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Almost Oscillation Criteria for Second-Order Neutral Difference Equations with Quasidifferences

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Abstract

Using the Riccati transformation techniques, we will extend some almost oscillation criteria for the second–order nonlinear neutral difference equation with quasidifferences

$$\Delta\left(r_n\left(\Delta\left(x_n + cx_{n-k}\right)\right)^{\gamma}\right) + q_n x_{n+1}^{\alpha} = e_n.$$

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1 Introduction

Recently there has been an increasing interest in the study of the qualitative behavior of solutions of neutral difference equations (see the monographs [1–3,9]). Particularly, the

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oscillation and nonoscillation of solutions of the second–order neutral difference equations attract attention; see the papers [4–8, 10–12, 14, 16–20] and the references therein. The interesting oscillatory results for first order and even order neutral difference equations can be found in [13] and [15].

In the presented paper, the results obtained in [21] by Thandapani, Vijaya and Győri for

$$\Delta^2 (x_n + cx_{n-k})^\gamma + q_n x_{n+1}^\alpha = e_n,$$

are generalized to the second–order nonlinear neutral difference equation with quasidifference of the form

$$\Delta \left(r_n \left(\Delta \left(x_n + cx_{n-k}\right)\right)^\gamma\right) + q_n x_{n+1}^\alpha = e_n. \tag{1.1}$$

Here k is a nonnegative integer, Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, c is a real nonnegative constant, $\alpha > \gamma \geq 1$ are ratios of odd positive integers, (r_n) and (q_n) and (e_n) are positive sequences defined on $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Finally, as a corollary of our main result, almost oscillation property of solutions of a special case of equation (1.1) in the form

$$\Delta \left(r_n \Delta x_n\right) + q_n x_{n+1}^\alpha = e_n \tag{1.2}$$

is studied. For $\alpha = 1$, equation (1.2) is known as the forced second–order Sturm–Liouville difference equation. Some oscillation results for equation (1.2) were investigated among others by Došlý, Graef and Jaroš in [4].

By a solution of equation (1.1) we mean a real valued sequence (x_n) defined on $\mathbb{N}_k := \{k, k+1, \ldots\}$, which satisfies (1.1) for every $n \in \mathbb{N}_k$.

Sequence (x_n) is said to be oscillatory, if for every integer $n_k \in \mathbb{N}_k$, there exists $n \geq n_k$ such that $x_n x_{n+1} \leq 0$; otherwise, it is called nonoscillatory.

Definition 1.1. Solution (x_n) of equation (1.1) is said to be almost oscillatory if either (x_n) is oscillatory, or (Δx_n) is oscillatory, or $x_n \to 0$ as $n \to \infty$.

We begin with some lemmas which will be used for proving the main result.

Lemma 1.2. *Set*

$$F(x) = a\, x^{\alpha-\gamma} + \frac{b}{x^\gamma} \text{ for } x > 0. \tag{1.3}$$

If $a > 0$, $b > 0$ and

$$\alpha > \gamma \geq 1, \tag{1.4}$$

then $F(x)$ attains its minimum

$$F_{\min} = \frac{\alpha a^{\frac{\gamma}{\alpha}} b^{1-\frac{\gamma}{\alpha}}}{\gamma^{\frac{\gamma}{\alpha}} (\alpha-\gamma)^{1-\frac{\gamma}{\alpha}}}.$$

Lemma 1.3. *For all $x \geq y \geq 0$ and $\gamma \geq 1$ we have the following inequality*

$$x^\gamma - y^\gamma \geq (x-y)^\gamma.$$

2 Almost Oscillation Criterion

In this section, by using the Riccati substitution we will establish new almost oscillation criterion for equation (1.1).

Theorem 2.1. *Let*

$$r, q, e\colon \mathbb{N} \to \mathbb{R}_+, \tag{2.1}$$

and there exists positive real constant R such that

$$r_n \le R \text{ for } n \in \mathbb{N}. \tag{2.2}$$

Assume also that

$$\alpha > \gamma \ge 1 \text{ are ratios of odd positive integers.} \tag{2.3}$$

If there exists positive sequence (p_n) such that

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \left(p_i Q_i - \frac{R\,(\Delta p_i)^2}{4p_i} \right) = \infty, \tag{2.4}$$

where

$$Q_n = \min\{Q_n^*, Q_n^{**}\}, \tag{2.5}$$

$$Q_n^* = \frac{d^{\alpha-\gamma} q_n}{(1+c)^\alpha} - d^{-\gamma} e_n,$$

$$Q_n^{**} = \frac{\alpha q_n^{\frac{\gamma}{\alpha}} e_n^{1-\frac{\gamma}{\alpha}}}{\gamma^{\frac{\gamma}{\alpha}} (\alpha-\gamma)^{1-\frac{\gamma}{\alpha}} (1+c)^\gamma}$$

and

$$\sum_{i=1}^{\infty} \sum_{j=1}^{i-1} (M q_j \pm e_j)^{\frac{1}{\gamma}} = \infty, \tag{2.6}$$

(here $d > 0$ and $M > 0$ are suitable constants), then every solution of equation (1.1) *is almost oscillatory.*

Proof. Set

$$z_n = x_n + c x_{n-k} \tag{2.7}$$

hence equation (1.1) takes the following form

$$\Delta\left(r_n (\Delta z_n)^\gamma\right) = -q_n x_{n+1}^\alpha + e_n.$$

Suppose, for the contrary, that sequence (x_n) is a solution eventually of one sign of equation (1.1) such that (Δx_n) is eventually of one sign as well.

Assume first that (x_n) is an eventually positive sequence. It means that there exists $n_0 \in \mathbb{N}$ such that $x_{n-k} > 0$ for all $n \geq n_0$. We have two possibilities to consider:

(Ia) $\Delta x_n > 0$ eventually or

(Ib) $\Delta x_n < 0$ eventually.

Case (Ia): Assume that $\Delta x_n > 0$. Thus $\Delta z_n > 0$. We have $x_n \geq \dfrac{z_n}{1+c}$. From equation (1.1), we get

$$\Delta\left(r_n\left(\Delta z_n\right)^\gamma\right) \leq \frac{-q_n}{(1+c)^\alpha} z_{n+1}^\alpha + e_n. \tag{2.8}$$

Let us denote by (w_n) the following sequence

$$w_n :\, = p_n \frac{r_n\left(\Delta z_n\right)^\gamma}{z_{n+1}^\gamma}, \tag{2.9}$$

where z_n is defined by (2.7). Thus $w_n > 0$ for $n \geq n_0$. We have

$$\begin{aligned}
\Delta w_n &= p_{n+1} \frac{r_{n+1}\left(\Delta z_{n+1}\right)^\gamma}{z_{n+2}^\gamma} - p_n \frac{r_n\left(\Delta z_n\right)^\gamma}{z_{n+1}^\gamma} = \\
&= p_n \frac{\Delta\left(r_n\left(\Delta z_n\right)^\gamma\right)}{z_{n+1}^\gamma} + p_{n+1} \frac{r_{n+1}\left(\Delta z_{n+1}\right)^\gamma}{z_{n+2}^\gamma} - p_n \frac{r_{n+1}\left(\Delta z_{n+1}\right)^\gamma}{z_{n+2}^\gamma} + \\
&+ p_n \frac{r_{n+1}\left(\Delta z_{n+1}\right)^\gamma}{z_{n+2}^\gamma} - p_n \frac{r_{n+1}\left(\Delta z_{n+1}\right)^\gamma}{z_{n+1}^\gamma} \\
&= p_n \frac{\Delta\left(r_n\left(\Delta z_n\right)^\gamma\right)}{z_{n+1}^\gamma} + \frac{\Delta p_n}{p_{n+1}} w_{n+1} + \frac{p_n r_{n+1}\left(\Delta z_{n+1}\right)^\gamma}{z_{n+2}^\gamma z_{n+1}^\gamma}\left[z_{n+1}^\gamma - z_{n+2}^\gamma\right]
\end{aligned}$$

and finally

$$\Delta w_n = p_n \frac{\Delta r_n\left(\Delta z_n\right)^\gamma}{z_{n+1}^\gamma} + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}} w_{n+1} \frac{\Delta z_{n+1}^\gamma}{z_{n+1}^\gamma}. \tag{2.10}$$

From the above and (2.8), we get

$$\Delta w_n \leq -p_n\left(\frac{q_n}{(1+c)^\alpha} z_{n+1}^{\alpha-\gamma} - \frac{e_n}{z_{n+1}^\gamma}\right) + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}} w_{n+1} \frac{\Delta z_{n+1}^\gamma}{z_{n+1}^\gamma}. \tag{2.11}$$

Let $G(x) = \dfrac{q_n}{(1+c)^\alpha} x^{\alpha-\gamma} - \dfrac{e_n}{x^\gamma}$. It is easy to verify that function G is increasing for positive arguments. Since x is increasing, there is a constant $d > 0$ such that $x \geq d > 0$ and

$$G(x) \geq \frac{q_n}{(1+c)^\alpha} d^{\alpha-\gamma} - e_n d^{-\gamma} :\, = Q_n^*. \tag{2.12}$$

From (2.11) and (2.12), we get the following inequality

$$\Delta w_n \le -p_n Q_n^* + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}} w_{n+1} \frac{(\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma}.$$

For $\Delta z_n > 0$ we have $z_{n+2} > z_{n+1}$ and $z_{n+2}^\gamma > z_{n+1}^\gamma$. Because of positivity of the sequence (z_n) for large n, say $n \ge n_1 \ge n_0$, we obtain $\frac{1}{z_{n+2}^\gamma} < \frac{1}{z_{n+1}^\gamma}$ for $n \ge n_1$. From (2.9) and by Lemma 1.3, we get

$$\Delta w_n \le -p_n Q_n^* + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}^2 r_{n+1}} w_{n+1}^2.$$

This and (2.2) imply that

$$\begin{aligned} \Delta w_n &\le -p_n Q_n^* + \frac{(\Delta p_n)^2 r_{n+1}}{4 p_n} - \left[\sqrt{\frac{p_n}{r_{n+1}}} \frac{1}{p_{n+1}} w_{n+1} - \frac{\sqrt{r_{n+1}} \Delta p_n}{2\sqrt{p_n}} \right]^2 \\ &\le -p_n Q_n^* + \frac{(\Delta p_n)^2 r_{n+1}}{4 p_n} \le - \left[p_n Q_n^* - \frac{(\Delta p_n)^2 R}{4 p_n} \right]. \end{aligned}$$

Summing both sides of the above inequality from $i = n_1$ to $n-1$, we obtain

$$w_n - w_{n_1} < - \sum_{i=n_1}^{n-1} \left(p_i Q_i^* - \frac{R(\Delta p_i)^2}{4 p_i} \right).$$

Because of $w_n \ge 0$ for $n \in \mathbb{N}$, we have $w_{n_1} > w_{n_1} - w_n$. Hence

$$w_{n_1} > \sum_{i=n_1}^{n-1} \left(p_i Q_i^* - \frac{R(\Delta p_i)^2}{4 p_i} \right).$$

Letting n into infinity we obtain

$$w_{n_1} \ge \limsup_{n\to\infty} \sum_{i=n_1}^{n-1} \left(p_i Q_i^* - \frac{R(\Delta p_i)^2}{4 p_i} \right).$$

From (2.5), we get

$$w_{n_1} > \limsup_{n\to\infty} \sum_{i=n_1}^{n-1} \left(p_i Q_i - \frac{R(\Delta p_i)^2}{4 p_i} \right).$$

This is a contradiction with (2.4).

Case (Ib): If $\Delta x_n < 0$, then $\Delta z_n < 0$. From $x_n > 0$ and $\Delta x_n < 0$, we get $\lim_{n\to\infty} x_n = l > 0$. Thus $x_{n+1}^{\alpha} \to l^{\alpha} > 0$ as $n \to \infty$. Hence, there exists $n_2 \in \mathbb{N}$ such that $x_{n+1}^{\alpha} \geq l^{\alpha}$ for $n \geq n_2$. Therefore, we have

$$\Delta\left(r_n\left(\Delta z_n\right)^{\gamma}\right) \leq -q_n l^{\alpha} + e_n.$$

Set $l^{\alpha} = M$. Summing the last inequality from n_2 to $n-1$, we obtain

$$r_n\left(\Delta z_n\right)^{\gamma} < r_n\left(\Delta z_n\right)^{\gamma} - r_{n_2}\left(\Delta z_{n_2}\right)^{\gamma} \leq -\left(\sum_{i=n_2}^{n-1} Mq_i - e_i\right)$$

and

$$\Delta z_n \leq -\left(\sum_{i=n_2}^{n-1} Mq_i - e_i\right)^{\frac{1}{\gamma}} r_n^{-\frac{1}{\gamma}}, \text{ for } n \geq n_2.$$

Summing again the above inequality from n_2 to n, we obtain

$$z_{n+1} \leq z_{n_2} - \sum_{i=n_2}^{n}\left(\sum_{j=n_2}^{i-1} Mq_j - e_j\right)^{\frac{1}{\gamma}} r_i^{-\frac{1}{\gamma}}.$$

From (2.2), we get

$$z_{n+1} \leq z_{n_2} - R^{-\frac{1}{\gamma}} \sum_{i=n_2}^{n}\left(\sum_{j=n_2}^{i-1} Mq_j - e_j\right)^{\frac{1}{\gamma}}.$$

Letting n into ∞, from condition (2.6) we obtain that the right side of the above inequality is negative. So, z_n is eventually negative, too. This contradiction ended the proof in this case.

Finally, we assume that (x_n) is an eventually negative sequence. It means that there exists $n_3 \in \mathbb{N}$ such that $x_n < 0$ for all $n \geq n_3$. We use the transformation $y_n = -x_n$ in the equation (1.1). Equation (1.1) takes the following form

$$\Delta\left(r_n\left(\Delta\left(y_n + cy_{n-k}\right)\right)^{\gamma}\right) + q_n y_{n+1}^{\alpha} = -e_n. \tag{2.13}$$

Here sequence (y_n) is an eventually positive solution of equation (2.13).

We have two possibilities to consider:

(IIa) $\Delta y_n > 0$ eventually or

(IIb) $\Delta y_n < 0$ eventually.

Case (IIa): Assume that $\Delta y_n > 0$. From (2.10), by (2.13), we have

$$\Delta w_n \leq -p_n\left(\frac{q_n}{(1+c)^{\alpha}} z_{n+1}^{\alpha-\gamma} + \frac{e_n}{z_{n+1}^{\gamma}}\right) + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}} w_{n+1} \frac{\Delta z_{n+1}^{\gamma}}{z_{n+1}^{\gamma}}.$$

Putting $a = \dfrac{q_n}{(1+c)^\alpha}$, $b = e_n$ and $x = z_{n+1}$ in (1.3), we have

$$F(z_{n+1}) = \frac{q_n}{(1+c_{n+1})^\alpha} z_{n+1}^{\alpha-\gamma} + \frac{e_n}{z_{n+1}^\gamma}.$$

By Lemma 1.2, we get

$$F(z_n) \geq \frac{\alpha q_n^{\frac{\gamma}{\alpha}} e_n^{1-\frac{\gamma}{\alpha}}}{\gamma^{\frac{\gamma}{\alpha}} (\alpha-\gamma)^{1-\frac{\gamma}{\alpha}} (1+c_{n+1})^\gamma} = Q_n^{**}.$$

and

$$\Delta w_n \leq -p_n Q_n^{**} + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}^2 r_{n+1}} w_{n+1}^2$$

is satisfied. The rest of the proof is similar to proof of case (Ia) and hence is omitted.

Case (IIb): Assume that $\Delta y_n < 0$. Hence sequence y_n has positive limit and the proof of this case is similar to case (Ib) and hence is omitted. The proof is now complete. □

We illustrate Theorem 2.1 by the following examples.

Example 2.2. Let us consider the difference equation

$$\Delta\left((2-\frac{(-1)^n}{n})\left(\Delta\left(x_n + \frac{1}{2}x_{n-1}\right)\right)^3\right) + 4x_{n+1}^5 = \frac{1}{n(n+1)}.$$

Here $r_n = 2 - \dfrac{(-1)^n}{n}$, $c = \dfrac{1}{2}$, $\gamma = 3$, $q_n = 4$, $\alpha = 5$ and $e_n = \dfrac{1}{n(n+1)}$. For $p_n = 1$, all assumptions of Theorem 2.1 are satisfied. Hence, any solution of the above equation is almost oscillatory. Sequence $x_n = (-1)^{n+1}$ is one of such solutions. Here, (x_n) is oscillatory.

Example 2.3. Let us consider the difference equation

$$\Delta\left((2+(-1)^n)\left(\Delta\left(x_n + 2x_{n-2}\right)\right)\right) + x_{n+1}^3 = 14 + 11(-1)^{n+1}.$$

Here $r_n = 2 + (-1)^n$, $c = 2$, $\gamma = 1$, $q_n = 1$, $\alpha = 3$ and $e_n = 14 + 11(-1)^{n+1}$. For $p_n = 1$, all assumptions of Theorem 2.1 are satisfied. Hence, any solution of the above equation is almost oscillatory. Sequence $x_n = 2 + (-1)^{n+1}$ is one of such solutions. Here, (x_n) is nonoscillatory but (Δx_n) oscillates.

Example 2.4. Let us consider the difference equation

$$\Delta\left(\frac{1}{3n+4}\left(\Delta\left(x_n + 2x_{n-1}\right)\right)\right) + n\,(n+2)^2\, x_{n+1}^3 = \frac{3 + n^2\,(n+1)\,(n+3)}{n\,(n+1)\,(n+2)\,(n+3)}.$$

Here $r_n = \dfrac{1}{3n+4}$, $c = 2$, $k = 1$, $\gamma = 1$, $q_n = n\,(n+2)^2$, $\alpha = 3$ and

$$e_n = \frac{3 + n^2\,(n+1)\,(n+3)}{n\,(n+1)\,(n+2)\,(n+3)}.$$

For $p_n = 1$, all assumptions of Theorem 2.1 are satisfied. Hence, any solution of the above equation is almost oscillatory. In fact, sequence $x_n = \dfrac{1}{n+1}$ is one of such solutions. Here, (x_n) tends to zero.

Assuming that $c = 0$ and $\gamma = 1$ equation (1.1) takes the form (1.2).

Corollary 2.5. *Assume that conditions* (2.1)*,* (2.2) *and* (2.3) *are satisfied. If there exists positive sequence* (p_n) *such that*

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \left(p_i Q_i - \frac{R\,(\Delta p_i)^2}{4p_i} \right) = \infty,$$

where

$$Q_n = \min\{Q_n^*, Q_n^{**}\},$$

$$Q_n^* = \frac{d^\alpha q_n - e_n}{d},$$

$$Q_n^{**} = \frac{\alpha q_n^{\frac{1}{\alpha}} e_n^{1-\frac{1}{\alpha}}}{(\alpha-1)^{1-\frac{1}{\alpha}}}$$

and

$$\sum_{i=1}^{\infty} \sum_{j=1}^{i-1} (M q_j \pm e_j) = \infty,$$

(here $d > 0$ *and* $M > 0$ *are suitable constants), then every solution of equation* (1.2) *is almost oscillatory.*

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