The \(q\)-deformation of Hyperbolic and Trigonometric Potentials

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Abstract

We present solutions of \(q\)-deformed Schrödinger equation for the hyperbolic and trigonometric potentials given by a factorization method. Their various properties including the correspondence \(q \to 1\) to the non–deformed Schrödinger operators are discussed.

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1 Introduction

The paper deals with a second order \(q\)-difference equation such that in the limit \(q \to 1\) we get a differential equation related to the stationary Schrödinger equation with the Scarf potentials [14].

We shall consider eigenproblems

\[ H_k \psi_k = \lambda_k \psi_k, \]  

(1.1)

for the discrete family \(k \in \mathbb{N} \cup \{0\}\) of the second order \(q\)-difference operators

\[ H_k = Z_k(x)\partial_q Q^{-1}\partial_q + W_k(x)\partial_q + V_k(x), \quad k \in \mathbb{N} \cup \{0\} \]  

(1.2)

acting in the Hilbert spaces \(\mathcal{H}_k\), where \(0 < q < 1\) and \(\partial_q\) is the \(q\)-derivative (see [9])

\[ \partial_q \psi(x) = \frac{\psi(x) - \psi(qx)}{(1-q)x}. \]  

(1.3)
By definition, $\mathcal{H}_k$ consists of the complex valued functions $\psi : [a, b]_q \rightarrow \mathbb{C}$ defined on the $q$-interval $[a, b]_q := \{ q^n a : n \in \mathbb{N} \cup \{ 0 \} \} \cup \{ q^n b : n \in \mathbb{N} \cup \{ 0 \} \}$, $a, b \in \mathbb{R}$, $a < b$, $a \neq q^n b$, which are square–integrable with respect to the scalar products

$$
\langle \psi | \varphi \rangle_k := \int_a^b \psi(x) \varphi(x) \varrho_k(x) dx.
$$

(1.4)

Let us recall that the $q$-integral is given by

$$
\int_a^b \psi(x) d_q x := \sum_{n=0}^{\infty} (1 - q^n) q^n (b\psi(q^n b) - a\psi(q^n a)),
$$

(1.5)

$$
\int_{-\infty}^{\infty} \psi(x) d_q x = \sum_{n=-\infty}^{\infty} (1 - q^n) q^n \psi(q^n) + \sum_{n=-\infty}^{\infty} (1 - q^n) q^n \psi(-q^n)
$$

and the shift operators $Q, Q^{-1}$ are given by

$$
Q \psi(x) = \psi(qx),
$$

(1.6)

$$
Q^{-1} \psi(x) = \psi(q^{-1}x).
$$

(1.7)

We postulate some conditions which must be satisfied by weight functions,

$$
g_{k-1} = Q (B_k \varrho_k),
$$

(1.8)

$$
\partial_q (B_k \varrho_k) = A_k \varrho_k,
$$

(1.9)

where $A_k, B_k$ are real valued functions on $[a, b]_q$. The equation (1.9) corresponds in the limit $q \rightarrow 1$ to the Pearson equation, which is important for the theory of classical orthogonal polynomials. Additionally we impose the boundary conditions

$$
B_k(a) \varrho_k(a) = B_k(b) \varrho_k(b) = 0.
$$

(1.10)

The basic idea of the factorization method is well known, see [10–12]. Some results concerning the factorization method for the second order $q$-difference equations were presented for example in papers [3, 13]. In this work, we apply our previous results obtained in the paper [6]. We will factorize the chain of second order $q$-difference operators (1.2) using the first order $q$-difference operators of the form

$$
A_k = \partial_q + f_k,
$$

(1.11)

$$
A_k^* = (\partial_q + f_k)^* = B_k (-\partial_q Q^{-1} + f_k) - A_k (1 + (1 - q)x f_k).
$$

(1.12)

The operator $A_k^* : \mathcal{H}_{k-1} \rightarrow \mathcal{H}_k$ is adjoint to $A_k : \mathcal{H}_k \rightarrow \mathcal{H}_{k-1}$ with respect to the scalar products (1.4). We say that the operators $\mathcal{H}_k$ admit a factorization if there exist sequences of operators $A_k, A_k^*$ and constants $a_k$ such that

$$
\mathcal{H}_k = A_k^* A_k + a_k = A_{k+1}^* A_{k+1} + a_{k+1} \quad \text{for} \quad k \in \mathbb{N} \cup \{ 0 \}.
$$

(1.13)
In papers [5, 7] we considered the \( q \)-deformation of the Schrödinger operator with the following potentials: shifted oscillator, isotropic oscillator, Rosen–Morse II, Eckart II, Poschl–Teller I and II. Next in the paper [8] we considered the case of \( q \)-deformed Schrödinger equation for the Morse potential. In this work we will study the case of \( q \)-deformed Schrödinger equation for the \( q \)-hyperbolic and \( q \)-trigonometric Scarf potentials.

2 The \( q \)-hyperbolic and \( q \)-trigonometric Potentials

We obtain the case of the \( q \)-deformation of hyperbolic and trigonometric potentials if we consider the operators \( H_k \) in the following forms:

\[
H_k = -q^k \left( b^2 x^2 + b_0 \right) \sqrt{\frac{\left( q b_2 + \frac{(a_0 - a_1)}{(1-q)^{-1}} \right) q^{-2k-1} x^2 + (1-q) h q^{-k-1} x + b_0}{b_2 q^{-2k-2} x^2 + b_0}} \partial_q Q^{-1} \partial_q
\]

\[
+ \frac{q^k}{(1-q)x}
\]

\[
\times \left( \left( \frac{b_2 + \frac{(a_0 - a_1)}{(1-q)^{-1}}}{q^{-2k+1} x^2 + (1-q) h q^{-k} x + b_0} \right) \left( b_2 q^{-2k} x^2 + b_0 \right) + \frac{q^{k+1} \left( b^2 x^2 + b_0 \right)}{(1-q)^2 x^2} \right)
\]

\[
\times \sqrt{\left( \left( q b_2 + \frac{(a_0 - a_1)}{(1-q)^{-1}} \right) q^{-2k-1} x^2 + (1-q) h q^{-k-1} x + b_0 \right) \left( b_2 q^{-2k-2} x^2 + b_0 \right)} - \frac{q^k}{(1-q)^2 x^2}
\]

\[
\times \left( \left( \frac{b_2 + \frac{(a_0 - a_1)}{(1-q)^{-1}}}{q^{-2k+1} x^2 + (1-q) h q^{-k} x + b_0} \right) \left( b_2 q^{-2k} x^2 + b_0 \right) + \frac{q^k}{(1-q)^2 x^2} \right)
\]

\[
\times \left( \left( \frac{b_2 + \frac{(a_0 - a_1)}{(1-q)^{-1}}}{q^{-2k+1} x^2 + (1-q) h q^{-k} x + b_0} \right) \left( b_2 q^{-2k} x^2 + b_0 \right) + \frac{q^{k+1} \left( b^2 x^2 + b_0 \right)}{(1-q)^2 x^2} + q^{1-k} \left( a_1[k]q - a_0[k - 1]q - q b_2[k - 1]q[k]q \right) \right).
\]
where

\[ [k]_q = \frac{1 - q^k}{1 - q}. \]

This chain is factorized using first order \( q \)-difference operators

\[ A_k = -\frac{1}{(1 - q)x}Q + \sqrt{\frac{(qb_2 + \frac{(a_0 - a_1)}{(1 - q)}q^{-2k+1}x^2 + (1 - q)hq^{-k}x + b_0)}{(1 - q)^2x^2(b_2q^{-2k}x^2 + b_0)}}, \]  
(2.2)

\[ A_k^* = -q^k(q^{2k} - b_2x^2 + b_0)(1 - q)_x Q^{-1} - \frac{q^k}{(1 - q)_x}\sqrt{b_2q^{-2k}x^2 + b_0} \]
\[ \times \sqrt{\left(qb_2 + \frac{(a_0 - a_1)}{(1 - q)}\right)q^{-2k+1}x^2 + (1 - q)hq^{-k}x + b_0}. \]  
(2.3)

The constants \( a_k \) in the factorization relations (1.13) are given by the following formula

\[ a_k = q^{1-k}(a_k[k]_q - a_0[k - 1]_q - qb_2[k - 1]_q[k]_q). \]  
(2.4)

In this case the functions \( B_k, A_k \) and \( f_k \) are given by

\[ B_k(x) = q^k\left(b_2x^2 + b_0\right), \]  
(2.5)

\[ A_k(x) = q^k[-2k]_q b_2x, \]  
(2.6)

\[ f_k(x) = \sqrt{\left(qb_2 + \frac{(a_0 - a_1)}{(1 - q)}\right)q^{-2k+1}x^2 + (1 - q)hq^{-k}x + b_0} \]
\[ \frac{1}{(1 - q)_x} - \frac{q^k}{(1 - q)_x}\sqrt{b_2q^{-2k}x^2 + b_0}. \]  
(2.7)

Solving the equation (1.9) we obtain the weight function

\[ \varrho_k(x) = \frac{1}{(-q^{-2k}x^2b_2^2; q^2)_{k+1}}, \]  
(2.8)

where

\[ (x; q^2)_{k+1} = (1 - x)(1 - q^2x)\ldots(1 - q^{2k}x). \]

Substituting (2.5) and (2.8) into the boundary condition (1.10) we choose as the \( q \)-interval \( [a, b]_q = [-\infty, \infty]_q \).

It is easy to see that any solution of the equation

\[ A_k\psi_0^k(x) = 0 \]  
(2.9)

is automatically a solution of equation (1.1) with the eigenvalue \( \lambda_k = a_k \). Solving the equation (2.9) we obtain

\[ \psi_0^k(x) = C\sqrt{\prod_{i=-k}^{\infty}(qb_2 + \frac{(a_0 - a_1)}{(1 - q)}q^{2i}x^2 + (1 - q)hq^i x + b_0)}, \]  
(2.10)
where \( C \in \mathbb{R} \). The solutions (2.10) can be used to construct solutions

\[
\psi_{k+n} = \mathbf{A}_{k+n}^* \cdots \mathbf{A}_{k+1}^* \psi_k^0
\]

of the eigenvalue problems for the operators \( H_{k+n} \) given by (2.1).

3 Limit Case

The second order \( q \)-difference operator (2.1) in the limit case as \( q \to 1 \) becomes the second order differential operator

\[
H_k = -\left( b_2 x^2 + b_0 \right) \frac{d^2}{dx^2} + 2kb_2 x \frac{d}{dx} + \frac{-2kb_2^2 x^2 + kb_2(a_0 - a_1)x^2 + kb_2hx}{b_2 x^2 + b_0} \\
+ \frac{(a_0 - a_1)x^2 + 2h(a_0 - a_1)x + h^2 + 4b_2 b_0}{4(b_2 x^2 + b_0)} - \frac{1}{2}(a_0 - a_1)
\]

(3.1)

\[
+ ka_1 - (k - 1)a_0 - k(k - 1)b_2.
\]

The chains of operators (1.11) and (1.12) in the limit are given by

\[
\mathbf{A}_k = \frac{d}{dx} + \frac{-2b_2x + (a_0 - a_1)x + h}{2(b_2 x^2 + b_0)},
\]

(3.2)

\[
\mathbf{A}_k^* = -(b_2 x^2 + b_0) \frac{d}{dx} + (2k - 1)b_2 x + \frac{a_0 - a_1}{2} x + \frac{h}{2}.
\]

(3.3)

The constants \( a_k \) (2.4) are transformed into

\[
a_k = ka_1 - (k - 1)a_0 - k(k - 1)b_2.
\]

(3.4)

The ground state (2.10) tends to

\[
\psi^0_k(x) = |b_2 x^2 + b_0|^{\frac{2b_2 + a_1 - a_0}{4b_2}} \exp \left( -\frac{h}{2b_0} \int \frac{dx}{1 + \frac{b_2}{b_0} x^2} \right)
\]

and the weight function (2.8) to

\[
\varrho_k(x) = |b_2 x^2 + b_0|^{-(k+1)}.
\]

In order to systematize the class of the potentials given by \( q \)-deformation we shall transform the considered differential operator (3.1) into the standard form

\[
\tilde{H}_k = -\frac{d^2}{dy^2} + V_k(x(y)),
\]

(3.5)

where

\[
V_k(x) = \frac{d_2 x + d_1}{b_2 x^2 + b_0} + d_0
\]

(3.6)
and

\[ d_2 = \hbar \left( kb_2 + \frac{a_0 - a_1}{2} \right), \quad (3.7) \]
\[ d_1 = \frac{h^2}{4} + \frac{1}{4} b_2 b_0 - \left( \left( k^2 + \frac{3}{4} \right) b_2 + (a_0 - a_1) \left( k + \frac{1}{4 b_2} \right) \right) b_0, \quad (3.8) \]
\[ d_0 = \frac{a_0 - a_1}{4 b_2} + \frac{a_1 + a_0}{2} + \frac{1}{4} b_2. \quad (3.9) \]

The transformation connecting these operators and these eigenvalue problems are following

\[ \psi_k(x) = (b_2 x^2 + b_0) \frac{2k+1}{2} \varphi_k(y), \quad (3.10) \]
\[ dy = \frac{dx}{\sqrt{b_2 x^2 + b_0}}, \quad (3.11) \]

where \( \varphi_k \) is eigenfunction of the operator (3.5). These functions belong to the standard Hilbert space \( L^2(\mathbb{R}, dx) \).

For the different values of the parameters \( b_2, b_0 \) we have the following possibilities:

1. If \( b_2, b_0 > 0 \) from (3.11) we obtain

\[ x = \sqrt{\frac{b_0}{b_2}} \sinh \left( \sqrt{b_2} (y - c) \right), \]

where \( c \) is constant. In this subcase the potential is given by

\[ V_k(y) = \frac{d_1}{\cosh^2(y - c)} + \frac{d_2 \sinh(y - c)}{\cosh^2(y - c)} + d_0, \]

where we put \( b_2 = b_0 = 1 \). We see that in the limit we obtain the hyperbolic potential \([1,2,4]\). This potential was proposed and solved by F. Scarf. It is usually called the hyperbolic Scarf potential. The domain for this potential is \(( -\infty, \infty )\).

2. If \( b_2 < 0 \) and \( b_0 > 0 \) from (3.11) we obtain

\[ x = \sqrt{-\frac{b_0}{b_2}} \sin \sqrt{|b_2|} (y - c). \]

In this subcase the potential is given by

\[ V_k(y) = \frac{d_1}{\cos^2(y - c)} + \frac{d_2 \sin(y - c)}{\cos^2(y - c)} + d_0, \]

where we put \( b_2 = -b_0 = -1 \). We see that in the limit we obtain the trigonometric potential \([1,2,4]\). In the literature it is know as trigonometric Scarf potential. The domain for this potential is \(( -\frac{\pi}{2} + c, \frac{\pi}{2} + c )\).
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References


