An Inverse Problem of the Calculus of Variations on Arbitrary Time Scales

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Abstract

We consider an inverse extremal problem for variational functionals on arbitrary time scales. Using the Euler–Lagrange equation and the strengthened Legendre condition, we derive a general form for a variational functional that attains a local minimum at a given point of the vector space.

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1 Introduction

We study an inverse problem associated with the following fundamental problem of the calculus of variations: to minimize

$$\mathcal{L}(y) = \int_a^b L\left(t, y(t), y^\Delta(t)\right) \Delta t$$

subject to the boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$, on a given time scale $\mathbb{T}$. More precisely, we describe a general form of a variational functional (1.1) having an extremum at a given function $y_0$ under the Euler–Lagrange and strengthened Legendre conditions on time scales [1]. Throughout the paper we assume the reader to be familiar with the basic definitions and results from the time scale theory [3, 4, 9]. For a review on general approaches to the calculus of variations on time scales see [1, 5–8, 12, 13, 17]. For analogous results in $\mathbb{T} = \mathbb{R}$ see [15, 16]. The results here obtained are new even for simple (but important) time scales like $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = q^\mathbb{N}_0$, $q > 1$.

The paper is organized as follows. In Section 2 we collect some necessary definitions and results of the delta calculus on time scales, which are used throughout the text. The main results are presented in Section 3. We find a general form of the variational functional (1.1) that solves the inverse extremal problem (Theorem 3.2). In order to illustrate our results, we present the form of the Lagrangian $L$ on an isolated time scale (Corollary 3.4). We end by presenting the form of the Lagrangian $L$ in the periodic time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$ (Example 3.6) and in the $q$-scale $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ (Example 3.7).

2 Preliminaries

In this section we introduce basic definitions and theorems that will be useful in the sequel. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Let $a, b \in \mathbb{T}$ with $a < b$. We define the interval $[a, b]$ in $\mathbb{T}$ by $[a, b] := [a, b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

**Definition 2.1** (See [3]). The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) := \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is given by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense or left-scattered if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, respectively. We say that $t$ is isolated if $\rho(t) < t < \sigma(t)$, that $t$ is dense if $\rho(t) = t = \sigma(t)$.

**Example 2.2.** The two classical time scales are $\mathbb{R}$ and $\mathbb{Z}$, representing the continuous and the purely discrete time, respectively. The other standard examples are $h\mathbb{Z}$, $h > 0$, and $q^\mathbb{N}_0$, $q > 1$. For the case of a periodic time scale $\mathbb{T} = \mathbb{Z}$, the graininess function $\mu(t) = 1$ for all $t \in \mathbb{T}$.
and \( q^{\N_0}, q > 1 \). It follows from Definition 2.1 that if \( T = \mathbb{R} \), then \( \sigma(t) = t \) and \( \mu(t) = 0 \) for all \( t \in T \); if \( T = h\mathbb{Z} \), then \( \sigma(t) = t + h \) and \( \mu(t) = h \) for all \( t \in T \); if \( T = q^{\N_0} \), then \( \sigma(t) = qt \) and \( \mu(t) = t(q - 1) \) for all \( t \in T \).

**Definition 2.3** (See [3]). A time scale \( T \) is said to be an isolated time scale provided given any \( t \in T \), there is a \( \delta > 0 \) such that \((t - \delta, t + \delta) \cap T = \{t\}\).

**Remark 2.4.** If the graininess function is bounded from below by a strictly positive number, then the time scale is isolated [2]. Therefore, \( h\mathbb{Z}, h > 0 \), and \( q^{\N_0}, q > 1 \), are examples of isolated time scales. Note that the converse is not true. For example, \( T = \log(\mathbb{N}) \) is an isolated time scale but its graininess function is not bounded from below by a strictly positive number.

To simplify the notation, one usually uses \( f^\sigma(t) := f(\sigma(t)) \). The delta derivative is defined for points from the set

\[
T^\kappa := \begin{cases} 
T \setminus \{\sup T\} & \text{if } \rho(\sup T) < \sup T < \infty, \\
T & \text{otherwise.}
\end{cases}
\]

**Definition 2.5** (See [3]). A function \( f : T \to \mathbb{R} \) is \( \Delta \)-differentiable at \( t \in T^\kappa \) if there is a number \( f^\Delta(t) \) such that for all \( \varepsilon > 0 \) there exists a neighborhood \( U \) of \( t \) (i.e., \( U = (t - \delta, t + \delta) \cap T \) for some \( \delta > 0 \)) such that

\[
|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.
\]

We call \( f^\Delta(t) \) the \( \Delta \)-derivative of \( f \) at \( t \).

**Example 2.6.** If \( T = h\mathbb{Z} \), then \( f : T \to \mathbb{R} \) is delta differentiable at \( t \in T \) if, and only if,

\[
f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t + h) - f(t)}{h} =: \Delta_h f(t).
\]

In the particular case \( h = 1 \), \( f^\Delta(t) = \Delta f(t) \), where \( \Delta \) is the usual forward difference operator. If \( T = q^{\N_0} = \{q^k : q > 1, k \in \N_0\} \), then \( f^\Delta(t) = \frac{f(qt) - f(t)}{(q - 1)t} =: \Delta_q f(t) \), i.e., we get the usual Jackson derivative of quantum calculus [11].

**Theorem 2.7** (See [3]). Let \( f : T \to \mathbb{R} \) and \( t \in T^\kappa \). If \( f \) is delta differentiable at \( t \), then

\[
f^\sigma(t) = f(t) + \mu(t)f^\Delta(t).
\]

**Definition 2.8** (See [3]). A function \( f : T \to \mathbb{R} \) is called rd-continuous provided it is continuous at right-dense points in \( T \) and its left-sided limits exists (finite) at all left-dense points in \( T \).
The set of all rd-continuous functions \( f : \mathbb{T} \to \mathbb{R} \) is denoted by \( C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}) \). The set of functions \( f : \mathbb{T} \to \mathbb{R} \) that are \( \Delta \)-differentiable and whose derivative is rd-continuous is denoted by \( C^{1}_{rd} = C^{1}_{rd}(\mathbb{T}) = C^{1}_{rd}(\mathbb{T}, \mathbb{R}) \).

A function \( F : \mathbb{T} \to \mathbb{R} \) is called an antiderivative of \( f : \mathbb{T} \to \mathbb{R} \) provided that \( F(\Delta) = f(t) \) for all \( t \in \mathbb{T} \). Let \( \mathbb{T} \) be a time scale and \( a, b \in \mathbb{T} \). If \( F \) is an antiderivative of \( f \), then the Cauchy \( \Delta \)-integral is defined by

\[
\int_{a}^{b} f(t) \Delta t := F(b) - F(a).
\]

**Theorem 2.9** (See [3]). Every rd-continuous function has an antiderivative. In particular, if \( t_0 \in \mathbb{T} \), then \( F \) defined by

\[
F(t) := \int_{t_0}^{t} f(\tau) \Delta \tau,
\]

\( t \in \mathbb{T} \), is an antiderivative of \( f \).

**Example 2.10.** If \( \mathbb{T} = h\mathbb{Z}, h > 0 \), and \( a, b \in \mathbb{T} \) with \( a < b \), then

\[
\int_{a}^{b} f(t) \Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(\frac{kh}{h}) h.
\]

If \( \mathbb{T} = q^{\mathbb{N}_0}, q > 1 \), then

\[
\int_{a}^{b} f(t) \Delta t = (q - 1) \sum_{t \in (a,b) \cap \mathbb{T}} tf(t).
\]

Let \( \mathbb{T} \) be a given time scale with at least three points. Consider the following variational problem on the time scale \( \mathbb{T} \):

\[
\mathcal{L}(y) = \int_{a}^{b} L \left( t, y^{\sigma}(t), y^{\Delta}(t) \right) \Delta t \longrightarrow \text{min}, \quad y(a) = \alpha, \quad y(b) = \beta,
\]

where \( a, b \in \mathbb{T} \) with \( a < b \), \( \alpha, \beta \in \mathbb{R}^n \) with \( n \in \mathbb{N} \), and \( L : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R} \).

**Definition 2.11.** We say that \( y \in C^{1}_{rd}(\mathbb{T}) \) is admissible for problem (2.1) if it satisfies the boundary conditions \( y(a) = \alpha \) and \( y(b) = \beta \).

**Definition 2.12.** An admissible function \( \hat{y} \) is called a local minimizer of problem (2.1) provided there exists \( \delta > 0 \) such that \( \mathcal{L}(\hat{y}) \leq \mathcal{L}(y) \) for all admissible \( y \) with \( \| y - \hat{y} \|_{C^{1}_{rd}} < \delta \), where

\[
\|f\|_{C^{1}_{rd}} = \sup_{t \in [a,b]^{\mathbb{T}}} \|f^{\sigma}(t)\| + \sup_{t \in [a,b]^{\mathbb{T}}} \|f^{\Delta}(t)\|
\]

with \( \| \cdot \| \) a norm in \( \mathbb{R}^n \).
In what follows the Lagrangian $L$ is understood as a function $(t, x, v) \rightarrow L(t, x, v)$ and by $L_x$ and $L_v$ we denote the partial derivatives of $L$ with respect to $x$ and $v$, respectively. Similar notation is used for second order partial derivatives.

**Theorem 2.13** (The Euler–Lagrange equation [10]). Assume that $L(t, \cdot, \cdot)$ is differentiable in $(x, v)$ and $L(t, \cdot, \cdot)$, $L_x(t, \cdot, \cdot)$, $L_v(t, \cdot, \cdot)$ are continuous at $(y^\sigma, y^\Delta)$, uniformly in $t$ and rd-continuous in $t$ for any admissible $y$. If $\hat{y}(t)$ is a local minimizer of the variational problem (2.1), then there exists a vector $c \in \mathbb{R}^n$ such that the Euler–Lagrange equation

$$L_v(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) = \int_a^t L_x(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta \tau + c^T$$

holds for $t \in [a, b]$.

**Theorem 2.14** (The Legendre condition [1]). If $\hat{y}$ is a local minimizer of the variational problem (2.1), then

$$A(t) + \mu(t) \left\{ C(t) + C^T(t) + \mu(t)B(t) + (\mu(\sigma(t)))^\dagger A(\sigma(t)) \right\} \geq 0,$$

for $t \in [a, b]^{\infty}_T$, where $A(t) = L_{vv}(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))$, $B(t) = L_{xx}(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))$, $C(t) = L_{xx}(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))$, and where $\alpha^\dagger = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$.

**Remark 2.15.** If (2.3) holds with the strict inequality $>$, then it is called the strengthened Legendre condition.

**Definition 2.16** (See [3]). We say that a function $p : T \rightarrow \mathbb{R}$ is regressive provided

$$1 + \mu(t) p(t) \neq 0$$

holds for all $t \in T^n$. The set of all regressive and rd-continuous functions $f : T \rightarrow \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(T) = \mathcal{R}(T, \mathbb{R})$.

**Theorem 2.17** (See [4]). Let $p \in \mathcal{R}$, $f \in C_{rd}$, $t_0 \in T$ and $y_0 \in \mathbb{R}$. Then, the unique solution of the initial value problem

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0,$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^t e_p(t, \sigma(\tau)) f(\tau) \Delta \tau,$$

where $e_p(\cdot, \cdot)$ denotes the exponential function on time scales.
Remark 2.18 (See [3]). An alternative form of the solution of the initial value problem (2.4) is given by

\[ y(t) = e_p(t, t_0) \left[ y_0 + \int_{t_0}^{t} e_p(t_0, \sigma(\tau)) f(\tau) \Delta \tau \right]. \]

For more properties of the delta exponential function we refer the reader to [3, 4].

3 Main Results

The problem under our consideration is to find a general form of the variational functional

\[ \mathcal{L}(y) = \int_{a}^{b} L(t, y^\sigma(t), y^\Delta(t)) \Delta t, \quad (3.1) \]

\( L : [a, b]_T \times \mathbb{R}^2 \to \mathbb{R}, \) subject to the boundary conditions \( y(a) = y(b) = 0, \) possessing a local minimum at zero, under the Euler–Lagrange and the strengthened Legendre conditions. We assume that \( L(t, \cdot, \cdot) \) is a \( C^2 \)-function with respect to \( (x,v) \) uniformly in \( t, \) and \( L, L_x, L_v, L_{vv} \in C_{rd} \) for any admissible path \( y(\cdot). \) Observe that under our assumptions, by Taylor’s theorem, we may write \( L, \) with the big \( O \) notation, in the form

\[ L(t,x,v) = P(t,x) + Q(t,x)v + \frac{1}{2} R(t,x,v)v^2 + O(v^3), \quad (3.2) \]

where

\[ P(t,x) = L(t,x,0), \]
\[ Q(t,x) = L_v(t,x,0), \]
\[ R(t,x,0) = L_{vv}(t,x,0). \quad (3.3) \]

Let \( R(t,x,v) = R(t,x,0) + O(v). \) Then, one can write (3.2) as

\[ L(t,x,v) = P(t,x) + Q(t,x)v + \frac{1}{2} R(t,x,v)v^2. \quad (3.4) \]

Now the idea is to find general forms of \( P(t,y^\sigma(t)), Q(t,y^\sigma(t)) \) and \( R(t,y^\sigma(t),y^\Delta(t)) \) using the Euler–Lagrange and the strengthened Legendre conditions. Note that the Euler–Lagrange equation (2.2) at the null extremal, with notation (3.3), is

\[ Q(t,0) = \int_{a}^{t} P_x(\tau,0) \Delta \tau + C, \quad (3.5) \]
We have just proved the following result.

With notation (3.3), the strengthened Legendre condition (2.3) at the null extremal has the form

$$R(t, 0, 0) + \mu(t) \left\{ 2Q_x(t, 0) + \mu(t)P_{xx}(t, 0) + (\mu^\alpha(t))^\dagger R(\sigma(t), 0, 0) \right\} > 0,$$

(3.7)

$t \in [a, b)^2_\mathbb{T}$, where $\alpha^\dagger = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$. Hence, we set

$$R(t, 0, 0) + \mu(t) \left\{ 2Q_x(t, 0) + \mu(t)P_{xx}(t, 0) + (\mu^\alpha(t))^\dagger R(\sigma(t), 0, 0) \right\} = p(t)$$

(3.8)

with $p \in C^1_{rd}([a, b]_\mathbb{T})$, $p(t) > 0$ for all $t \in [a, b)^2_\mathbb{T}$, chosen arbitrarily. Note that there exists a unique solution of (3.8) with respect to $R(t, 0, 0)$. If $t$ is a right-dense point, then $\mu(t) = 0$ and $R(t, 0, 0) = p(t)$. Otherwise, $\mu(t) \neq 0$, and using Theorem 2.7 with $f(t) = R(t, 0, 0)$ we modify equation (3.8) into a first order delta dynamic equation, which has a unique solution $R(t, 0, 0)$ in agreement with Theorem 2.17 (see details in the proof of Corollary 3.4). We derive a general form of $R$ from Legendre’s condition (3.7), as a sum of the solution $R(t, 0, 0)$ of equation (3.8) and function $w$, which is chosen arbitrarily in such a way that $w(t, \cdot, \cdot) \in C^2$ with respect to the second and the third variable, uniformly in $t$; $w_x, w_v$ and $w_{vv}$ are rd-continuous in $t$ for all admissible $y$.

Concluding: a general form of the integrand $L$ for functional (3.1) follows from (3.4), (3.6) and (3.8), and is given by

$$L(t, y^\alpha(t), y^{\Delta}(t)) = P(t, y^\alpha(t))$$

$$\quad + \left( C + \int_a^t P_x(\tau, 0) \Delta \tau + q(t, y^\alpha(t)) - q(t, 0) \right) y^{\Delta}(t)$$

$$\quad + \left( p(t) - \mu(t) \left\{ 2Q_x(t, 0) + \mu(t)P_{xx}(t, 0) + (\mu^\alpha(t))^\dagger R(\sigma(t), 0, 0) \right\}$$

$$\quad + w(t, y^\alpha(t), y^{\Delta}(t)) - w(t, 0, 0) \right) \frac{y^{\Delta}(t)^2}{2}.$$  

(3.9)

We have just proved the following result.
Theorem 3.1. Let $\mathbb{T}$ be an arbitrary time scale. If functional (3.1) with boundary conditions $y(a) = y(b) = 0$ attains a local minimum at $\dot{y}(t) \equiv 0$ under the strengthened Legendre condition, then its Lagrangian $L$ takes the form (3.9), where $R(t, 0, 0)$ is a solution of equation (3.8), $C \in \mathbb{R}$, $\alpha = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$. Functions $P$, $p$, $q$ and $w$ are arbitrary functions satisfying:

(i) $P(t, \cdot), q(t, \cdot) \in C^2$ with respect to the second variable uniformly in $t$; $P$, $P_x$, $q$, $q_x$ are rd-continuous in $t$ for all admissible $y$; $P_{xx}(\cdot, 0)$ is rd-continuous in $t$; $p \in C_{rd}^1$ with $p(t) > 0$ for all $t \in [a, b]_T$;

(ii) $w(t, \cdot, \cdot) \in C^2$ with respect to the second and the third variable, uniformly in $t$, $w_x, w_v, w_{vv}$ are rd-continuous in $t$ for all admissible $y$.

Now we consider the general situation when the variational problem consists in minimizing (3.1) subject to arbitrary boundary conditions $y(a) = y_0(a)$ and $y(b) = y_0(b)$, for a certain given function $y_0 \in C^2_{rd}([a, b]_T)$.

Theorem 3.2. Let $\mathbb{T}$ be an arbitrary time scale. If the variational functional (3.1) with boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$, attains a local minimum for a certain given function $y_0(\cdot) \in C^2_{rd}([a, b]_T)$ under the strengthened Legendre condition, then its Lagrangian $L$ has the form

$$L(t, y^a(t), y^\Delta(t)) = P(t, y^a(t) - y_0^a(t)) + (y^\Delta(t) - y_0^\Delta(t))$$

$$\times \left( C + \int_a^t P_x(\tau, -y_0^a(\tau)) \Delta\tau + q(t, y^a(t) - y_0^a(t)) - q(t, -y_0^a(t)) \right)$$

$$+ \frac{1}{2} \left( p(t) - \mu(t) \left\{ 2Q_x(t, 0) + \mu(t)P_{xx}(t, 0) + (\mu^a(t))^\dagger R(\sigma(t), 0, 0) \right\} \right)$$

$$+ w(t, y^a(t) - y_0^a(t), y^\Delta(t) - y_0^\Delta(t)) - w(t, -y_0^a(t), -y_0^\Delta(t)) \right) (y^\Delta(t) - y_0^\Delta(t))^2,$$

where $R(t, 0, 0)$ is the solution of equation (3.8), $C \in \mathbb{R}$ and functions $P$, $p$, $q$, $w$ satisfy conditions (i) and (ii) of Theorem 3.1.

Proof. The result follows as a corollary of Theorem 3.1. In order to reduce the problem to the case of null boundary conditions $y(a) = 0$ and $y(b) = 0$, we introduce the auxiliary variational functional

$$\tilde{L}(y) := \mathcal{L}(y + y_0) = \int_a^b L(t, y^a(t) + y_0^a(t), y^\Delta(t) + y_0^\Delta(t)) \Delta t$$

$$= : \int_a^b \tilde{L}(t, y^a(t), y^\Delta(t)) \Delta t$$
subject to boundary conditions $y(a) = 0$ and $y(b) = 0$. The result follows by application of Theorem 3.1 to the auxiliary Lagrangian $\tilde{L}$.

For the classical situation $\mathbb{T} = \mathbb{R}$, Theorem 3.2 gives a recent result of [15].

**Corollary 3.3** (See [15, Theorem 4]). If the variational functional

$$\mathcal{L}(y) = \int_{a}^{b} L(t, y(t), y'(t))dt$$

attains a local minimum at $y_0(\cdot) \in C^2([a, b])$ satisfying boundary conditions $y(a) = y_0(a)$ and $y(b) = y_0(b)$ and the classical Legendre condition $R(t, y_0(t), y'_0(t)) > 0$, $t \in [a, b]$, then its Lagrangian $L$ has the form

$$L(t, y(t), y'(t)) = P(t, y(t) - y_0(t))$$

$$+ \left( y'(t) - y'_0(t) \right) \left( C + \int_{a}^{t} P_x(\tau, -y_0(\tau))d\tau + q(t, y(t) - y_0(t)) - q(t, -y_0(t)) \right)$$

$$+ \frac{1}{2} \left( p(t) + w(t, y(t) - y_0(t), y'(t) - y'_0(t)) - w(t, -y_0(t), -y'_0(t)) \right) (y'(t) - y'_0(t))^2,$$

where $C \in \mathbb{R}$.

**Proof.** Follows from Theorem 3.2 with $\mathbb{T} = \mathbb{R}$. \hfill $\square$

Theorem 3.2 seems to be new for any time scale other than $\mathbb{T} = \mathbb{R}$. In the particular case of an isolated time scale, where $\mu(t) \neq 0$ for all $t \in \mathbb{T}$, we get the following corollary.

**Corollary 3.4.** Let $\mathbb{T}$ be an isolated time scale. If functional (3.1) subject to the boundary conditions $y(a) = y(b) = 0$ attains a local minimum at $\hat{y}(t) \equiv 0$ under the strengthened Legendre condition, then the Lagrangian $L$ has the form

$$L(\hat{t}, y^\sigma(\hat{t}), y^\Delta(\hat{t})) = P(\hat{t}, y^\sigma(\hat{t}))$$

$$+ \left( C + \int_{a}^{\hat{t}} P_x(\tau, 0) \Delta \tau + q(\hat{t}, y^\sigma(\hat{t})) - q(\hat{t}, 0) \right) y^\Delta(\hat{t})$$

$$+ \left( e_r(t, a) R_0 + \int_{a}^{\hat{t}} e_r(\tau, \sigma(\tau)) s(\tau) \Delta \tau + w(\hat{t}, y^\sigma(\hat{t}), y^\Delta(\hat{t})) - w(\hat{t}, 0, 0) \right) \frac{y^\Delta(\hat{t})^2}{2},$$

where $C, R_0 \in \mathbb{R}$ and $r(t)$ and $s(t)$ are given by

$$r(t) := -\frac{1 + \mu(t)(\mu^\sigma(t))^t}{\mu^2(t)(\mu^\sigma(t))^t}, \quad s(t) := \frac{p(t) - \mu(t)[2Q_x(t, 0) + \mu(t)P_{xx}(t, 0)]}{\mu^2(t)(\mu^\sigma(t))^t},$$

(3.10)
with $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$, where functions $P$, $p$, $q$, $w$ satisfy assumptions of Theorem 3.1.

**Proof.** In the case of an isolated time scale $\mathbb{T}$, we may obtain the form of function $Q$ in the same way as it was done in the proof of Theorem 3.1. We derive a general form for $R$ from Legendre’s condition. By relation $f^\sigma = f + \mu f^\Delta$ (Theorem 2.7), one may write equation (3.8) as

$$R(t, 0, 0) + \mu(t)(\mu^\sigma(t))^\dagger \left( R(t, 0, 0) + \mu(t)R^\Delta(t, 0, 0) \right)$$

$$+ \mu(t) \{2Q_x(t, 0) + \mu(t)P_{xx}(t, 0)\} - p(t) = 0.$$

Hence,

$$\mu^2(t)(\mu^\sigma(t))^\dagger R^\Delta(t, 0, 0) + \left[1 + \mu(t)(\mu^\sigma(t))^\dagger\right] R(t, 0, 0)$$

$$+ \mu(t)[2Q_x(t, 0) + \mu(t)P_{xx}(t, 0)] - p(t) = 0.$$  (3.12)

For an isolated time scale $\mathbb{T}$, equation (3.12) is a first order delta dynamic equation of the following form:

$$R^\Delta(t, 0, 0) + \frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu^2(t)(\mu^\sigma(t))^\dagger} R(t, 0, 0) + \frac{\mu(t)[2Q_x(t, 0) + \mu(t)P_{xx}(t, 0)]}{\mu^2(t)(\mu^\sigma(t))^\dagger} - p(t) = 0.$$

With notation (3.11) we have

$$R^\Delta(t, 0, 0) = r(t)R(t, 0, 0) + s(t).$$  (3.13)

Observe that $r(t)$ is regressive. Indeed, if $\mu(t) \neq 0$, then

$$1 + \mu(t)r(t) = 1 - \frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu(t)(\mu^\sigma(t))^\dagger} = 1 - \frac{\mu^\sigma(t) + \mu(t)}{\mu(t)} = -\frac{\mu^\sigma(t)}{\mu(t)} \neq 0$$

for all $t \in [a, b]^\kappa$. Therefore, by Theorem 2.17, there is a unique solution to equation (3.13) with initial condition $R(a, 0, 0) = R_0 \in \mathbb{R}$:

$$R(t, 0, 0) = e_r(t, a)R_0 + \int_a^t e_r(t, \sigma(\tau))s(\tau)\Delta\tau.$$  (3.14)

Thus, a general form of the integrand $L$ for functional (3.1) is given by (3.10).

**Remark** 3.5. Instead of (3.14), we can use an alternative form of the solution of the initial value problem (3.13) subject to $R(a, 0, 0) = R_0$ (cf. Remark 2.18):

$$R(t, 0, 0) = e_r(t, a) \left[ R_0 + \int_a^t e_r(a, \sigma(\tau))s(\tau)\Delta\tau \right].$$
Then the Lagrangian $L$ (3.10) can be written as

$$\begin{align*}
L(t, y^\sigma(t), y^\Delta(t)) &= P(t, y^\sigma(t)) \\
&+ \left( C + \int_a^t P_x(\tau, 0) \Delta \tau + q(t, y^\sigma(t)) - q(t, 0) \right) y^\Delta(t) \\
&+ \left( e_r(t, a) \left[ R_0 + \int_a^t e_r(a, \sigma(\tau)) s(\tau) \Delta \tau \right] \right) \frac{y^\Delta(t)^2}{2} \\
&+ \left( w(t, y^\sigma(t), y^\Delta(t)) - w(t, 0, 0) \right) \frac{y^\Delta(t)^2}{2}
\end{align*}$$

Based on Corollary 3.4, we present the form of Lagrangian $L$ in the periodic time scale $T = h\mathbb{Z}$.

**Example 3.6.** Let $T = h\mathbb{Z}$, $h > 0$, and $a, b \in h\mathbb{Z}$ with $a < b$. Then $\mu(t) \equiv h$. We consider the variational functional

$$\mathcal{L}(y) = h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} L(kh, y(kh + h), \Delta_h y(kh))$$

subject to the boundary conditions $y(a) = y(b) = 0$, which attains a local minimum at $\dot{y}(kh) \equiv 0$ under the strengthened Legendre condition

$$R(kh, 0, 0) + 2hQ_x(kh, 0) + h^2 P_{xx}(kh, 0) + R(kh + h, 0, 0) > 0,$$

$kh \in [a, b - 2h] \cap h\mathbb{Z}$. Functions $r(t)$ and $s(t)$ (see (3.11)) have the following form:

$$r(t) = \frac{-2}{h} \in \mathcal{R}, \quad s(t) = \frac{p(t)}{h} - (2Q_x(t, 0) + hP_{xx}(t, 0)).$$

Hence,

$$\int_a^t P_x(\tau, 0) \Delta \tau = h \sum_{i=\frac{a}{h}}^{\frac{t}{h}-1} P_x(ih, 0),$$

$$\int_a^t e_r(t, \sigma(\tau)) s(\tau) \Delta \tau = \sum_{i=\frac{a}{h}}^{\frac{t}{h}-1} (-1)^{\frac{t}{h}-i-1} \left( p(ih) - 2hQ_x(ih, 0) - h^2 P_{xx}(ih, 0) \right).$$
Therefore, the Lagrangian $L$ of the variational functional (3.15) on $\mathbb{T} = h\mathbb{Z}$ has the form
\[
L(kh, y(kh + h), \Delta h y(kh)) = P(kh, y(kh + h)) + \left( C + \sum_{i=\frac{n}{k}}^{k-1} h P_{x}(ih, 0) + q(kh, y(kh + h)) - q(kh, 0) \right) \Delta h y(kh)
+ \frac{1}{2} \left( (-1)^{k-\frac{1}{2}} R_{0} + \sum_{i=\frac{n}{k}}^{k-1} (-1)^{k-i-1} \left( p(ih) - 2h Q_{x}(ih, 0) - h^{2} P_{xx}(ih, 0) \right) \right)
+ w(kh, y(kh + h), \Delta h y(kh)) - w(kh, 0, 0) \right) (\Delta h y(kh))^{2},
\]
where functions $P, p, q, w$ are arbitrary but satisfy assumptions of Theorem 3.1.

Now we consider the $q$-scale $\mathbb{T} = q^{N_{0}}, q > 1$. In order to present the form of Lagrangian $L$, we use Remark 3.5.

**Example 3.7.** Let $\mathbb{T} = q^{N_{0}} = \{ q^{k} : q > 1, k \in \mathbb{N}_{0} \}$ and $a, b \in \mathbb{T}$ with $a < b$. We consider the variational functional
\[
\mathcal{L}(y) = (q - 1) \sum_{t \in [a, b)} t L(t, y(qt), \Delta_{q} y(t))
\]
subject to the boundary conditions $y(a) = y(b) = 0$, which attains a local minimum at $\hat{y}(t) \equiv 0$ under the strengthened Legendre condition
\[
R(t, 0, 0) + (q - 1)t \{ 2Q_{x}(t, 0) + (q - 1)t P_{xx}(t, 0) \} + \frac{1}{q} R(qt, 0, 0) > 0
\]
at the null extremal, $t \in \left[ \frac{a}{q}, \frac{b}{q} \right] \cap q^{N_{0}}$. Functions given by (3.11) may be written as
\[
r(t) = \frac{q + 1}{t(1 - q)}, \quad s(t) = \frac{q p(t)}{t(q - 1)} - 2q Q_{x}(t, 0) - q(q - 1)t P_{xx}(t, 0).
\]
Hence,
\[
\int_{a}^{t} P_{x}(\tau, 0) \Delta \tau = (q - 1) \sum_{\tau \in [a, t)} \tau P_{x}(\tau, 0), \quad e_{r}(t, a) = \prod_{s \in [a, t)} (-q),
\]
\[
\int_{a}^{t} e_{r}(a, \sigma(\tau)) s(\tau) \Delta \tau
= \sum_{\tau \in [a, t)} \frac{(1 - q)\tau}{q} \prod_{s \in [a, \tau)} (-q) \left[ \frac{q p(\tau)}{\tau(q - 1)} - 2q Q_{x}(\tau, 0) - q(q - 1)\tau P_{xx}(\tau, 0) \right].
\]
Therefore, the Lagrangian \( L \) of the variational functional (3.16) has the form

\[
L(t, y(qt), \Delta_q y(t)) = P(t, y(qt)) + \left( C + (q - 1) \sum_{\tau \in [a, t]} \tau P_x(\tau, 0) + q(t, y(qt)) - q(t, 0) \right) \Delta_q y(t) + \left\{ \prod_{s \in [a, t]} (-q) \right\}
\]

\[
\times \left[ R_0 + \sum_{\tau \in [a, t]} \frac{(1-q)\tau}{q} \prod_{s \in [a, \tau]} (-q) \left( \frac{qp(\tau)}{\tau(q-1)} - 2qQ_x(\tau, 0) - q(q-1)\tau P_{xx}(\tau, 0) \right) \right]
\]

\[
+ w(t, y(qt), \Delta_q y(t)) - w(t, 0, 0) \left\{ \frac{(\Delta_q y(t))^2}{2} \right\},
\]

where functions \( P, p, r, w \) are arbitrary but satisfy assumptions of Theorem 3.1.

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