Almost Oscillation Criteria for Second-Order Neutral Difference Equations with Quasidifferences

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Abstract

Using the Riccati transformation techniques, we will extend some almost oscillation criteria for the second–order nonlinear neutral difference equation with quasidifferences

$$
\Delta \left( r_n \left( \Delta \left( x_n + cx_{n-k} \right) \right)^\gamma \right) + q_n x_{n+1}^\alpha = e_n.
$$

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1 Introduction

Recently there has been an increasing interest in the study of the qualitative behavior of solutions of neutral difference equations (see the monographs [1–3,9]). Particularly, the
oscillation and nonoscillation of solutions of the second–order neutral difference equations attract attention; see the papers [4–8, 10–12, 14, 16–20] and the references therein. The interesting oscillatory results for first order and even order neutral difference equations can be found in [13] and [15].

In the presented paper, the results obtained in [21] by Thandapani, Vijaya and Győri

$$\Delta^2 (x_n + cx_{n-k})^\gamma + q_n x_{n+1}^\alpha = e_n,$$

are generalized to the second–order nonlinear neutral difference equation with quasidifference of the form

$$\Delta (r_n (\Delta (x_n + cx_{n-k}))^\gamma) + q_n x_{n+1}^\alpha = e_n. \quad (1.1)$$

Here \( k \) is a nonnegative integer, \( \Delta \) is the forward difference operator defined by \( \Delta x_n = x_{n+1} - x_n \), \( c \) is a real nonnegative constant, \( \alpha > \gamma \geq 1 \) are ratios of odd positive integers, \( (r_n) \) and \( (q_n) \) and \( (e_n) \) are positive sequences defined on \( \mathbb{N} = \{1, 2, 3, \ldots\} \).

Finally, as a corollary of our main result, almost oscillation property of solutions of a special case of equation (1.1) in the form

$$\Delta (r_n \Delta x_n) + q_n x_{n+1}^\alpha = e_n \quad (1.2)$$

is studied. For \( \alpha = 1 \), equation (1.2) is known as the forced second–order Sturm–Liouville difference equation. Some oscillation results for equation (1.2) were investigated among others by Došlý, Graef and Jaros in [4].

By a solution of equation (1.1) we mean a real valued sequence \( (x_n) \) defined on \( \mathbb{N}_k := \{k, k+1, \ldots\} \), which satisfies (1.1) for every \( n \in \mathbb{N}_k \).

Sequence \( (x_n) \) is said to be oscillatory, if for every integer \( n_k \in \mathbb{N}_k \), there exists \( n \geq n_k \) such that \( x_n x_{n+1} \leq 0 \); otherwise, it is called nonoscillatory.

**Definition 1.1.** Solution \( (x_n) \) of equation (1.1) is said to be almost oscillatory if either \( (x_n) \) is oscillatory, or \( (\Delta x_n) \) is oscillatory, or \( x_n \to 0 \) as \( n \to \infty \).

We begin with some lemmas which will be used for proving the main result.

**Lemma 1.2.** Set

$$F(x) = a x^{\alpha-\gamma} + \frac{b}{x^\gamma} \text{ for } x > 0. \quad (1.3)$$

If \( a > 0, b > 0 \) and

$$\alpha > \gamma \geq 1, \quad (1.4)$$

then \( F(x) \) attains its minimum

$$F_{\min} = \frac{\alpha a^{\frac{\gamma}{\alpha}} b^{1-\frac{\gamma}{\alpha}}}{\gamma^{\frac{\gamma}{\alpha}} (\alpha - \gamma)^{1-\frac{\gamma}{\alpha}}}.$$

**Lemma 1.3.** For all \( x \geq y \geq 0 \) and \( \gamma \geq 1 \) we have the following inequality

$$x^\gamma - y^\gamma \geq (x-y)^\gamma.$$
2 Almost Oscillation Criterion

In this section, by using the Riccati substitution we will establish new almost oscillation criterion for equation (1.1).

**Theorem 2.1.** Let
\[ r, q, e : N \to \mathbb{R}_+, \tag{2.1} \]
and there exists positive real constant \( R \) such that
\[ r_n \leq R \text{ for } n \in \mathbb{N}. \tag{2.2} \]
Assume also that \( \alpha > \gamma \geq 1 \) are ratios of odd positive integers. \( \tag{2.3} \)

If there exists positive sequence \( (p_n) \) such that
\[
\limsup_{n \to \infty} \sum_{i=1}^{n} \left( p_i Q_i - \frac{R (\Delta p_i)^2}{4p_i} \right) = \infty, \tag{2.4}
\]
where
\[
Q_n = \min\{Q^*_n, Q^{**}_n\}, \tag{2.5}
\]
\[
Q^*_n = \frac{d^{\alpha-\gamma} q_n}{(1 + c)^\alpha} - d^{-\gamma} e_n,
\]
\[
Q^{**}_n = \frac{\alpha q_n^{\frac{\alpha}{\gamma}} e_n^{\frac{1-\gamma}{\gamma}}}{\gamma^{\frac{\alpha}{\gamma}} (\alpha - \gamma)^{1-\frac{\alpha}{\gamma}} (1 + c)^\gamma}
\]
and
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{i-1} (Mq_j \pm e_j)^{\frac{1}{\gamma}} = \infty, \tag{2.6}
\]
(here \( d > 0 \) and \( M > 0 \) are suitable constants), then every solution of equation (1.1) is almost oscillatory.

**Proof.** Set
\[ z_n = x_n + cx_{n-k} \tag{2.7} \]
hence equation (1.1) takes the following form
\[
\Delta (r_n (\Delta z_n)^\gamma) = -q_n x_n^{\alpha} + e_n.
\]

Suppose, for the contrary, that sequence \( (x_n) \) is a solution eventually of one sign of equation (1.1) such that \( (\Delta x_n) \) is eventually of one sign as well.
Assume first that \((x_n)\) is an eventually positive sequence. It means that there exists \(n_0 \in \mathbb{N}\) such that \(x_{n-k} > 0\) for all \(n \geq n_0\). We have two possibilities to consider:

\((Ia)\) \(\Delta x_n > 0\) eventually or

\((Ib)\) \(\Delta x_n < 0\) eventually.

Case \((Ia)\): Assume that \(\Delta x_n > 0\). Thus \(\Delta z_n > 0\). We have \(x_n \geq \frac{z_n}{1+c}\). From equation (1.1), we get

\[
\Delta (r_n (\Delta z_n)^\gamma) \leq \frac{-q_n}{(1+c)^\alpha} z_{n+1}^\alpha + e_n.
\]

Let us denote by \((w_n)\) the following sequence

\[
w_n := p_n \frac{r_n (\Delta z_n)^\gamma}{z_{n+1}^\gamma},
\]

where \(z_n\) is defined by (2.7). Thus \(w_n > 0\) for \(n \geq n_0\). We have

\[
\Delta w_n = p_{n+1} \frac{r_{n+1} (\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma} - p_n \frac{r_n (\Delta z_n)^\gamma}{z_{n+1}^\gamma} =
\]

\[
= p_n \frac{\Delta (r_n (\Delta z_n)^\gamma)}{z_{n+1}^\gamma} + p_{n+1} \frac{r_{n+1} (\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma} - p_n \frac{r_n (\Delta z_n)^\gamma}{z_{n+1}^\gamma} +
\]

\[
+ p_n \frac{r_{n+1} (\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma} - p_n \frac{r_n (\Delta z_n)^\gamma}{z_{n+1}^\gamma}
\]

\[
= p_n \frac{\Delta (r_n (\Delta z_n)^\gamma)}{z_{n+1}^\gamma} + \Delta p_n \frac{w_{n+1}}{p_{n+1}} + \frac{p_n r_{n+1} (\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma \gamma} \left[ z_{n+1}^\gamma - z_{n+2}^\gamma \right]
\]

and finally

\[
\Delta w_n = p_n \frac{\Delta r_n (\Delta z_n)^\gamma}{z_{n+1}^\gamma} + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}} w_{n+1} \frac{\Delta z_{n+1}^\gamma}{z_{n+1}^\gamma}.
\]

From the above and (2.8), we get

\[
\Delta w_n \leq -p_n \left( \frac{q_n}{(1+c)^\alpha} z_{n+1}^{\alpha-\gamma} - \frac{e_n}{z_{n+1}^\gamma} \right) + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}} w_{n+1} \frac{\Delta z_{n+1}^\gamma}{z_{n+1}^\gamma}.
\]

Let \(G(x) = \frac{q_n}{(1+c)^\alpha} x^{\alpha-\gamma} - \frac{e_n}{x^\gamma}\). It is easy to verify that function \(G\) is increasing for positive arguments. Since \(x\) is increasing, there is a constant \(d > 0\) such that \(x \geq d > 0\) and

\[
G(x) \geq \frac{q_n}{(1+c)^\alpha} d^{\alpha-\gamma} - \frac{e_n}{d^\gamma} = Q_n^*.
\]

From (2.11) and (2.12), we get the following inequality
Almost Oscillation Criteria for Second-Order Neutral Difference Equation

\[
\Delta w_n \leq -p_n Q_n^* + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}} w_{n+1} \frac{(\Delta z_{n+1})^\gamma}{z_{n+2}^\gamma}.
\]

For \( \Delta z_n > 0 \) we have \( z_{n+2} > z_{n+1} \) and \( z_{n+2}^\gamma > z_{n+1}^\gamma \). Because of positivity of the sequence \((z_n)\) for large \(n\), say \(n \geq n_1 \geq n_0\), we obtain \( \frac{1}{z_{n+2}^\gamma} < \frac{1}{z_{n+1}^\gamma} \) for \(n \geq n_1\). From (2.9) and by Lemma 1.3, we get

\[
\Delta w_n \leq -p_n Q_n^* + \frac{\Delta p_n}{p_{n+1}} w_{n+1} - \frac{p_n}{p_{n+1}^2 p_{n+1}} w_{n+1}^2.
\]

This and (2.2) imply that

\[
\Delta w_n \leq -p_n Q_n^* + \frac{(\Delta p_n)^2 r_{n+1}}{4p_n} - \left[ \sqrt{\frac{p_n}{r_{n+1} p_{n+1}}} w_{n+1} - \frac{\sqrt{r_{n+1} \Delta p_n}}{2\sqrt{p_n}} \right]^2
\]

\[
\leq -p_n Q_n^* + \frac{(\Delta p_n)^2 r_{n+1}}{4p_n} \leq - \left[ p_n Q_n^* - \frac{(\Delta p_n)^2 R}{4p_n} \right].
\]

Summing both sides of the above inequality from \(i = n_1\) to \(n - 1\), we obtain

\[
w_n - w_{n_1} < - \sum_{i=n_1}^{n-1} \left( p_i Q_i^* - \frac{R(\Delta p_i)^2}{4p_i} \right).
\]

Because of \(w_n \geq 0\) for \(n \in \mathbb{N}\), we have \(w_{n_1} > w_{n_1} - w_n\). Hence

\[
w_{n_1} > \sum_{i=n_1}^{n-1} \left( p_i Q_i^* - \frac{R(\Delta p_i)^2}{4p_i} \right).
\]

Letting \(n\) into infinity we obtain

\[
w_{n_1} \geq \limsup_{n \to \infty} \sum_{i=n_1}^{n-1} \left( p_i Q_i^* - \frac{R(\Delta p_i)^2}{4p_i} \right).
\]

From (2.5), we get

\[
w_{n_1} > \limsup_{n \to \infty} \sum_{i=n_1}^{n-1} \left( p_i Q_i^* - \frac{R(\Delta p_i)^2}{4p_i} \right).
\]

This is a contradiction with (2.4).
Case (IIb): If $\Delta x_n < 0$, then $\Delta z_n < 0$. From $x_n > 0$ and $\Delta x_n < 0$, we get 
$$\lim_{n \to \infty} x_n = l > 0.$$ 
Thus $x^\alpha_{n+1} \to l^\alpha > 0$ as $n \to \infty$. Hence, there exists $n_2 \in \mathbb{N}$ such that 
$x^\alpha_{n+1} \geq l^\alpha$ for $n \geq n_2$. Therefore, we have
$$\Delta (r_n (\Delta z_n)^\gamma) \leq -q_n l^\alpha + e_n.$$ 
Set $l^\alpha = M$. Summing the last inequality from $n_2$ to $n - 1$, we obtain
$$r_n (\Delta z_n)^\gamma < r_n (\Delta z_n)^\gamma - r_{n_2} (\Delta z_{n_2})^\gamma \leq -\left(\sum_{i=n_2}^{n-1} Mq_i - e_i\right)$$
and
$$\Delta z_n \leq -\left(\sum_{i=n_2}^{n-1} Mq_i - e_i\right)^\frac{1}{\gamma} r_n^{-\frac{1}{\gamma}}, \text{ for } n \geq n_2.$$ 
Summing again the above inequality from $n_2$ to $n$, we obtain
$$z_{n+1} \leq z_{n_2} - \sum_{i=n_2}^{n} \left(\sum_{j=n_2}^{i-1} Mq_j - e_j\right)^\frac{1}{\gamma} r_i^{-\frac{1}{\gamma}}.$$ 
From (2.2), we get
$$z_{n+1} \leq z_{n_2} - R^{-\frac{1}{\gamma}} \sum_{i=n_2}^{n} \left(\sum_{j=n_2}^{i-1} Mq_j - e_j\right)^\frac{1}{\gamma}.$$ 
Letting $n$ into $\infty$, from condition (2.6) we obtain that the right side of the above inequality is negative. So, $z_n$ is eventually negative, too. This contradiction ended the proof in this case.

Finally, we assume that $(x_n)$ is an eventually negative sequence. It means that there exists $n_3 \in \mathbb{N}$ such that $x_n < 0$ for all $n \geq n_3$. We use the transformation $y_n = -x_n$ in the equation (1.1). Equation (1.1) takes the following form
$$\Delta (r_n (\Delta (y_n + cy_{n-k})^\gamma) + q_n y_{n+1}^\alpha = -e_n. \quad (2.13)$$ 
Here sequence $(y_n)$ is an eventually positive solution of equation (2.13).

We have two possibilities to consider:

(IIa) $\Delta y_n > 0$ eventually or

(IIb) $\Delta y_n < 0$ eventually.

Case (IIa): Assume that $\Delta y_n > 0$. From (2.10), by (2.13), we have
$$\Delta w_n \leq -p_n \left(\frac{q_n}{(1+c)^\alpha} z_{n+1}^{\alpha-\gamma} + \frac{e_n}{z_{n+1}^\gamma}\right) + \frac{\Delta p_n}{p_{n+1}} w_{n+1} + \frac{p_n}{p_{n+1}} w_{n+1} \frac{\Delta z_{n+1}^{\gamma} - e_n}{z_{n+1}^\gamma}.$$
Putting \( a = \frac{q_n}{(1 + c)^n}, b = e_n \) and \( x = z_{n+1} \) in (1.3), we have

\[
F(z_{n+1}) = \frac{q_n}{(1 + c_{n+1})^\alpha} z_n^{-\gamma} + \frac{e_n}{z_n^{-\gamma}}.
\]

By Lemma 1.2, we get

\[
F(z_n) \geq \frac{\alpha q_n e_n^{1-\gamma}}{\gamma^\gamma (\alpha - \gamma)^{1-\gamma}} (1 + c_{n+1})^{-\gamma} = Q_n^{**},
\]

and

\[
\Delta w_n \leq -p_n Q_n^{**} + \frac{\Delta p_n w_{n+1} - p_n}{p_{n+1} r_{n+1}} w_{n+1}^2
\]

is satisfied. The rest of the proof is similar to proof of case (Ia) and hence is omitted.

Case (IIb): Assume that \( \Delta y_n < 0 \). Hence sequence \( y_n \) has positive limit and the proof of this case is similar to case (IIb) and hence is omitted. The proof is now complete.

We illustrate Theorem 2.1 by the following examples.

**Example 2.2.** Let us consider the difference equation

\[
\Delta \left( 2 - \left( -1 \right) \frac{x_n}{n} \right) \left( \Delta \left( x_n + \frac{1}{2} x_{n-1} \right) \right)^3 + 4 x_{n+1}^3 = \frac{1}{n(n+1)}.
\]

Here \( r_n = 2 - \frac{(-1)^n}{n}, c = \frac{1}{2}, \gamma = 3, q_n = 4, \alpha = 5 \) and \( e_n = \frac{1}{n(n+1)} \). For \( p_n = 1 \), all assumptions of Theorem 2.1 are satisfied. Hence, any solution of the above equation is almost oscillatory. Sequence \( x_n = (-1)^{n+1} \) is one of such solutions. Here, \( (x_n) \) is oscillatory.

**Example 2.3.** Let us consider the difference equation

\[
\Delta \left( 2 + (-1)^n \right) \left( \Delta \left( x_n + 2 x_{n-2} \right) \right) + x_{n+1}^3 = 14 + 11(-1)^{n+1}.
\]

Here \( r_n = 2 + (-1)^n, c = 2, \gamma = 1, q_n = 1, \alpha = 3 \) and \( e_n = 14 + 11(-1)^{n+1} \). For \( p_n = 1 \), all assumptions of Theorem 2.1 are satisfied. Hence, any solution of the above equation is almost oscillatory. Sequence \( x_n = 2 + (-1)^{n+1} \) is one of such solutions. Here, \( (x_n) \) is nonoscillatory but \( \Delta x_n \) oscillates.

**Example 2.4.** Let us consider the difference equation

\[
\Delta \left( \frac{1}{3n+4} \left( \Delta \left( x_n + 2 x_{n-1} \right) \right) \right) + n(n+2)^2 x_{n+1}^3 = \frac{3 + n^2 (n+1) (n+3)}{n(n+1)(n+2)(n+3)}.
\]
Here $r_n = \frac{1}{3n + 4}$, $c = 2$, $k = 1$, $\gamma = 1$, $q_n = n(n + 2)^2$, $\alpha = 3$ and

$$
e_n = \frac{3 + n^2(n + 1)(n + 3)}{n(n + 1)(n + 2)(n + 3)}.
$$

For $p_n = 1$, all assumptions of Theorem 2.1 are satisfied. Hence, any solution of the above equation is almost oscillatory. In fact, sequence $x_n = \frac{1}{n + 1}$ is one of such solutions. Here, $(x_n)$ tends to zero.

Assuming that $c = 0$ and $\gamma = 1$ equation (1.1) takes the form (1.2).

**Corollary 2.5.** Assume that conditions (2.1), (2.2) and (2.3) are satisfied. If there exists positive sequence $(p_n)$ such that

$$
limit_{n \to \infty} \sum_{i=1}^{n} \left( p_i Q_i - \frac{R(\Delta p_i)^2}{4p_i} \right) = \infty,
$$

where

$$Q_n = \min\{Q_n^*, Q_n^{**}\},
$$

$$Q_n^* = \frac{d^n q_n - e_n}{d},
$$

$$Q_n^{**} = \alpha q_n^{\frac{1}{\alpha}} e_n^{1 - \frac{1}{\alpha}}
$$

and

$$\sum_{i=1}^{\infty} \sum_{j=1}^{i-1} (M q_j \pm e_j) = \infty,
$$

(here $d > 0$ and $M > 0$ are suitable constants), then every solution of equation (1.2) is almost oscillatory.

**References**


Almost Oscillation Criteria for Second–Order Neutral Difference Equation


