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An Inverse Problem of the Calculus of Variations on Arbitrary Time Scales

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Abstract

We consider an inverse extremal problem for variational functionals on arbitrary time scales. Using the Euler–Lagrange equation and the strengthened Legendre condition, we derive a general form for a variational functional that attains a local minimum at a given point of the vector space.

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1 Introduction

We study an inverse problem associated with the following fundamental problem of the calculus of variations: to minimize

$$\mathcal{L}(y) = \int_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t \tag{1.1}$$

subject to the boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$, on a given time scale $\mathbb{T}$. More precisely, we describe a general form of a variational functional (1.1) having an extremum at a given function y_0 under the Euler–Lagrange and strengthened Legendre conditions on time scales [1]. Throughout the paper we assume the reader to be familiar with the basic definitions and results from the time scale theory [3, 4, 9]. For a review on general approaches to the calculus of variations on time scales see [1, 5–8, 12, 13, 17]. For analogous results in $\mathbb{T} = \mathbb{R}$ see [15, 16]. The results here obtained are new even for simple (but important) time scales like $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$.

The paper is organized as follows. In Section 2 we collect some necessary definitions and results of the delta calculus on time scales, which are used throughout the text. The main results are presented in Section 3. We find a general form of the variational functional (1.1) that solves the inverse extremal problem (Theorem 3.2). In order to illustrate our results, we present the form of the Lagrangian L on an isolated time scale (Corollary 3.4). We end by presenting the form of the Lagrangian L in the periodic time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$ (Example 3.6) and in the q-scale $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ (Example 3.7).

2 Preliminaries

In this section we introduce basic definitions and theorems that will be useful in the sequel. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Let $a, b \in \mathbb{T}$ with $a < b$. We define the interval $[a, b]$ in $\mathbb{T}$ by $[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

Definition 2.1 (See [3])**.** The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) := \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is given by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* or *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, respectively. We say that t is *isolated* if $\rho(t) < t < \sigma(t)$, that t is *dense* if $\rho(t) = t = \sigma(t)$.

Example 2.2. The two classical time scales are $\mathbb{R}$ and $\mathbb{Z}$, representing the continuous and the purely discrete time, respectively. The other standard examples are $h\mathbb{Z}$, $h > 0$, and $q^{\mathbb{N}_0}$, $q > 1$. It follows from Definition 2.1 that if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) = 0$ for all $t \in \mathbb{T}$; if $\mathbb{T} = h\mathbb{Z}$, then $\sigma(t) = t + h$ and $\mu(t) = h$ for all $t \in \mathbb{T}$; if $\mathbb{T} = q^{\mathbb{N}_0}$, then $\sigma(t) = qt$ and $\mu(t) = t(q-1)$ for all $t \in \mathbb{T}$.

Definition 2.3 (See [14])**.** A time scale $\mathbb{T}$ is said to be an isolated time scale provided given any $t \in \mathbb{T}$, there is a $\delta > 0$ such that $(t-\delta, t+\delta) \cap \mathbb{T} = \{t\}$.

Remark 2.4. If the graininess function is bounded from below by a strictly positive number, then the time scale is isolated [2]. Therefore, $h\mathbb{Z}$, $h > 0$, and $q^{\mathbb{N}_0}$, $q > 1$, are examples of isolated time scales. Note that the converse is not true. For example, $\mathbb{T} = \log(\mathbb{N})$ is an isolated time scale but its graininess function is not bounded from below by a strictly positive number.

To simplify the notation, one usually uses $f^\sigma(t) := f(\sigma(t))$. The delta derivative is defined for points from the set

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus \{\sup \mathbb{T}\} & \text{if } \rho(\sup \mathbb{T}) < \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

Definition 2.5 (See [3])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is Δ-*differentiable* at $t \in \mathbb{T}^\kappa$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

We call $f^\Delta(t)$ the Δ-*derivative* of f at t.

Example 2.6. If $\mathbb{T} = h\mathbb{Z}$, then $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}$ if, and only if,

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+h) - f(t)}{h} =: \Delta_h f(t).$$

In the particular case $h = 1$, $f^\Delta(t) = \Delta f(t)$, where Δ is the usual forward difference operator. If $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : q > 1, k \in \mathbb{N}_0\}$, then $f^\Delta(t) = \dfrac{f(qt) - f(t)}{(q-1)t} =: \Delta_q f(t)$, i.e., we get the usual Jackson derivative of quantum calculus [11].

Theorem 2.7 (See [3])**.** *Let* $f : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^\kappa$. *If* f *is delta differentiable at* t, *then*

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t).$$

Definition 2.8 (See [3])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exists (finite) at all left-dense points in $\mathbb{T}$.

The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are Δ-differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. If F is an antiderivative of f, then the Cauchy Δ-integral is defined by

$$\int_a^b f(t)\Delta t := F(b) - F(a).$$

Theorem 2.9 (See [3]). *Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^t f(\tau)\Delta\tau,$$

$t \in \mathbb{T}$, is an antiderivative of f.

Example 2.10. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and $a, b \in \mathbb{T}$ with $a < b$, then

$$\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h.$$

If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $\int_a^b f(t)\Delta t = (q-1) \sum_{t\in[a,b)\cap\mathbb{T}} tf(t)$.

Let $\mathbb{T}$ be a given time scale with at least three points. Consider the following variational problem on the time scale $\mathbb{T}$:

$$\mathcal{L}(y) = \int_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right)\Delta t \longrightarrow \min, \quad y(a) = \alpha, \quad y(b) = \beta, \tag{2.1}$$

where $a, b \in \mathbb{T}$ with $a < b$; $\alpha, \beta \in \mathbb{R}^n$ with $n \in \mathbb{N}$, and $L : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$.

Definition 2.11. We say that $y \in C^1_{rd}(\mathbb{T})$ is admissible for problem (2.1) if it satisfies the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$.

Definition 2.12. An admissible function $\hat{y}$ is called a *local minimizer* of problem (2.1) provided there exists $\delta > 0$ such that $\mathcal{L}(\hat{y}) \le \mathcal{L}(y)$ for all admissible y with $\|y - \hat{y}\|_{C^1_{rd}} < \delta$, where

$$\|f\|_{C^1_{rd}} = \sup_{t\in[a,b]^k_{\mathbb{T}}} \|f^\sigma(t)\| + \sup_{t\in[a,b]^k_{\mathbb{T}}} \|f^\triangle(t)\|$$

with $\|\cdot\|$ a norm in $\mathbb{R}^n$.

In what follows the Lagrangian L is understood as a function $(t, x, v) \to L(t, x, v)$ and by L_x and L_v we denote the partial derivatives of L with respect to x and v, respectively. Similar notation is used for second order partial derivatives.

Theorem 2.13 (The Euler–Lagrange equation [10])**.** *Assume that $L(t, \cdot, \cdot)$ is differentiable in (x, v) and $L(t, \cdot, \cdot)$, $L_x(t, \cdot, \cdot)$, $L_v(t, \cdot, \cdot)$ are continuous at (y^σ, y^Δ), uniformly in t and rd-continuous in t for any admissible y. If $\hat{y}(t)$ is a local minimizer of the variational problem* (2.1)*, then there exists a vector $c \in \mathbb{R}^n$ such that the Euler–Lagrange equation*

$$L_v\left(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\right) = \int\limits_a^t L_x(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau))\Delta\tau + c^T \tag{2.2}$$

holds for $t \in [a, b]_{\mathbb{T}}^\kappa$.

Theorem 2.14 (The Legendre condition [1])**.** *If $\hat{y}$ is a local minimizer of the variational problem* (2.1)*, then*

$$A(t) + \mu(t)\left\{C(t) + C^T(t) + \mu(t)B(t) + (\mu(\sigma(t)))^\dagger A(\sigma(t))\right\} \geq 0, \tag{2.3}$$

$t \in [a, b]_{\mathbb{T}}^{\kappa^2}$, where $A(t) = L_{vv}\left(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\right)$, $B(t) = L_{xx}\left(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\right)$, $C(t) = L_{xv}\left(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\right)$, and where $\alpha^\dagger = \dfrac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$.

Remark 2.15. If (2.3) holds with the strict inequality $>$, then it is called *the strengthened Legendre condition*.

Definition 2.16 (See [3])**.** We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0$$

holds for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Theorem 2.17 (See [4])**.** *Let $p \in \mathcal{R}$, $f \in C_{rd}$, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. Then, the unique solution of the initial value problem*

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0, \tag{2.4}$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int\limits_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau,$$

where $e_p(\cdot, \cdot)$ denotes the exponential function on time scales.

Remark 2.18 (See [3]). An alternative form of the solution of the initial value problem (2.4) is given by

$$y(t) = e_p(t, t_0) \left[y_0 + \int_{t_0}^{t} e_p(t_0, \sigma(\tau)) f(\tau) \Delta\tau \right].$$

For more properties of the delta exponential function we refer the reader to [3, 4].

3 Main Results

The problem under our consideration is to find a general form of the variational functional

$$\mathcal{L}(y) = \int_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t, \tag{3.1}$$

$L : [a, b]_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$, subject to the boundary conditions $y(a) = y(b) = 0$, possessing a local minimum at zero, under the Euler–Lagrange and the strengthened Legendre conditions. We assume that $L(t, \cdot, \cdot)$ is a C^2-function with respect to (x, v) uniformly in t, and L, L_x, L_v, $L_{vv} \in C_{rd}$ for any admissible path $y(\cdot)$. Observe that under our assumptions, by Taylor's theorem, we may write L, with the big O notation, in the form

$$L(t, x, v) = P(t, x) + Q(t, x)v + \frac{1}{2}R(t, x, 0)v^2 + O(v^3), \tag{3.2}$$

where

$$\begin{aligned} P(t, x) &= L(t, x, 0), \\ Q(t, x) &= L_v(t, x, 0), \\ R(t, x, 0) &= L_{vv}(t, x, 0). \end{aligned} \tag{3.3}$$

Let $R(t, x, v) = R(t, x, 0) + O(v)$. Then, one can write (3.2) as

$$L(t, x, v) = P(t, x) + Q(t, x)v + \frac{1}{2}R(t, x, v)v^2. \tag{3.4}$$

Now the idea is to find general forms of $P(t, y^\sigma(t))$, $Q(t, y^\sigma(t))$ and $R(t, y^\sigma(t), y^\Delta(t))$ using the Euler–Lagrange and the strengthened Legendre conditions. Note that the Euler–Lagrange equation (2.2) at the null extremal, with notation (3.3), is

$$Q(t, 0) = \int_a^t P_x(\tau, 0)\Delta\tau + C, \tag{3.5}$$

$t \in [a,b]^{\kappa}_{\mathbb{T}}$. Therefore, choosing an arbitrary function $P(t, y^\sigma(t))$ such that $P(t,\cdot) \in C^2$ with respect to the second variable, uniformly in t, P and P_x are rd-continuous in t for all admissible y, and by (3.5) we can write a general form of Q:

$$Q(t, y^\sigma(t)) = C + \int_a^t P_x(\tau, 0)\Delta\tau + q(t, y^\sigma(t)) - q(t, 0), \tag{3.6}$$

where $C \in \mathbb{R}$ and q is an arbitrary function such that $q(t,\cdot) \in C^2$ with respect to the second variable, uniformly in t, and q and q_x are rd-continuous in t for all admissible y. With notation (3.3), the strengthened Legendre condition (2.3) at the null extremal has the form

$$R(t,0,0) + \mu(t)\left\{2Q_x(t,0) + \mu(t)P_{xx}(t,0) + (\mu^\sigma(t))^\dagger R(\sigma(t),0,0)\right\} > 0, \tag{3.7}$$

$t \in [a,b]^{\kappa^2}_{\mathbb{T}}$, where $\alpha^\dagger = \dfrac{1}{\alpha}$ if $\alpha \in \mathbb{R}\setminus\{0\}$ and $0^\dagger = 0$. Hence, we set

$$R(t,0,0) + \mu(t)\left\{2Q_x(t,0) + \mu(t)P_{xx}(t,0) + (\mu^\sigma(t))^\dagger R(\sigma(t),0,0)\right\} = p(t) \tag{3.8}$$

with $p \in C^1_{rd}([a,b]_{\mathbb{T}})$, $p(t) > 0$ for all $t \in [a,b]^{\kappa^2}_{\mathbb{T}}$, chosen arbitrarily. Note that there exists a unique solution of (3.8) with respect to $R(t,0,0)$. If t is a right-dense point, then $\mu(t) = 0$ and $R(t,0,0) = p(t)$. Otherwise, $\mu(t) \neq 0$, and using Theorem 2.7 with $f(t) = R(t,0,0)$ we modify equation (3.8) into a first order delta dynamic equation, which has a unique solution $R(t,0,0)$ in agreement with Theorem 2.17 (see details in the proof of Corollary 3.4). We derive a general form of R from Legendre's condition (3.7), as a sum of the solution $R(t,0,0)$ of equation (3.8) and function w, which is chosen arbitrarily in such a way that $w(t,\cdot,\cdot) \in C^2$ with respect to the second and the third variable, uniformly in t; w_x, w_v and w_{vv} are rd-continuous in t for all admissible y. Concluding: a general form of the integrand L for functional (3.1) follows from (3.4), (3.6) and (3.8), and is given by

$$\begin{aligned} L\left(t, y^\sigma(t), y^\Delta(t)\right) &= P\left(t, y^\sigma(t)\right) \\ &+ \left(C + \int_a^t P_x(\tau,0)\Delta\tau + q(t,y^\sigma(t)) - q(t,0)\right) y^\Delta(t) \\ &+ \Bigg(p(t) - \mu(t)\left\{2Q_x(t,0) + \mu(t)P_{xx}(t,0) + (\mu^\sigma(t))^\dagger R(\sigma(t),0,0)\right\} \\ &+ w(t, y^\sigma(t), y^\Delta(t)) - w(t,0,0)\Bigg)\frac{y^\Delta(t)^2}{2}. \end{aligned} \tag{3.9}$$

We have just proved the following result.

Theorem 3.1. *Let $\mathbb{T}$ be an arbitrary time scale. If functional* (3.1) *with boundary conditions $y(a) = y(b) = 0$ attains a local minimum at $\hat{y}(t) \equiv 0$ under the strengthened Legendre condition, then its Lagrangian L takes the form* (3.9)*, where $R(t, 0, 0)$ is a solution of equation* (3.8)*, $C \in \mathbb{R}$, $\alpha^\dagger = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$. Functions P, p, q and w are arbitrary functions satisfying:*

(i) $P(t, \cdot), q(t, \cdot) \in C^2$ *with respect to the second variable uniformly in t; P, P_x, q, q_x are rd-continuous in t for all admissible y; $P_{xx}(\cdot, 0)$ is rd-continuous in t; $p \in C^1_{rd}$ with $p(t) > 0$ for all $t \in [a, b]^{\kappa^2}_{\mathbb{T}}$;*

(ii) $w(t, \cdot, \cdot) \in C^2$ *with respect to the second and the third variable, uniformly in t, w_x, w_v, w_{vv} are rd-continuous in t for all admissible y.*

Now we consider the general situation when the variational problem consists in minimizing (3.1) subject to arbitrary boundary conditions $y(a) = y_0(a)$ and $y(b) = y_0(b)$, for a certain given function $y_0 \in C^2_{rd}([a, b]_{\mathbb{T}})$.

Theorem 3.2. *Let $\mathbb{T}$ be an arbitrary time scale. If the variational functional* (3.1) *with boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$, attains a local minimum for a certain given function $y_0(\cdot) \in C^2_{rd}([a, b]_{\mathbb{T}})$ under the strengthened Legendre condition, then its Lagrangian L has the form*

$$
\begin{aligned}
&L\left(t, y^\sigma(t), y^\Delta(t)\right) = P\left(t, y^\sigma(t) - y_0^\sigma(t)\right) + \left(y^\Delta(t) - y_0^\Delta(t)\right) \\
&\times \left(C + \int_a^t P_x\left(\tau, -y_0^\sigma(\tau)\right) \Delta\tau + q\left(t, y^\sigma(t) - y_0^\sigma(t)\right) - q\left(t, -y_0^\sigma(t)\right) \right) \\
&+ \frac{1}{2}\Bigg(p(t) - \mu(t) \left\{ 2Q_x(t, 0) + \mu(t) P_{xx}(t, 0) + \left(\mu^\sigma(t)\right)^\dagger R(\sigma(t), 0, 0) \right\} \\
&+ w(t, y^\sigma(t) - y_0^\sigma(t), y^\Delta(t) - y_0^\Delta(t)) - w\left(t, -y_0^\sigma(t), -y_0^\Delta(t)\right) \Bigg) \left(y^\Delta(t) - y_0^\Delta(t)\right)^2,
\end{aligned}
$$

where $R(t, 0, 0)$ is the solution of equation (3.8)*, $C \in \mathbb{R}$ and functions P, p, q, w satisfy conditions (i) and (ii) of Theorem 3.1.*

Proof. The result follows as a corollary of Theorem 3.1. In order to reduce the problem to the case of null boundary conditions $y(a) = 0$ and $y(b) = 0$, we introduce the auxiliary variational functional

$$
\begin{aligned}
\tilde{\mathcal{L}}(y) := \mathcal{L}(y + y_0) &= \int_a^b L\left(t, y^\sigma(t) + y_0^\sigma(t), y^\Delta(t) + y_0^\Delta(t)\right) \Delta t \\
&=: \int_a^b \tilde{L}\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t
\end{aligned}
$$

subject to boundary conditions $y(a) = 0$ and $y(b) = 0$. The result follows by application of Theorem 3.1 to the auxiliary Lagrangian $\tilde{L}$. □

For the classical situation $\mathbb{T} = \mathbb{R}$, Theorem 3.2 gives a recent result of [15].

Corollary 3.3 (See [15, Theorem 4])**.** *If the variational functional*

$$\mathcal{L}(y) = \int_a^b L(t, y(t), y'(t))dt$$

attains a local minimum at $y_0(\cdot) \in C^2([a, b])$ *satisfying boundary conditions* $y(a) = y_0(a)$ *and* $y(b) = y_0(b)$ *and the classical Legendre condition* $R(t, y_0(t), y_0'(t)) > 0$, $t \in [a, b]$*, then its Lagrangian* L *has the form*

$$\begin{aligned} &L(t, y(t), y'(t)) = P(t, y(t) - y_0(t)) \\ &+ (y'(t) - y_0'(t)) \left(C + \int_a^t P_x(\tau, -y_0(\tau))d\tau + q(t, y(t) - y_0(t)) - q(t, -y_0(t)) \right) \\ &+\frac{1}{2} \left(p(t) + w(t, y(t) - y_0(t), y'(t) - y_0'(t)) - w(t, -y_0(t), -y_0'(t))\right) (y'(t)-y_0'(t))^2, \end{aligned}$$

where $C \in \mathbb{R}$.

Proof. Follows from Theorem 3.2 with $\mathbb{T} = \mathbb{R}$. □

Theorem 3.2 seems to be new for any time scale other than $\mathbb{T} = \mathbb{R}$. In the particular case of an isolated time scale, where $\mu(t) \neq 0$ for all $t \in \mathbb{T}$, we get the following corollary.

Corollary 3.4. *Let* $\mathbb{T}$ *be an isolated time scale. If functional* (3.1) *subject to the boundary conditions* $y(a) = y(b) = 0$ *attains a local minimum at* $\hat{y}(t) \equiv 0$ *under the strengthened Legendre condition, then the Lagrangian* L *has the form*

$$\begin{aligned} &L\left(t, y^\sigma(t), y^\Delta(t)\right) = P\left(t, y^\sigma(t)\right) \\ &+ \left(C + \int_a^t P_x(\tau, 0)\Delta\tau + q(t, y^\sigma(t)) - q(t, 0) \right) y^\Delta(t) \\ &+ \left(e_r(t, a)R_0 + \int_a^t e_r(t, \sigma(\tau))s(\tau)\Delta\tau + w(t, y^\sigma(t), y^\Delta(t)) - w(t, 0, 0) \right) \frac{y^\Delta(t)^2}{2}, \end{aligned} \tag{3.10}$$

where $C, R_0 \in \mathbb{R}$ *and* $r(t)$ *and* $s(t)$ *are given by*

$$r(t) := -\frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu^2(t)(\mu^\sigma(t))^\dagger}, \quad s(t) := \frac{p(t) - \mu(t)[2Q_x(t, 0) + \mu(t)P_{xx}(t, 0)]}{\mu^2(t)(\mu^\sigma(t))^\dagger}, \tag{3.11}$$

with $\alpha \in \mathbb{R} \setminus \{0\}$ *and* $0^\dagger = 0$*, where functions* P*,* p*,* q*,* w *satisfy assumptions of Theorem 3.1.*

Proof. In the case of an isolated time scale $\mathbb{T}$, we may obtain the form of function Q in the same way as it was done in the proof of Theorem 3.1. We derive a general form for R from Legendre's condition. By relation $f^\sigma = f + \mu f^\Delta$ (Theorem 2.7), one may write equation (3.8) as

$$R(t,0,0) + \mu(t)(\mu^\sigma(t))^\dagger \left(R(t,0,0) + \mu(t)R^\Delta(t,0,0)\right) \\ + \mu(t)\left\{2Q_x(t,0) + \mu(t)P_{xx}(t,0)\right\} - p(t) = 0.$$

Hence,

$$\mu^2(t)(\mu^\sigma(t))^\dagger R^\Delta(t,0,0) + \left[1 + \mu(t)(\mu^\sigma(t))^\dagger\right] R(t,0,0) \\ + \mu(t)[2Q_x(t,0) + \mu(t)P_{xx}(t,0)] - p(t) = 0. \quad (3.12)$$

For an isolated time scale $\mathbb{T}$, equation (3.12) is a first order delta dynamic equation of the following form:

$$R^\Delta(t,0,0) + \frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu^2(t)(\mu^\sigma(t))^\dagger} R(t,0,0) + \frac{\mu(t)[2Q_x(t,0) + \mu(t)P_{xx}(t,0)] - p(t)}{\mu^2(t)(\mu^\sigma(t))^\dagger} = 0.$$

With notation (3.11) we have

$$R^\Delta(t,0,0) = r(t)R(t,0,0) + s(t). \quad (3.13)$$

Observe that $r(t)$ is regressive. Indeed, if $\mu(t) \neq 0$, then

$$1 + \mu(t)r(t) = 1 - \frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu(t)(\mu^\sigma(t))^\dagger} = 1 - \frac{\mu^\sigma(t) + \mu(t)}{\mu(t)} = -\frac{\mu^\sigma(t)}{\mu(t)} \neq 0$$

for all $t \in [a,b]^\kappa$. Therefore, by Theorem 2.17, there is a unique solution to equation (3.13) with initial condition $R(a,0,0) = R_0 \in \mathbb{R}$:

$$R(t,0,0) = e_r(t,a)R_0 + \int_a^t e_r(t,\sigma(\tau))s(\tau)\Delta\tau. \quad (3.14)$$

Thus, a general form of the integrand L for functional (3.1) is given by (3.10). □

Remark 3.5. Instead of (3.14), we can use an alternative form of the solution of the initial value problem (3.13) subject to $R(a,0,0) = R_0$ (cf. Remark 2.18):

$$R(t,0,0) = e_r(t,a)\left[R_0 + \int_a^t e_r(a,\sigma(\tau))s(\tau)\Delta\tau\right].$$

Then the Lagrangian L (3.10) can be written as

$$\begin{aligned} L\left(t, y^\sigma(t), y^\Delta(t)\right) &= P\left(t, y^\sigma(t)\right) \\ &+ \left(C + \int_a^t P_x(\tau,0)\Delta\tau + q(t, y^\sigma(t)) - q(t,0)\right) y^\Delta(t) \\ &+ \left(e_r(t,a)\left[R_0 + \int_a^t e_r(a,\sigma(\tau))s(\tau)\Delta\tau\right]\right) \frac{y^\Delta(t)^2}{2} \\ &+ \left(w(t, y^\sigma(t), y^\Delta(t)) - w(t,0,0)\right) \frac{y^\Delta(t)^2}{2}. \end{aligned}$$

Based on Corollary 3.4, we present the form of Lagrangian L in the periodic time scale $\mathbb{T} = h\mathbb{Z}$.

Example 3.6. Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and $a, b \in h\mathbb{Z}$ with $a < b$. Then $\mu(t) \equiv h$. We consider the variational functional

$$\mathcal{L}(y) = h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} L\left(kh, y(kh+h), \Delta_h y(kh)\right) \tag{3.15}$$

subject to the boundary conditions $y(a) = y(b) = 0$, which attains a local minimum at $\hat{y}(kh) \equiv 0$ under the strengthened Legendre condition

$$R(kh,0,0) + 2hQ_x(kh,0) + h^2 P_{xx}(kh,0) + R(kh+h,0,0) > 0,$$

$kh \in [a, b-2h] \cap h\mathbb{Z}$. Functions $r(t)$ and $s(t)$ (see (3.11)) have the following form:

$$r(t) = \frac{-2}{h} \in \mathcal{R}, \quad s(t) = \frac{p(t)}{h} - \left(2Q_x(t,0) + hP_{xx}(t,0)\right).$$

Hence,

$$\int_a^t P_x(\tau,0)\Delta\tau = h \sum_{i=\frac{a}{h}}^{\frac{t}{h}-1} P_x(ih,0),$$

$$\int_a^t e_r(t,\sigma(\tau))s(\tau)\Delta\tau = \sum_{i=\frac{a}{h}}^{\frac{t}{h}-1} (-1)^{\frac{t}{h}-i-1} \left(p(ih) - 2hQ_x(ih,0) - h^2 P_{xx}(ih,0)\right).$$

Therefore, the Lagrangian L of the variational functional (3.15) on $\mathbb{T} = h\mathbb{Z}$ has the form

$$\begin{aligned} L\left(kh, y(kh+h), \Delta_h y(kh)\right) &= P\left(kh, y(kh+h)\right) \\ &+ \left(C + \sum_{i=\frac{a}{h}}^{k-1} hP_x(ih,0) + q(kh, y(kh+h)) - q(kh,0) \right) \Delta_h y(kh) \\ &+ \frac{1}{2}\Bigg((-1)^{k-\frac{a}{h}} R_0 + \sum_{i=\frac{a}{h}}^{k-1} (-1)^{k-i-1} \left(p(ih) - 2hQ_x(ih,0) - h^2 P_{xx}(ih,0) \right) \\ &+ w(kh, y(kh+h), \Delta_h y(kh)) - w(kh,0,0) \Bigg) \left(\Delta_h y(kh)\right)^2, \end{aligned}$$

where functions P, p, q, w are arbitrary but satisfy assumptions of Theorem 3.1.

Now we consider the q-scale $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. In order to present the form of Lagrangian L, we use Remark 3.5.

Example 3.7. Let $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : q > 1, k \in \mathbb{N}_0\}$ and $a, b \in \mathbb{T}$ with $a < b$. We consider the variational functional

$$\mathcal{L}(y) = (q-1) \sum_{t \in [a,b)} tL\left(t, y(qt), \Delta_q y(t)\right) \tag{3.16}$$

subject to the boundary conditions $y(a) = y(b) = 0$, which attains a local minimum at $\hat{y}(t) \equiv 0$ under the strengthened Legendre condition

$$R(t,0,0) + (q-1)t\{2Q_x(t,0) + (q-1)tP_{xx}(t,0)\} + \frac{1}{q}R(qt,0,0) > 0$$

at the null extremal, $t \in \left[a, \frac{b}{q^2}\right] \cap q^{\mathbb{N}_0}$. Functions given by (3.11) may be written as

$$r(t) = \frac{q+1}{t(1-q)}, \quad s(t) = \frac{qp(t)}{t(q-1)} - 2qQ_x(t,0) - q(q-1)tP_{xx}(t,0).$$

Hence,

$$\int_a^t P_x(\tau,0)\Delta\tau = (q-1) \sum_{\tau \in [a,t)} \tau P_x(\tau,0), \quad e_r(t,a) = \prod_{s \in [a,t)} (-q),$$

$$\begin{aligned} &\int_a^t e_r(a, \sigma(\tau)) s(\tau) \Delta\tau \\ &\qquad = \sum_{\tau \in [a,t)} \frac{(1-q)\tau}{q \prod\limits_{s \in [a,\tau)} (-q)} \left[\frac{qp(\tau)}{\tau(q-1)} - 2qQ_x(\tau,0) - q(q-1)\tau P_{xx}(\tau,0) \right]. \end{aligned}$$

Therefore, the Lagrangian L of the variational functional (3.16) has the form

$$
\begin{aligned}
&L(t, y(qt), \Delta_q y(t)) = P(t, y(qt)) \\
&+ \left(C + (q-1) \sum_{\tau \in [a,t)} \tau P_x(\tau, 0) + q(t, y(qt)) - q(t, 0) \right) \Delta_q y(t) + \Bigg\{ \prod_{s \in [a,t)} (-q) \\
&\times \left[R_0 + \sum_{\tau \in [a,t)} \frac{(1-q)\tau}{q \prod\limits_{s \in [a,\tau)} (-q)} \left(\frac{qp(\tau)}{\tau(q-1)} - 2qQ_x(\tau, 0) - q(q-1)\tau P_{xx}(\tau, 0) \right) \right] \\
&\qquad + w\left(t, y(qt), \Delta_q y(t)\right) - w(t, 0, 0) \Bigg\} \frac{(\Delta_q y(t))^2}{2},
\end{aligned}
$$

where functions P, p, r, w are arbitrary but satisfy assumptions of Theorem 3.1.

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