \ge n_0$. The original differential model of (4.1) is considered by Ladas and Qian [10]. Let

$$f(n, y) = a(n) + b(n)y^p - c(n)y^q$$

and

$$F_1(y) = a^l + b^l y^p - c^L y^q, \quad F_2(y) = a^L + b^L y^p - c^l y^q.$$

We note that $F_1(y) \le f(n, y) \le F_2(y)$ for $n \ge n_0,\ y \in [0, \infty)$. It is easy to see that all the assumptions (H_1), (H_2) and (H_3) are satisfied. We denote by y_* and y^* the unique positive solution of the equations $F_1(y) = 0$ and $F_2(y) = 0$, respectively. Then, any positive solution $N(n)$ of (4.1) satisfies eventually for all large n

$$m_1 \le N(n) \le M_1,$$

where

$$M_1 = \begin{cases} y^* e^{a^L r^L}, & b(n) \le 0, \\ y^* \exp\left[r^L \left(a^L + b^L \left(\dfrac{b^L p}{c^l q}\right)^{p/(q-p)} - c^l \left(\dfrac{b^L p}{c^l q}\right)^{q/(q-p)} \right) \right], & b(n) > 0 \end{cases}$$

and

$$m_1 = y_* \exp[r^L(a^l + b^l M_1^p - c^L M_1^q)]. \tag{4.4}$$

If $a(n) \equiv a,\ b(n) \equiv b,\ c(n) \equiv c$ and $r(n) \equiv r$ for $n \ge n_0$ and y_0 is the unique positive solution of the equation

$$F(y) = a + by^p - cy^q, \qquad y \in [0, \infty),$$

then $N(n)$ satisfies (eventually for all large n)

$$m_2 \le N(n) \le M_2$$

in which

$$M_2 = \begin{cases} y_0 e^{ar}, & b \le 0, \\ y_0 \exp\left[r \left(a + b \left(\dfrac{bp}{cq}\right)^{p/(q-p)} - c \left(\dfrac{bp}{cq}\right)^{q/(q-p)} \right) \right], & b(n) > 0 \end{cases}$$

and

$$m_2 = y_0 \exp[r(a + bM_2^p - cM_2^q)].$$

According to the above result and Theorem 3.5, we can obtain the following corollary for equation (4.1). This equation has been studied recently [9] treated as a delay differential equation, and the conditions of the following corollary offer an improvement of these known results for differential equations.

Corollary 4.2. *Let* $a(n), b(n)$ *and* $c(n)$ *be almost periodic functions. Suppose that* (4.2) *and* (4.3) *hold. If*

$$\limsup_{n\to\infty} \sum_{s=n-r(n)}^{n-1} [c(s)qM_1^q - b(s)pM_1^p] < 1 + \frac{r^l + 2}{2(r^L+1)} \ \textit{for} \ b(n) < 0$$

and

$$\limsup_{n\to\infty} \sum_{s=n-r(n)}^{n-1} [c(s)qM_1^q - b(s)pm_1^p] < 1 + \frac{r^l + 2}{2(r^L+1)} \ \textit{for} \ b(n) \geq 0$$

in which m_1 *and* M_1 *are defined by* (4.4)*, then* (4.1) *has a unique almost periodic solution which is global uniformly asymptotically stable.*

Proof. Let $N(n)$ be any positive solution of (4.1). Let $e^{u(n)} = N(n)$. Note that

$$g(n,u) = -\frac{\partial f(n, e^u)}{\partial u} = -b(n)pN^p + c(n)qN^q. \tag{4.5}$$

We have from Example 4.1 that $N(n)$ satisfies

$$m_1 \leq N(n) \leq M_1,$$

eventually for all large n which, together with (4.5), implies that $g(n,u) \leq G(n)$, where

$$G(n) = \begin{cases} c(n)qM_1^q - b(n)pM_1^p & \text{for } b(n) < 0 \\ c(n)qM_1^q - b(n)pm_1^p & \text{for } b(n) \geq 0. \end{cases}$$

Then, from Theorem 3.5, the conclusion follows. □

We consider the following example. Applying Theorem 3.5 to the logistic delay equation

$$N(n+1) = N(n)\exp\{a(n) - b(n)N(n - r(n))\} \qquad \text{for } n \geq n_0, \tag{4.6}$$

we obtain the following result which is a generalization to the delay difference equation of that in [5].

Corollary 4.3. *Assume that* $a(n), b(n)$ *and* $r(n)$ *are positive almost periodic functions satisfying*

$$0 < a^l \le a(n) \le a^L, \qquad b^l \le b(n) \le b^L, \qquad r^l \le r(n) \le r^L,$$

where a^l, a^L, b^l, b^L, r^l *and* r^L *are defined similarly as in* (4.2) *and* (4.3)*, and*

$$\mu \equiv \frac{a^L}{b^l} b^L r^L \eta^{a^l r^L} < 1 + \frac{r^l + 2}{2(r^L + 1)}$$

for $n \ge n_0$*, then* (4.6) *has a unique almost periodic solution which is uniformly asymptotically stable.*

Remark 4.4. The previously known result obtained for delay differential equations in [4] required $\mu < 1$. Then the sufficient condition given in Corollary 4.3 improves the previously known result $\mu < 1$.

5 Almost Periodic Models with Infinite Delay

We finally consider the existence of an almost periodic solution of the following almost periodic Ricker type difference equation in the case where f of (2.1) has an infinite delay (see, e.g., [4])

$$N(n+1) = N(n) \exp\left\{ a(n) - b(n) \sum_{s=0}^{\infty} K(s) N(n-s) \right\}, \tag{5.1}$$

where a and b are positive almost periodic functions in n, and K is a nonnegative kernel on $[0, \infty)$ such that there exists a $\sigma > 0$ satisfying

$$\sum_{s=0}^{\infty} K(s) = 1, \quad \sum_{s=0}^{\infty} sK(s) \le \sigma < \infty \quad \text{and} \quad \sum_{s=0}^{\sigma} K(s) > 0. \tag{5.2}$$

Moreover, we assume that

$$\begin{aligned} 0 < a^l = \liminf_{n\to\infty} a(n) \le a(n) \le \limsup_{n\to\infty} a(n) = a^L, \\ 0 < b^l = \liminf_{n\to\infty} b(n) \le b(n) \le \limsup_{n\to\infty} b(n) = b^L. \end{aligned} \tag{5.3}$$

In the following we are concerned with positive solutions of (5.1) corresponding to initial conditions of the form

$$N(s) = \phi(s) \ge 0, \;\; s \in (-\infty, 0], \phi(0) > 0 \;\text{ and }\; \phi \in \mathrm{BS}((-\infty, 0], \mathbb{R}^+). \tag{5.4}$$

The following lemma provides an a priori upper estimate of any positive solution of (5.1).

Lemma 5.1. *If the assumptions* (5.2)*,* (5.3) *and* (5.4) *hold, then there exists a* $T^* > 0$ *such that any positive solution* $N(n)$ *of* (5.1) *satisfies*

$$N(n) \leq N^*, \quad \textit{where } N^* = \frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)} e^{a^L \sigma} \textit{ for } n > T^*. \tag{5.5}$$

Furthermore, if

$$a^l > b^L N^*,$$

then there exist a $T_* \geq T^*$ *and an* $(N^* >)N_* > 0$ *such that*

$$N(n) \geq N_* \quad \textit{for} \quad n \geq T_*.$$

Proof. By the positive solution $N = N(n)$ of (5.1) and equation (5.1), we have

$$\begin{aligned} N(n+1) &\leq N(n) \exp\left\{ a^L - b^l \sum_{s=0}^{\infty} K(s) N(n-s) \right\} \\ &\leq N(n) \exp\left\{ a^L - b^l \sum_{s=0}^{\sigma} K(s) N(n-s) \right\} \text{ for } n > 0. \end{aligned}$$

Suppose N is not oscillatory about $\dfrac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)}$. Then, there exists a $T_1 > 0$ such that either

$$N(n) > \frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)} \text{ for } n > T_1 \tag{5.6}$$

or

$$N(n) < \frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)} \text{ for } n > T_1. \tag{5.7}$$

If (5.7) holds, then (5.5) follows with $T^* = T_1$. Suppose now that (5.6) holds. Then

$$N(n+1) \leq N(n) \qquad \text{for } n > T_1 + \sigma,$$

and hence for some $N_0 > 0$,

$$N(n) \to N_0 \qquad \text{as } n \to \infty. \tag{5.8}$$

From this, we can show that

$$N(n) \leq \frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)}. \tag{5.9}$$

The conclusion (5.5) now follows from (5.8) and (5.9). Suppose that N is oscillatory about $\frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)}$. Let $N(n^*)$ denote an arbitrary local maximum of N. We can see from (5.1) that

$$0 = N(n^* + 1) \le N(n^*) \exp \left\{ a^L - b^l \sum_{s=0}^{\sigma} K(s) N(n^* - s) \right\},$$

and this implies

$$\sum_{s=0}^{\sigma} K(s) N(n^* - s) \le \frac{a^L}{b^l}. \tag{5.10}$$

If $N(n) > \frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)}$ for all $n \in [n^* - \sigma, n^*]$, then $\sum_{s=0}^{\sigma} K(s) N(n^* - s) > \frac{a^L}{b^l}$, which contradicts (5.10). Thus, by the oscillation of N about $\frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)}$, there exists a $n_1 \in [n^* - \sigma, n^*]$ such that

$$N(n_1) = \frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)}.$$

By the same argument as in (2.4) and (2.5) for the interval $[n_1, n^*]$, we have

$$\begin{aligned} \log \frac{N(n^*)}{N(n_1)} &\le \sum_{s=n_1}^{n^*-n_1} \left\{ a^L - b^l \sum_{s=0}^{\sigma} K(s) N(n^* - s) \right\} \\ &\le a^L \sum_{s=n_1}^{n^*-n_1} 1 \le a^L \sum_{s=n^*-\sigma}^{n^*} 1 = a^L \sigma \end{aligned}$$

so that

$$N(n^*) < \frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)} e^{a^L \sigma}.$$

Since $N(n^*)$ is arbitrary local maximum of N, we can conclude that

$$N(n) \le N(n^*) \le \frac{a^L}{b^l \sum_{s=0}^{\sigma} K(s)} e^{a^L \sigma} \quad \text{for } n > T_1,$$

where $T^* = T_1$ is the first zero of the oscillatory N. We next show that

$$\liminf_{n \to \infty} N(n) \ge N_*. \tag{5.11}$$

According to above assertion, there exists a $k^* \geq T^*$ such that $N(n) \leq N^*$ for all $n \geq k^*$. We assume that there exists an $l_0 \geq k^*$ such that $N(l_0+1) \leq N(l_0)$. Note that for $n \geq l_0$,

$$\begin{aligned} N(n+1) &= N(n)\exp\left\{a(n) - b(n)\sum_{s=-\infty}^{n} K(n-s)N(s)\right\} \\ &\geq N(n)\exp\{a^l - b^L N^*\}. \end{aligned}$$

In particular, with $n = l_0$, we have

$$a^l - b^L N^* \leq 0$$

which implies that

$$N(l_0) \geq (a^l - b^L N^*).$$

Then,

$$N(l_0+1) \geq (a^l - b^L N^*)\exp(a^l - b^L N^*) =: N_1.$$

We assert that

$$N(n) \geq N_1, \quad \text{for all } \ n \geq l_0. \tag{5.12}$$

By way of contradiction, we assume that there exists a $p_0 \geq l_0$ such that $N(p_0) < N_1$. Then $p_0 \geq l_0 + 2$. Let $\hat{p}_0 + 2$ be the smallest integer such that $N(\hat{p}_0) < N_1$. Then $N(\hat{p}_0) \leq N(\hat{p}_0 - 1)$. Then above argument yields $N(\hat{p}_0) \geq N_1$, which is a contradiction. This proves our claim. We now assume that $N(n+1) < N(n)$ for all $n \geq T^*$. Then $\lim_{n\to\infty} N(n)$ exists, which is denoted by $\underline{N}_1$. We claim that $\underline{N}_1 \geq (a^l - b^L N^*)$. Suppose to the contrary that $\underline{N}_1 < (a^l - b^L N^*)$. Taking limits in (5.1), we set that

$$0 = \lim_{n\to\infty}\left(a(n) - b(n)\sum_{s=-\infty}^{n} K(n-s)N(s)\right) \geq a^l - b^L N^* > 0,$$

which is a contradiction. It follows that (5.11) holds, and then $0 < N_* \leq N(n)$ for all $n \geq T_* = l_0$ from (5.11) and (5.12). This completes the proof. □

Theorem 5.2. *Assume that* a, b *and* K *in* (5.1) *satisfy* (5.2), (5.3) *and* (5.4). *Then equation* (5.1) *has a unique positive almost periodic solution say* $p(n)$ *such that any other positive solution* $N(n)$ *of* (5.1) *satisfies*

$$\lim_{n\to\infty}\{N(n) - p(n)\} = 0. \tag{5.13}$$

Proof. For (5.1), we first introduce the change of variables

$$N(n) = \exp\{v(n)\}, \ \ p(n) = \exp\{y(n)\}.$$

Then, (5.1) can be written as

$$v(n+1) - v(n) = a(n) - b(n) \sum_{s=-\infty}^{n} K(n-s) \exp\{v(s)\}. \tag{5.14}$$

We first consider Liapunov functional

$$\begin{aligned} V_1 &= V_1(v(n), y(n)) \\ &= |v(n) - y(n)| + \sum_{s=0}^{\infty} K(s) \sum_{l=n-s}^{n-1} b(s+l)|\exp\{v(l)\} - \exp\{y(l)\}|, \end{aligned}$$

where $y(n)$ and $v(n)$ are solutions of (5.14) which remain in the bounded set $B := \{x \in \mathbb{R} | N_* \leq x \leq N^*\}$. We have

$$\begin{aligned} \Delta V_1(v(n), y(n)) &\leq |v(n+1) - v(n)| - |y(n+1) - y(n)| \\ &\quad + \sum_{s=0}^{\infty} K(s)[b(s+n)|\exp\{v(n)\} - \exp\{y(n)\}| \\ &\qquad - b(n)|\exp\{v(n-s)\} - \exp\{y(n-s)\}|] \\ &= \left| a(n) - b(n) \sum_{s=0}^{\infty} K(s) \exp\{v(n-s)\} \right| \\ &\quad - \left| a(n) - b(n) \sum_{s=0}^{\infty} K(s) \exp\{y(n-s)\} \right| \\ &\quad + \sum_{s=0}^{\infty} K(s)[b(s+n)|\exp\{v(n)\} - \exp\{y(n)\}| \\ &\qquad - b(n)|\exp\{v(n-s)\} - \exp\{y(n-s)\}|] \\ &\leq |a(n) - b(n)\exp\{v(n-s)\}| - |a(n) - b(n)\exp\{y(n-s)\}| \\ &\quad + b(s+n)|\exp\{v(n)\} - \exp\{y(n)\}| - b(n)|\exp\{v(n-s)\} - \exp\{y(n-s)\}| \\ &\leq -2b(n)|\exp\{v(n-s)\} - \exp\{y(n-s)\}| \\ &\quad + b(s+n)|\exp\{v(n)\} - \exp\{y(n)\}|. \end{aligned}$$

Secondly we consider

$$V_2 = V_2(v(n), y(n)) = 2|v(n+s) - y(n+s)|.$$

Similarly, we can calculate

$$\begin{aligned} \Delta V_2(v(n), y(n)) &\leq 2(|v(n+s+1) - v(n+s)| - |y(n+s+1) - y(n+s)|) \\ &\leq -2b(s+n)|\exp\{v(n)\} - \exp\{y(n)\}|. \end{aligned}$$

We take $V = V_1 + V_2$. Then, we have

$$\begin{aligned}\Delta V(v(n), y(n)) &\leq -2b(n)|\exp\{v(n-s)\} - \exp\{y(n-s)\}| \\ &\quad -b(n+s)|\exp\{v(n)\} - \exp\{y(n)\}| \\ &\leq -b(n+s)|\exp\{v(n)\} - \exp\{y(n)\}| \\ &\leq -b^l|\exp\{v(n)\} - \exp\{y(n)\}|.\end{aligned}$$

From the mean value theorem, we have

$$|\exp\{v(n)\} - \exp\{y(n)\}| = \exp\{\theta(n)\}|v(n) - y(n)|,$$

where $\theta(n)$ lies between $v(n)$ and $y(n)$. Then, we have

$$\Delta V(v(n), y(n)) \leq -b^l D|v(n) - y(n)|,$$

where we set $D = \exp(N_*)$. Suppose the solution $z(n)$ of (5.14) is such that $N^* \geq z(n) \geq N_*$ for $n \geq T_*$. Thus $|v(n) - y(n)| \to 0$ as $n \to \infty$, and hence $V = V(v(n), y(n)) \to 0$ as $n \to \infty$. Moreover, we can show that $|v^*(n) - y^*(n)| \to 0$ as $n \to \infty$, where v^* and y^* are solutions in the hull of (5.14) by the same argument as in [10]. By using similar a Liapunov functional V^* of V, we can show that $V^* \to 0$ as $n \to \infty$. Note that this Liapunov functional V^* is a nonincreasing functional on Z, and hence $V^* = V^*(v^*(n), y^*(n)) \equiv 0$. Thus, $v^*(n) = y^*(n)$ for all $n \in \mathbb{Z}$. Therefore, each hull equation of (5.14) has a unique strictly positive bounded solution. By the equivalence between (5.1) and (5.14), it follows from results in [14] that (5.1) has an almost periodic solution $p(n)$ such that $N_* \leq p(n) \leq N^*$ for all $n \in \mathbb{Z}$, and (5.13) follows. The proof is complete. □

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