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Utilization of Circulant Matrix Theory in Periodic Autonomous Difference Equations

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Abstract

In this paper we develop easily verifiable tests that we can apply to determine whether or not a higher order autonomous difference equation has a p-periodic solution. One of the main tools in our investigations is a transformation, recently introduced by the authors, which formulates a given higher order difference equation as a first order recursion. The second important tool is the theory of circulant matrices. The periodicity conditions are formulated in terms of the coefficients of the higher order equation, along with examples showing that they have nontrivial applications.

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1 Introduction

The problem of the existence of periodic solutions of difference equations has attracted the attention of several authors for theoretical as well as practical reasons. There are a lot of techniques that are used in the study of the existence of periodic solutions (see, for instance [1, 2, 4, 6, 9]).

In a recent book [6] the authors E. A. Grove and G. Ladas present a series of results and many questions and open problems for higher order difference equations having

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periodic solutions. Based on this book and some connected recent works (see, for instance [1, 2, 4, 9]) one may formulate the following question for the higher order linear difference equation

$$x(n)=\sum_{k=1}^{s}A_kx(n-k),\quad n\geq 0, \tag{1.1}$$

where $s\geq 1$ is a given integer, and $A_k\in\mathbb{R}$ for every $1\leq k\leq s$.

Is there any easily verifiable explicit formula that we can apply to determine whether or not equation (1.1) has a p-periodic solution? The construction of the periodic initial sequences and finding the prime periods are further important questions.

The question on the existence of p-periodic solutions may be answered only partially based on the fact, that equation (1.1) has a nontrivial p-periodic solution if and only if its characteristic equation

$$\lambda^s-\sum_{k=1}^{s}A_k\lambda^{s-k}=0$$

has a zero which is the pth root of the unity. But it is more effective and also convenient to study the existence of p-periodic solutions and initial sequences based on a fundamental result on the circulant matrix theory. This result gives both the eigenvalues and eigenvectors in explicit forms.

In fact, in this paper we obtain necessary and sufficient conditions for the existence of nontrivial periodic solutions in terms of the coefficients A_k $(1\leq k\leq s)$, and the order of the equation s. Among others, we also give in explicit form initial vectors $(x(-s),\ldots,x(-1))$ which generate p-periodic solutions, and we determine the prime periods.

The main tools for the proof of the main results are Theorems 4.1-4.3 in our earlier paper [7] and some basic results on circulant matrices given in [3]. The essence of our foregoing results are summarized in Theorem 2.1 and Theorem 2.2. Theorem 2.1 gives a construction of a p by p matrix from the coefficients of (1.1) such that 1 is an eigenvalue of it if and only if equation (1.1) has a nontrivial p-periodic solution. Theorem 2.2 claims that the eigenvectors belonging to the eigenvalue 1 of the matrix given in Theorem 2.1 are p-periodic initial vectors of (1.1).

The main theorems from the circulant matrix theory given in Section 2.2 are related to the easily verifiable forms of the eigenvalues and the related eigenvectors which we need in our proofs.

Special examples are presented to illustrate the applicability and effectiveness of the main results.

A selection of possible applications is now discussed on the simple three-term difference equation

$$x(n)-x(n-1)=A_sx(n-s),\quad n\geq 0, \tag{1.2}$$

where $s\geq 2$ is an integer and $A_s\in\mathbb{R}$.

In our first application the following set is needed:

$$U := \left\{ 2\,(-1)^{k+1} \sin\left(\frac{\pi\,(2k+1)}{2\,(2s-1)} \right) \mid 0 \le k \le s-1 \right\}.$$

Now we can try to characterize those equations of type (1.2) which have a nonconstant periodic solution.

Theorem 1.1. *Consider equation* (1.2)*, and let* $s \ge 2$ *be an integer. Then*

(a) Equation (1.2) *has a nonconstant periodic solution if and only if* $A_s \in U$.

(b) $A_s \in U$ *yields that equation* (1.2) *has a nonconstant* $(4s-2)$*-periodic solution.*

(c) The prime period, say q*, of a nontrivial periodic solution obeys one of the following relations:*

- *(*c_1*)* $A_s = 2\,(-1)^s$ $(k = s-1$ *in* $U)$ *yields* $q = 2$.
- *(*c_2*)* $A_s \in U$ $(0 \le k \le s-2)$ *yields that* $q \ge 6$ *is an even divisor of* $4s-2$*, furthermore if* $2s-1$ *is a prime, then* $q = 4s-2$.

Levin and May [8] showed that the three-term difference equation

$$y(n+1) - y(n) + by(n-m) = 0, \quad n \ge 0, \tag{1.3}$$

where $m \ge 1$ is an integer and $b \in \mathbb{R}$, is asymptotically stable if and only if

$$0 < b < 2\cos\left(\frac{m}{2m+1}\pi \right).$$

Since the asymptotic stability of the equations (1.2) and (1.3) are equivalent if $b = A_s$ and $m = s-1$,we have that equation (1.2) is asymptotically stable if and only if

$$-2\cos\left(\frac{s-1}{2s-1}\pi \right) < A_s < 0.$$

At the same time Theorem 1.1 yields that equation (1.2) (and hence equation (1.3)) has a nonconstant periodic solution on the boundary of the asymptotic stability domain.

Corollary 1.2. *Consider equation* (1.2)*, and let* $s \ge 2$ *be an integer. If*

$$A_s = -2\cos\left(\frac{s-1}{2s-1}\pi \right),$$

then equation (1.2) *has a* $4s-2$*-periodic nonconstant solution.*

It is clear that the solutions of (1.1) are uniquely determined by their initial values

$$x(n) = \varphi(n), \quad -s \leq n \leq -1, \tag{1.4}$$

where $\varphi(n) \in \mathbb{R}$. The unique solution of (1.1) and (1.4) is denoted by

$$x(\varphi) = (x(\varphi)(n))_{n \geq -s},$$

where $\varphi := (\varphi(-s), \ldots, \varphi(-1))^T \in \mathbb{R}^s$.

The basic periodicity notions are given in the next definition.

Definition 1.3. Consider equation (1.1). Let $\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in \mathbb{R}^s$ be a given initial vector.

(a) The solution $x(\varphi) = (x(\varphi)(n))_{n \geq -s}$ of (1.1) and (1.4) is called periodic if there exists a positive integer p such that $x(\varphi)(n + p) = x(\varphi)(n)$ for all $n \geq -s$. In this case we say that $x(\varphi)$ is p-periodic.

(b) φ is said to be a p-periodic initial vector of (1.1) if the solution $x(\varphi)$ of (1.1) and (1.4) is p-periodic.

(c) We say that p is the prime period of the solution $x(\varphi)$ of (1.1) and (1.4) if it is p-periodic and p is the smallest positive integer having this property.

(d) A periodic solution of (1.1) is called nontrivial if it is different from the zero solution.

2 Preliminary Results

2.1 Algebraic Conditions for Periodicity

The following two results contain the essence of [7, Theorems 4.1, 4.2, and 4.3] for equation (1.1).

If $a \in \mathbb{R}$, then $[a]$ denotes the greatest integer that does not exceed a.

Theorem 2.1. *Assume* p *is a positive integer. Equation* (1.1) *has a nontrivial* p*-periodic solution if and only if* 1 *is an eigenvalue of the circulant matrix*

$$\mathcal{B}_{p,s} = \begin{pmatrix} B_p & B_{p-1} & \ldots & B_2 & B_1 \\ B_1 & B_p & \ldots & B_3 & B_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{p-1} & B_{p-2} & \ldots & B_1 & B_p \end{pmatrix}, \tag{2.1}$$

where $\mathcal{B}_{p,s}$ *depends on the relation between* s *and* p*:*

(a) Suppose $p < s$*, and let* $u := \left[\dfrac{s}{p}\right]$ *and* $v := s - up$ $(0 \leq v \leq p-1)$.

If $v = 0$*, define*

$$B_i := \sum_{j=0}^{u-1} A_{jp+i}, \quad 1 \leq i \leq p,$$

while if $v \neq 0$*, define*

$$B_i := \begin{cases} \sum_{j=0}^{u} A_{jp+i}, & 1 \leq i \leq v \\ \sum_{j=0}^{u-1} A_{jp+i}, & v+1 \leq i \leq p \end{cases}.$$

(b) If $p = s$*, define*

$$B_i := A_i, \quad 1 \leq i \leq s.$$

(c) If $p > s$*, define*

$$B_i := \begin{cases} A_i, & 1 \leq i \leq s \\ 0, & s+1 \leq i \leq p \end{cases}.$$

Theorem 2.2. *Assume* p *is a positive integer, and suppose that* 1 *is an eigenvalue of the circulant matrix* $\mathcal{B}_{p,s}$ *in* (2.1)*. Let* $\psi := (\psi(-p), \ldots, \psi(-1))^T \in \mathbb{R}^p$ *be an eigenvector corresponding to the eigenvalue* 1.

(a) If $p < s$*, then* $\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in \mathbb{R}^s$*, where* $\varphi(i) := \psi(j)$ *if* $i \equiv j \pmod p$ $(i = -s, \ldots, -1)$*, is a* p*-periodic initial vector of equation* (1.1).

(b) If $p \geq s$*, then* $\varphi = (\psi(-s), \ldots, \psi(-1))^T \in \mathbb{R}^s$ *is a* p*-periodic initial vector of equation* (1.1)*, and*

$$\psi = (x(\varphi)(0), \ldots, x(\varphi)(p-s-1), \varphi(-s), \ldots, \varphi(-1))^T.$$

2.2 Circulant Matrices

In Section 3, we shall apply circulant matrices to obtain necessary and sufficient conditions for the periodicity of linear autonomous higher order difference equations in terms of their coefficients. Our aim in this subsection is to review some basic facts about circulant matrices which will be needed later on.

A square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called circulant matrix. P. J. Davis in [3] denotes it as

$$B := \text{circ}\,(b_0, b_1, \ldots, b_{n-1}) = \begin{pmatrix} b_0 & b_1 & \ldots & b_{n-2} & b_{n-1} \\ b_{n-1} & b_0 & \ldots & b_{n-3} & b_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 & b_2 & \ldots & b_{n-1} & b_0 \end{pmatrix}, \tag{2.2}$$

where n is a positive integer and b_k $(0 \leq k \leq n-1)$ is a complex number.

For a positive integer n denote $\rho_0, \ldots, \rho_{n-1}$ be the nth roots of 1, that is

$$\rho_k := \exp\left(\frac{2\pi}{n}ki\right) = \cos\left(\frac{2\pi}{n}k\right) + i\sin\left(\frac{2\pi}{n}k\right), \quad k = 0, \ldots, n-1,$$

where i is the imaginary unit.

It is readily verified (see [3] or [10]) that B has n orthogonal eigenvectors and n eigenvalues

$$w_k = \left(1, \rho_k, \rho_k^2, \ldots, \rho_k^{n-1}\right)^T, \quad \lambda_k = b_0 + b_1\rho_k + b_2\rho_k^2 \ldots + b_{n-1}\rho_k^{n-1} \tag{2.3}$$

for $k = 0, 1, \ldots, n-1$.

The following two results will be useful.

Theorem 2.3 (See [5, Theorem 2]). *Let n be a positive prime number. Assume that the entries of the circulant matrix B in* (2.2) *are rational numbers. Then* 1 *is an eigenvalue of B if and only if either $\lambda_0 = 1$ or $b_0 - 1 = b_1 = \ldots = b_{n-1}$.*

Theorem 2.4. *Consider the circulant matrix B in* (2.2)*. Then*

(a) (see [5, Corollary 8]) If either for some $1 \leq k \leq n-1$

$$|b_k| > |b_0 - 1| + \sum_{\substack{j=1 \\ j \neq k}}^{n-1} |b_j|,$$

or

$$|b_0 - 1| > \sum_{j=1}^{n-1} |b_j|,$$

then 1 *is not an eigenvalue of B.*

(b) (see [5, Corollary 10]) Let $d \geq 2$ be a divisor of n, and assume that the vector $(b_0 - 1, b_1, \ldots, b_{n-1})$ consists of $\frac{n}{d}$ consecutive constant blocks of length d (i.e., $\tilde{b}_{kd+j} = \tilde{b}_{kd}$ for $k = 0, \ldots, \frac{n}{d} - 1$ and $j = 0, \ldots, d-1$, where $\tilde{b}_0 := b_0 - 1$ and $\tilde{b}_k := b_k$ $(1 \leq k \leq n-1)$). Then $\lambda_k = 1$ whenever $k \neq 0$ and $k \equiv 0 \mod \left(\frac{n}{d}\right)$, hence $B - I$ is singular and its nullity is $\geq d - 1$.

3 Main Results

We present some necessary and sufficient conditions for determining the existence of nontrivial periodic solutions of equation (1.1).

We begin with a trivial case. If $s = 1$, then the solution of (1.1) and (1.4) is

$$x(n) = A_1^{n+1}\varphi(0), \quad n \geq -1,$$

and therefore equation (1.1) has a nontrivial p-periodic solution (p is a positive integer) if and only if

(i) either $A_1 = 1$; all solutions have prime period 1 (constant solutions),

(ii) or p is even and $A_1 = -1$; all solutions different from the zero solution have prime period 2.

Henceforth it is enough to study the existence of nontrivial p-periodic solutions of equation (1.1) if $s \geq 2$.

Throughout this paper the empty sum means 0 as usual. The greatest common divisor of two integers a and b will be denoted by $\gcd(a, b)$.

First we present our result for the existence of s-periodic solutions of equation (1.1).

Theorem 3.1. *Consider equation* (1.1) *with* $s \geq 2$.

(a) Equation (1.1) *has a nontrivial* s*-periodic solution if and only if*

(a_1) either

$$\sum_{k=1}^{s} A_k = 1, \tag{3.1}$$

(a_2) or s *is even and*

$$\sum_{k=1}^{s} (-1)^k A_k = 1, \tag{3.2}$$

(a_3) or $s > 2$ *and there is an integer* $1 \leq k_0 < \dfrac{s}{2}$ *such that*

$$A_s = 1 - \frac{1}{\sin\left(\frac{2\pi}{s}k_0\right)} \sum_{k=1}^{s-2} A_k \sin\left(\frac{2\pi}{s}(k+1)k_0\right)$$

and

$$A_{s-1} = \frac{1}{\sin\left(\frac{2\pi}{s}k_0\right)} \sum_{k=1}^{s-2} A_k \sin\left(\frac{2\pi}{s}kk_0\right),$$

where $A_1, \ldots, A_{s-2}$ *are free parameters.*

(b) If equation (1.1) *has a nontrivial* s*-periodic solution, then*

(b_1)
$$(1,\dots,1)^T\in\mathbb{R}^s$$
is an s-periodic initial vector of equation (1.1)*, if (a_1) holds,*

(b_2)
$$(1,-1,\dots,1,-1)^T\in\mathbb{R}^s$$
is an s-periodic initial vector of equation (1.1)*, if (a_2) holds,*

(b_3)
$$\left(1,\cos\left(\frac{2\pi}{s}k_0\right),\dots,\cos\left(\frac{2\pi}{s}(s-1)k_0\right)\right)^T\in\mathbb{R}^s$$
and
$$\left(0,\sin\left(\frac{2\pi}{s}k_0\right),\dots,\sin\left(\frac{2\pi}{s}(s-1)k_0\right)\right)^T\in\mathbb{R}^s,$$
are s-periodic initial vectors of equation (1.1)*, if (a_3) holds. The prime period of the solutions corresponding to these initial vectors is*
$$\frac{s}{\gcd(k_0,s)}.$$

In the next result the existence of p-periodic solutions of equation (1.1) is discussed with $p>s$.

Theorem 3.2. *Consider equation* (1.1) *with $s\geq 2$. Assume $p>s$ is an integer.*

(a) Equation (1.1) *has a nontrivial p-periodic solution if and only if*

(a_1) either (3.1) *holds,*

(a_2) or p is even and (3.2) *is satisfied,*

(a_3) or for some integer $1\leq k_0<\frac{p}{2}$
$$A_s=-\frac{\sin\left(\frac{2\pi}{p}(s-1)k_0\right)}{\sin\left(\frac{2\pi}{p}k_0\right)}+\frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)}\sum_{k=1}^{s-2}A_k\sin\left(\frac{2\pi}{p}(s-k-1)k_0\right)$$
and
$$A_{s-1}=\frac{\sin\left(\frac{2\pi}{p}sk_0\right)}{\sin\left(\frac{2\pi}{p}k_0\right)}-\frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)}\sum_{k=1}^{s-2}A_k\sin\left(\frac{2\pi}{p}(s-k)k_0\right),$$
where $A_1,\dots,A_{s-2}$ are free parameters.

(b) *If equation* (1.1) *has a nontrivial* p*-periodic solution, then*

(b_1)
$$(1,\dots,1)^T\in\mathbb{R}^s$$
is a p*-periodic initial vector of equation* (1.1)*, if (a_1) holds,*

(b_2)
$$\left((-1)^s,(-1)^{s-1},\dots,1,-1\right)^T\in\mathbb{R}^s$$
is a p*-periodic initial vector of equation* (1.1)*, if (a_2) holds,*

(b_3)
$$\left(\cos\left(\frac{2\pi}{p}(p-s)k_0\right),\dots,\cos\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T\in\mathbb{R}^s$$
and
$$\left(\sin\left(\frac{2\pi}{p}(p-s)k_0\right),\dots,\sin\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T\in\mathbb{R}^s,$$
are p*-periodic initial vectors of equation* (1.1)*, if (a_3) holds. The prime period of the solutions corresponding to these initial vectors is*
$$\frac{p}{\gcd(k_0,p)}.$$

Finally, we present a necessary and sufficient condition for the existence of p-periodic solutions of equation (1.1) with $1\le p<s$.

Theorem 3.3. *Consider equation* (1.1) *with* $s\ge 2$*. Assume* $1\le p<s$ *is an integer, and let* $u:=\left[\frac{s}{p}\right]$ *and* $v:=s-up$ $(0\le v\le p-1)$*.*

(a) *Equation* (1.1) *has a nontrivial* p*-periodic solution if and only if*

(a_1) *either* (3.1) *holds,*

(a_2) *or* p *is even and* (3.2) *is satisfied,*

(a_3) *or* $p>2$, $0\le v<p-1$ *and there is an integer* $1\le k_0<\frac{p}{2}$ *such that*
$$\sum_{j=0}^{u-1}A_{jp+p}=1-\frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)}\left(\sum_{l=1}^{v}\left(\sum_{j=0}^{u}A_{jp+l}\right)\sin\left(\frac{2\pi}{p}(l+1)k_0\right)\right.$$
$$\left.+\sum_{l=v+1}^{p-2}\left(\sum_{j=0}^{u-1}A_{jp+l}\right)\sin\left(\frac{2\pi}{p}(l+1)k_0\right)\right),$$

and

$$\sum_{j=0}^{u-1} A_{jp+p-1} = \frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)} \left(\sum_{l=1}^{v} \left(\sum_{j=0}^{u} A_{jp+l} \right) \sin\left(\frac{2\pi}{p} l k_0\right) + \sum_{l=v+1}^{p-2} \left(\sum_{j=0}^{u-1} A_{jp+l} \right) \sin\left(\frac{2\pi}{p} l k_0\right) \right),$$

(a_4) or $p > 2$*,* $v = p-1$ *and there is an integer* $1 \le k_0 < \frac{p}{2}$ *such that*

$$\sum_{j=0}^{u-1} A_{jp+p} = 1 - \frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)} \sum_{l=1}^{p-2} \left(\sum_{j=0}^{u} A_{jp+l} \right) \sin\left(\frac{2\pi}{p}(l+1) k_0\right)$$

and

$$\sum_{j=0}^{u} A_{jp+p-1} = \frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)} \sum_{l=1}^{p-2} \left(\sum_{j=0}^{u} A_{jp+l} \right) \sin\left(\frac{2\pi}{p} l k_0\right).$$

(b) If equation (1.1) *has a nontrivial* p*-periodic solution, then*

(b_1)

$$(1, \ldots, 1)^T \in \mathbb{R}^s$$

is a p*-periodic initial vector of equation* (1.1)*, if (a_1) holds,*

(b_2)

$$\left((-1)^s, (-1)^{s-1}, \ldots, 1, -1\right)^T \in \mathbb{R}^s$$

is a p*-periodic initial vector of equation* (1.1)*, if (a_2) holds,*

(b_3) Let $\psi \in \mathbb{R}^p$ *be either*

$$\left(1, \cos\left(\frac{2\pi}{p}k_0\right), \ldots, \cos\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T \in \mathbb{R}^p,$$

or

$$\left(0, \sin\left(\frac{2\pi}{p}k_0\right), \ldots, \sin\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T \in \mathbb{R}^p.$$

$\varphi = (\varphi(-s), \ldots, \varphi(-1))^T \in \mathbb{R}^s$*, where* $\varphi(i) := \psi(j)$ *if* $i \equiv j \pmod p$ $(i = -s, \ldots, -1)$*, is a* p*-periodic initial vector of equation* (1.1)*, if either (a_3) or (a_4) holds. The prime period of the solutions corresponding to these initial vectors is*

$$\frac{p}{\gcd(k_0, p)}.$$

Remark 3.4. In Theorem 3.3 (a_3) and (a_4) we can express A_p and A_{p-1} with the other coefficients as free parameters.

4 Auxiliary Results and Proofs of Main Results

Theorem 2.1 and Theorem 2.2 lead us to investigate those circulant matrices which has 1 as an eigenvalue.

Lemma 4.1. *Assume p is a positive integer.*

(a) Then 1 is an eigenvalue of the circulant matrix (2.1) if and only if

(a_1) either

$$\sum_{k=1}^{p} B_k = 1,$$

(a_2) or p is even and

$$\sum_{k=1}^{p} (-1)^k B_k = 1,$$

(a_3) or $p > 2$ and there is an integer $1 \le k_0 < \frac{p}{2}$ such that

$$B_s = \frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)} \left(-\sin\left(\frac{2\pi}{p}(s-1)k_0\right) - \sum_{k=s+1}^{p} B_k \sin\left(\frac{2\pi}{p}(k-s+1)k_0\right) + \sum_{k=1}^{s-2} B_k \sin\left(\frac{2\pi}{p}(s-k-1)k_0\right)\right)$$

and

$$B_{s-1} = \frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)} \left(\sin\left(\frac{2\pi}{p}sk_0\right) + \sum_{k=s+1}^{p} B_k \sin\left(\frac{2\pi}{p}(k-s)k_0\right) - \sum_{k=1}^{s-2} B_k \sin\left(\frac{2\pi}{p}(s-k)k_0\right)\right),$$

where $2 \le s \le p$.

(b) If 1 is an eigenvalue of the circulant matrix (2.1), then

(b_1) An eigenvector corresponding to 1 is

$$(1, \ldots, 1)^T \in \mathbb{R}^p,$$

if (a_1) holds.

(b_2) An eigenvector corresponding to 1 *is*

$$(1, -1, \ldots, 1, -1)^T \in \mathbb{R}^p,$$

if (a_2) holds.

(b_3) Two eigenvectors corresponding to 1 *are*

$$\left(1, \cos\left(\frac{2\pi}{p}k_0\right), \ldots, \cos\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T \in \mathbb{R}^p \tag{4.1}$$

and

$$\left(0, \sin\left(\frac{2\pi}{p}k_0\right), \ldots, \sin\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T \in \mathbb{R}^p, \tag{4.2}$$

if (a_3) holds.

Proof. Based on (2.3), one can easily see that the complex eigenvalues of (2.1) are

$$\lambda_k := B_p + B_{p-1}\rho_k + B_{p-2}\rho_k^2 + \ldots + B_1\rho_k^{p-1}, \quad k = 0, \ldots, p-1, \tag{4.3}$$

and the only eigenvector corresponding to λ_k is

$$\left(1, \rho_k, \ldots, \rho_k^{p-1}\right)^T,$$

where

$$\rho_k := \exp\left(\frac{2\pi}{p}ki\right), \quad k = 0, \ldots, p-1.$$

The proof is complete. □

Lemma 4.2. *Suppose* u *and* v *are positive integers such that* $u < v$ *and* $\gcd(u, v) = 1$.

(a) The function

$$l \to \cos\left(2\pi\frac{u}{v}l\right), \quad l \in \mathbb{Z}$$

has prime period v.

(b) If $v \geq 3$*, then the function*

$$l \to \sin\left(2\pi\frac{u}{v}l\right), \quad l \in \mathbb{Z}$$

has also prime period v.

Proof. It is obvious that both functions are periodic with period v.

(a) Since $\cos(0) = 1$, and

$$\cos\left(2\pi\frac{u}{v}l\right) = 1$$

if and only if $\frac{u}{v}l$ is an integer, $\gcd(u, v) = 1$ implies that v is a divisor of l.

(b) We can prove in a similar way.

The proof is complete. □

With these tools, we can prove Theorem 3.1–3.3.

Proof. Theorem 2.1 and Theorem 2.2 can be applied by adverting Lemma 4.1 and Lemma 4.2. □

5 Applications

We begin with equation (1.1) with $s = 2$

$$x(n) = A_1x(n-1) + A_2x(n-2), \quad n \geq 0. \tag{5.1}$$

In this case our main results give the following corollary.

Corollary 5.1. *Assume $p \geq 1$ is an integer. Equation* (5.1) *has a nontrivial p-periodic solution if and only if*

(a) either

$$A_1 + A_2 = 1,$$

(b) or p is even and

$$A_2 - A_1 = 1, \tag{5.2}$$

(c) or $p > 2$ and there is an integer $1 \leq k_0 < \frac{p}{2}$ such that

$$A_2 = -1 \quad \textit{and} \quad A_1 = 2\cos\left(\frac{2\pi}{p}k_0\right). \tag{5.3}$$

(d)

$$\left(\cos\left(\frac{2\pi}{p}(p-2)k_0\right), \cos\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T \in \mathbb{R}^2$$

and

$$\left(\sin\left(\frac{2\pi}{p}(p-2)k_0\right), \sin\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T \in \mathbb{R}^2,$$

are p-periodic initial vectors of equation (5.1)*, if (c) holds. The prime period of the solutions corresponding to these initial vectors is*

$$\frac{p}{\gcd(k_0, p)}.$$

Proof. It comes from Theorems 3.1-3.3. □

Next, we consider the following special case of equation (1.1)

$$x(n) = A_1 x(n-1) + A_2 x(n-2) + A_3 x(n-3), \quad n \geq 0. \tag{5.4}$$

First, we summarize the main results for the equation (5.4).

Corollary 5.2.

(a) Equation (5.4) *has a nontrivial* 1*-periodic solution if and only if*

$$A_1 + A_2 + A_3 = 1. \tag{5.5}$$

(b) Equation (5.4) *has a nontrivial* 2*-periodic solution if and only if either* (5.5) *holds or*

$$-A_1 + A_2 - A_3 = 1. \tag{5.6}$$

(c) Assume $p \geq 3$ *is an integer. Equation* (5.4) *has a nontrivial* p*-periodic solution if and only if*

(c_1) either (5.5) *holds,*

(c_2) or p *is even and* (5.6) *is satisfied,*

(c_3) or for some integer $1 \leq k_0 < \frac{p}{2}$

$$A_3 = A_1 - 2\cos\left(\frac{2\pi}{p}k_0\right)$$

and

$$A_2 = 4\cos^2\left(\frac{2\pi}{p}k_0\right) - 2A_1\cos\left(\frac{2\pi}{p}k_0\right) - 1.$$

(d)

$$\left(\cos\left(\frac{2\pi}{p}(p-3)k_0\right), \cos\left(\frac{2\pi}{p}(p-2)k_0\right), \cos\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T \in \mathbb{R}^3$$

and

$$\left(\sin\left(\frac{2\pi}{p}(p-3)k_0\right), \sin\left(\frac{2\pi}{p}(p-2)k_0\right), \sin\left(\frac{2\pi}{p}(p-1)k_0\right)\right)^T \in \mathbb{R}^3,$$

are p*-periodic initial vectors of equation* (1.1)*, if (c_3) holds. The prime period of the solutions corresponding to these initial vectors is*

$$\frac{p}{\gcd(k_0, p)}.$$

Proof. They are immediate consequences of Theorem 3.1-3.3. □

Remark 5.3. Equation (5.4) has a nontrivial 3-periodic solution if and only if either (5.5) holds or

$$A_2 = A_1 \quad \text{and} \quad A_3 = 1 + A_1.$$

By using the previous result we construct two difference equations of type (5.4) which have periodic solutions with two different prime periods.

Example 5.4. If we would like to get a difference equation of type (5.4) which has periodic solutions with prime period 2 and prime period 3, the following equations must be satisfied by Corollary 5.2 (b) and (c)

$$A_1 - A_2 + A_3 = -1 \quad \text{and} \quad A_2 = A_1 \quad \text{and} \quad A_3 = 1 + A_1.$$

This implies that $A_1 = A_2 = -2$ and $A_3 = -1$. Consider the equation

$$x(n) = -2x(n-1) - 2x(n-2) - x(n-3), \quad n \geq 0. \tag{5.7}$$

Equation (5.7) has 2-periodic solutions with prime period 2. The 2-periodic initial vectors belonging to these solutions are $\varphi = (\alpha, -\alpha, \alpha)$, where $\alpha \neq 0$. Equation (5.7) has also 3-periodic solutions with prime period 3. The 3-periodic initial vectors belonging to these solutions are $\varphi = (\alpha, \beta, -\alpha - \beta)$, where $\alpha\beta \neq 0$.

Example 5.5. In this example we seek a difference equation of type (5.4) which has periodic solutions with prime period 2 and prime period 5. By Corollary 5.2 (b) and (d), a possible alternative

$$-A_1 + A_2 - A_3 = 1$$

and

$$A_3 = A_1 - 2\cos\left(\frac{2\pi}{5}\right) \quad \text{and} \quad A_2 = 4\cos^2\left(\frac{2\pi}{5}\right) - 2A_1\cos\left(\frac{2\pi}{5}\right) - 1.$$

By solving this system of linear equations, we have

$$A_1 = A_2 = \frac{\sqrt{5} - 3}{2}, \quad A_3 = -1,$$

and therefore consider the equation

$$x(n) = \frac{\sqrt{5} - 3}{2} x(n-1) + \frac{\sqrt{5} - 3}{2} x(n-2) - x(n-3), \quad n \geq 0. \tag{5.8}$$

Equation (5.8) has 2-periodic solutions with prime period 2. The 2-periodic initial vectors belonging to these solutions are $\varphi = (\alpha, -\alpha, \alpha)$, where $\alpha \neq 0$. Equation (5.8) has also 5-periodic solutions with prime period 5. The 5-periodic initial vectors belonging to these solutions are

$$\varphi = \left(\frac{\sqrt{5}-1}{2}\beta - \alpha, -\frac{1}{2}\left(\sqrt{5}-1\right)(\beta+\alpha), \frac{\sqrt{5}-1}{2}\alpha - \beta \right),$$

where $\alpha\beta \neq 0$.

Next, we consider a special difference equation of type (1.1), where s is arbitrary.

Example 5.6. Let $s \geq 2$ be an integer. Then the equation

$$x(n) = \sum_{k=1}^{s-1} \binom{s}{s-k} x(n-k) + 2x(n-s), \quad n \geq 0$$

has an s-periodic solution if and only if $s = 6m$ with some $m \geq 1$.

By (4.3), the eigenvalues of the circulant matrix

$$\mathcal{B}_{s,s} = \begin{pmatrix} 2 & \binom{s}{1} & \cdots & \binom{s}{s-2} & \binom{s}{s-1} \\ \binom{s}{s-1} & 2 & \cdots & \binom{s}{s-3} & \binom{s}{s-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{s}{1} & \binom{s}{2} & \cdots & \binom{s}{s-1} & 2 \end{pmatrix}$$

are

$$\lambda_k := \binom{s}{0} 2 + \binom{s}{1}\rho_k + \binom{s}{2}\rho_k^2 + \ldots + \binom{s}{s-1}\rho_k^{s-1}$$
$$= (1+\rho_k)^s, \quad k = 0, \ldots, s-1.$$

One of them is 1 if and only if $s = 6m$ for some $m \geq 1$ (see [3]), and thus the result follows from Theorem 2.1.

When $s = 6$ our equation becomes

$$x(n) = 6x(n-1) + 15x(n-2) + 20x(n-3)$$

$$+15x(n-4) + 6x(n-5) + 2x(n-6), \quad n \geq 0.$$

The next result gives the full characterization of the existence of periodic solutions of equation (1.2), when the period is greater or equal than s. We shall use this assertion to prove Theorem 1.1.

Theorem 5.7. *Consider equation* (1.2)*, and let $p \geq s \geq 2$ be integers.*

(a) *Equation* (1.2) *has a nonconstant* p*-periodic solution if and only if* p *is even and either*

(a$_1$)

$$A_s = (-1)^s\, 2, \tag{5.9}$$

or

(a$_2$) *or* $p > s$ *and there is an integer* $1 \le k_0 < \frac{p}{2}$ *such that*

$$l_0 := \frac{k_0\,(2s-1)}{p} - \frac{1}{2}$$

is a nonnegative integer and

$$A_s = 2\,(-1)^{l_0+1} \sin\left(\frac{\pi}{p} k_0\right). \tag{5.10}$$

(b) *The prime period, say* q*, of a nontrivial* p*-periodic solution obeys one of the following relations:*

(b$_1$)

$$q = 2, \quad \text{if } (a_1) \text{ holds;}$$

(b$_2$) q *is an even divisor of* p *greater or equal than* 4*, if* (a$_2$) *holds;*

(b$_3$) $q = p$*, if* $p = 2r$ *with a prime* r*, and* (a$_2$) *is satisfied.*

Proof. If $A_s = 0$, then equation (1.2) has only nontrivial 1-periodic (constant) solutions, hence $A_s \neq 0$ can be supposed.

Then Theorem 3.1 shows that equation (1.2) has a nontrivial s-periodic solution if and only if one of the following conditions holds:

(A$_1$) s is even and

$$-1 + A_s = 1.$$

(A$_2$) $s > 2$ and there is an integer $1 \le k_0 < \frac{s}{2}$ such that

$$A_s = 1 - \frac{1}{\sin\left(\frac{2\pi}{s} k_0\right)} \sin\left(\frac{2\pi}{s} 2k_0\right)$$

and

$$A_{s-1} = \frac{1}{\sin\left(\frac{2\pi}{s} k_0\right)} \sin\left(\frac{2\pi}{s} k_0\right) = 0, \tag{5.11}$$

but (5.11) can not be satisfied.

If $p > s$, then Theorem 3.2 gives that equation (1.2) has a nontrivial p-periodic solution if and only if one of the following conditions holds:

(A_3) p is even and

$$-1 + (-1)^s A_s = 1.$$

(A_4) $s = 2$ and there is an integer $1 \le k_0 < \frac{p}{2}$ such that

$$A_2 = -\frac{\sin\left(\frac{2\pi}{p}k_0\right)}{\sin\left(\frac{2\pi}{p}k_0\right)} = -1$$

and

$$A_1 = \frac{\sin\left(\frac{2\pi}{p}2k_0\right)}{\sin\left(\frac{2\pi}{p}k_0\right)} = 2\cos\left(\frac{2\pi}{p}k_0\right) = 1. \tag{5.12}$$

(5.12) implies that

$$6k_0 = p.$$

(A_5) $s > 2$ and for some integer $1 \le k_0 < \frac{p}{2}$

$$A_s = -\frac{\sin\left(\frac{2\pi}{p}(s-1)k_0\right)}{\sin\left(\frac{2\pi}{p}k_0\right)} + \frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)}\sin\left(\frac{2\pi}{p}(s-2)k_0\right) \tag{5.13}$$

and

$$A_{s-1} = \frac{\sin\left(\frac{2\pi}{p}sk_0\right)}{\sin\left(\frac{2\pi}{p}k_0\right)} - \frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)}\sin\left(\frac{2\pi}{p}(s-1)k_0\right) = 0. \tag{5.14}$$

We have from (5.14) that

$$\sin\left(\frac{2\pi}{p}sk_0\right) - \sin\left(\frac{2\pi}{p}(s-1)k_0\right) = 2\cos\left(\frac{2\pi}{p}k_0\left(s-\frac{1}{2}\right)\right)\sin\left(\frac{\pi}{p}k_0\right) = 0$$

which is satisfied exactly if

$$\text{either } \frac{2}{p}k_0\left(s-\frac{1}{2}\right) - \frac{1}{2} \text{ or } \frac{1}{p}k_0$$

is an integer. Bringing in $1 \le k_0 < \frac{p}{2}$, we find that (5.14) holds if and only if

$$l_0 := \frac{k_0(2s-1)}{p} - \frac{1}{2} \tag{5.15}$$

is a nonnegative integer $(0 \le l_0 < s-1)$.

In light of (5.13), (5.14) and (5.15)

$$A_s = -\frac{1}{\sin\left(\frac{2\pi}{p}k_0\right)}\left(\sin\left(\frac{2\pi}{p}sk_0\right)\cos\left(\frac{2\pi}{p}k_0\right) - \sin\left(\frac{2\pi}{p}k_0\right)\cos\left(\frac{2\pi}{p}sk_0\right)\right.$$

$$\left. - \sin\left(\frac{2\pi}{p}(s-1)k_0\right)\cos\left(\frac{2\pi}{p}k_0\right) + \sin\left(\frac{2\pi}{p}k_0\right)\cos\left(\frac{2\pi}{p}(s-1)k_0\right)\right)$$

$$= \cos\left(\frac{2\pi}{p}sk_0\right) - \cos\left(\frac{2\pi}{p}(s-1)k_0\right) = -2\sin\left(\frac{\pi}{p}k_0(2s-1)\right)\sin\left(\frac{\pi}{p}k_0\right)$$

$$= -2\sin\left(\frac{\pi}{2}(1+2l_0)\right)\sin\left(\frac{\pi}{p}k_0\right) = 2(-1)^{l_0+1}\sin\left(\frac{\pi}{p}k_0\right).$$

Summarized the above assumptions we have that equation (1.2) has a nontrivial p-periodic solution if and only if

(B$_1$) either p is even and

$$-1 + (-1)^s A_s = 1,$$

(B$_2$) or $p > s = 2$ and

$$A_2 = -1,$$

and there is an integer k_0 such that

$$6k_0 = p.$$

(B$_3$) or $p > s > 2$ and there are integers $1 \le k_0 < \frac{p}{2}$ and $0 \le l_0 < s-1$ such that

$$2k_0(2s-1) = p(1+2l_0) \text{ and } A_s = 2(-1)^{l_0+1}\sin\left(\frac{\pi}{p}k_0\right).$$

It is easy to check that (B$_2$) and (B$_3$) can be contracted: if $p > s \ge 2$ in (B$_3$), then (B$_3$) contains (B$_2$). We can see that p is even if (B$_3$) is satisfied.

Since equation (1.2) has a nontrivial 1-periodic solution if and only if $A_s = 0$, and in (B$_1$) and (B$_3$) $A_s \ne 0$, so under the conditions (B$_1$) and (B$_3$) the nontrivial p-periodic solutions are nonconstant p-periodic solutions, and thus (a$_1$) and (a$_2$) hold.

(b) To prove (b$_1$) we shall use Lemma 4.1. We consider the case s is even and $A_s = 2$, the other case can be handled similarly. Then by Lemma 4.1, the eigenvalues of the matrix $\mathcal{B}_{p,s}$ are

$$2 + \rho_k^{s-1}, \quad k = 0, \ldots, p-1,$$

if $p = s$, and

$$2\rho_k^{p-s} + \rho_k^{p-1}, \quad k = 0, \ldots, p-1,$$

if $p > s$. It is not hard to verify that in both cases the algebraic multiplicity of the eigenvalue 1 is 1 and the eigenspace associated with 1 is

$$\alpha\,(1,-1,\ldots,1,-1)\in\mathbb{R}^p,\quad \alpha\in\mathbb{R},$$

and therefore the nonconstant p-periodic solutions have prime period 2.

To prove (b_2) it is enough to show that the nonconstant p-periodic solutions are not 2-periodic ((a) gives that it does not exist any nonconstant periodic solution with odd period). It follows from (B_1) $(s = 2)$ and from Theorem 3.3 (b) $(s > 2)$ that equation (1.2) has a nontrivial 2-periodic solution if and only if either s is even and $A_s = 2$, or s is odd and $A_s = -2$. According to (B_3), $A_s \neq \pm 2$.

(b_3) Since the nontrivial p-periodic solutions are nonconstant p-periodic solutions, (a_2) yields the result.

The proof is complete. □

Now we can prove Theorem 1.1.

Proof. Let $s \geq 2$ be fixed.

If equation (1.2) has a nonconstant periodic solution, then it also has a nonconstant p-periodic solution with $p \geq s$, and therefore all possible values of A_s are given in Theorem 5.7 (a_1) and (a_2). We have to pick and choose the different values of A_s.

Since $1 \leq k_0 < \dfrac{p}{2}$ in (5.10), the coefficient (5.9) can not be found between the numbers in (5.10). Thus we get $A_s = (-1)^s\, 2$.

Now we study the coefficients in (5.10). Suppose $p_0 := 2t_0 > s$ and $p_1 := 2t_1 > s$ are even integers for which there are integers $1 \leq k_0 < t_0$ and $1 \leq k_1 < t_1$ such that

$$l_0 := \frac{k_0\,(2s-1)}{2t_0} - \frac{1}{2} \quad \text{and} \quad l_1 := \frac{k_1\,(2s-1)}{2t_1} - \frac{1}{2}$$

are nonnegative integers, and let

$$A_{s,0} := 2\,(-1)^{l_0+1}\sin\left(\frac{\pi}{2t_0}k_0\right) \quad \text{and} \quad A_{s,1} := 2\,(-1)^{l_1+1}\sin\left(\frac{\pi}{2t_1}k_1\right).$$

If $A_{s,0} = A_{s,1}$, then obviously

$$\sin\left(\frac{\pi}{2t_0}k_0\right) = \sin\left(\frac{\pi}{2t_1}k_1\right),$$

and this yields that

$$\sin\left(\frac{\pi}{2t_0}k_0\right) - \sin\left(\frac{\pi}{2t_1}k_1\right)$$
$$= \cos\left(\frac{\pi}{4t_0}k_0 + \frac{\pi}{4t_1}k_1\right)\sin\left(\frac{\pi}{4t_0}k_0 - \frac{\pi}{4t_1}k_1\right) = 0.$$

Then because of

$$0 < \frac{\pi}{4t_0}k_0 + \frac{\pi}{4t_1}k_1 < \frac{\pi}{2} \quad \text{and} \quad \frac{-\pi}{2} < \frac{\pi}{4t_0}k_0 - \frac{\pi}{4t_1}k_1 < \frac{\pi}{2},$$

we have

$$\frac{\pi}{4t_0}k_0 - \frac{\pi}{4t_1}k_1 = 0.$$

Consequently,

$$\frac{k_0}{t_0} = \frac{k_1}{t_1}. \tag{5.16}$$

Conversely, if $p_0 := 2t_0 > s$ and $p_1 := 2t_1 > s$ are even integers for which there are integers $1 \leq k_0 < t_0$ and $1 \leq k_1 < t_1$ such that

$$l_0 := \frac{k_0\,(2s-1)}{2t_0} - \frac{1}{2}$$

is a nonnegative integer, and (5.16) is satisfied, then $l_0 = l_1$, and hence $A_{s,0} = A_{s,1}$.

It follows from the above assumptions and from

$$\frac{k_0}{t_0} = \frac{2l_0+1}{2s-1},$$

that the different coefficients in (5.10) can be derived in the next way:

$$A_s = 2\,(-1)^{l_0+1} \sin\left(\frac{\pi}{2}\frac{2l_0+1}{2s-1}\right), \quad l_0 = 0, \dots, l-2,$$

which give the set $U \setminus \{(-1)^s\, 2\}$.

We have proved (a).

(b) follows from Theorem 5.7 (b_1) and the fact that equation (1.2) has a nonconstant $2t_0$-periodic solution if $A_s \in U \setminus \{(-1)^s\, 2\}$.

Theorem 5.7 (b) implies (c).

The proof is complete. □

Proof of Corollary 1.2. By Theorem 1.1, we are ready if it is shown that

$$-2\cos\left(\frac{s-1}{2s-1}\pi\right) \in U.$$

Since

$$-2\cos\left(\frac{s-1}{2s-1}\pi\right) = -2\sin\left(\frac{s-1}{2s-1}\pi + \frac{\pi}{2}\right) = -2\sin\left(\frac{\pi}{4s-2}\,(4s-3)\right)$$
$$= -2\sin\left(\frac{\pi}{4s-2}\right),$$

and hence $k = 1$ can be chosen in the definition of the set U. The proof is complete. □

We close this section with a result which evidently follows from Theorem 2.3 and Theorem 2.4 given for circulant matrices. These results are less general than Theorem 3.1–3.3, but in some cases they give fast algorithm to make decision on the existence of periodic solutions.

Theorem 5.8. *Consider equation* (1.1) *with* $s \geq 2$*.*

(a) Suppose $p > s$ *is an integer. If either*

$$\sum_{k=1}^{s} |A_k| < 1$$

or for some $1 \leq k_0 \leq s$

$$1 + \sum_{\substack{k=1 \\ k \neq k_0}}^{s} |A_k| < |A_{k_0}|,$$

then equation (1.1) *has no* p*-periodic solution.*

(b) If $d \geq 2$ *is a divisor of* s*, and the vector* $(A_s - 1, A_{s-1}, \ldots, A_1)$ *consists of* $\frac{s}{d}$ *consecutive constant blocks of length* d *(that is,* $\tilde{A}_{kd+j} = \tilde{A}_{kd}$ *for* $k = 0, \ldots, \frac{n}{d} - 1$ *and* $j = 0, \ldots, d-1$*, where* $\tilde{A}_0 := A_s - 1$ *and* $\tilde{A}_k := A_{s-k}$ $(1 \leq k \leq s-1)$*), then equation* (1.1) *has a nontrivial* s*-periodic solution.*

(c) Let p *be a prime, and assume that* $A_1, \ldots, A_s$ *are rational numbers. Then*

(c_1) in case $p = s$ *equation* (1.1) *has a nontrivial* s*-periodic solution if and only if either*

$$\sum_{k=1}^{s} A_k = 1 \tag{5.17}$$

or

$$A_1 = \ldots = A_{s-1} = A_s - 1.$$

(c_2) in case $p > s$ *equation* (1.1) *has a nontrivial* p*-periodic solution if and only if either* (5.17) *holds or* $p = s + 1$ *and*

$$A_1 = \ldots = A_{s-1} = A_s = -1.$$

(c_3) in case $2 \leq p < s$ *equation* (1.1) *has a nontrivial* p*-periodic solution if and only if either* (5.17) *holds or*

$$B_1 = \ldots = B_{p-1} = B_p - 1,$$

where the numbers $B_1, \ldots, B_p$ *are defined in Theorem 2.1 (a).*

Proof. The results are immediate consequences of Theorem 2.1 and

(a) Theorem 2.4 (a).

(b) Theorem 2.4 (b).

(c) Theorem 2.3.

The proof is complete. □

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