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On Three Second-Order Rational Difference Equations with Period-Two Solutions

Maruta Avotina
University of Latvia
Department of Mathematics
Riga, LV-1002, Latvia
marutaavotina@inbox.lv

Abstract

The aim of this paper is to investigate the periodic behavior of three special cases of the second-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}$$

with non-negative real initial values. We address Open problem 2.9.4 given in the book by M.R.S. Kulenovic, G. Ladas Dynamics of Second Order Rational Difference Equations, with open problems and conjectures (Chapmann and Hall/CRC, 2008) and provide some relevant results and ideas with regard to these three special cases, whose solutions are either with period-two or converge to a period-two solution.

AMS Subject Classifications: 39A10, 39A23.
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1 Introduction

In the monograph [5] the authors present results on the boundedness, global stability, and periodicity of solutions of rational difference equations in the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \tag{1.1}$$

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where the parameters $\alpha, \beta, \gamma, A, B, C$ are non-negative real numbers and the initial conditions x_{-1} and x_0 are arbitrary non-negative real numbers such that

$$A + Bx_n + Cx_{n-1} > 0 \quad \text{for} \quad \text{all} \quad n \geq 0.$$

The authors of [5] maintain that the results for (1.1) are of paramount importance in the development of the basic theory of the global behaviour of solutions of non-linear difference equations of order greater than one. In [3] certain results on some special cases of (1.1) are summarized, as well as some facts regarding third-order rational difference equations whose forms are an extension of the form of equation (1.1) are stated.

In this paper we investigate the following open problem proposed by M.R.S. Kulenovic and G. Ladas in [5].

Open problem 2.9.4. It is known that every positive solution of each of the following three equations

$$x_{n+1} = 1 + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

$$x_{n+1} = \frac{1 + x_{n-1}}{1 + x_n}, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

$$x_{n+1} = \frac{x_n + 2x_{n-1}}{1 + x_n}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

converges to solutions with (not necessarily prime) period-two:

$$\ldots, \phi, \psi, \phi, \psi, \ldots. \tag{1.5}$$

In each case, determine ϕ and ψ in terms of the initial conditions x_{-1} and x_0. Conversely, if $\ldots, \phi, \psi, \phi, \psi, \ldots$ is a period-two solution for one of the equations (1.2) or (1.3) or (1.4), determine all initial conditions $(x_{-1}, x_0) \in (0; +\infty) \times (0; +\infty)$ for which the solution x_n converges to the period solution (1.5).

In [2] Basu and Merino showed that all solutions of (1.1) converge to the positive equilibrium or to a prime period-two solution. Many authors have investigated the behavior (boundedness, periodicity, stability) of solutions of difference equations that are similar to (1.2), e.g., in [8] Stevic studied the equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

where α is a negative number. In [1] Amleh, Grove, and Ladas investigated equation (1.6) with $\alpha > 0$. For $\alpha = 1$ the following statements were obtained:

1. Suppose $x_{-1} < x_1$. Then $x_{-1} < x_1 < x_3 < \ldots$ and $x_0 < x_2 < x_4 < \ldots$.
2. Suppose $x_{-1} = x_1$. Then $x_{-1} = x_1 = x_3 = \ldots$ and $x_0 = x_2 = x_4 = \ldots$.
3. Suppose $x_{-1} > x_1$. Then $x_{-1} > x_1 > x_3 > \ldots$ and $x_0 > x_2 > x_4 > \ldots$.

In [4], [6] and [7] the nonautonomous difference equations, $x_{n+1} = p_n + \dfrac{x_{n-1}}{x_n}$, $x_{n+1} = A_n + \left(\dfrac{x_{n-1}}{x_n}\right)^p$ and $x_{n+1} = A_n + \dfrac{x_{n-1}^p}{x_n^q}$, respectively, were studied.

There are very few papers in which the solutions of some difference equation are studied with respect to initial values, that is, how the solutions change if initial values are changed or how to determine the solutions in terms of initial values. In [9] Sun and Xi obtained a set of initial values such that the positive solutions of (1.2) converge to the unique equilibrium $\bar{x} = 2$.

Equations (1.3) and (1.4) have not been investigated so widely as equation (1.2). Nevertheless equations (1.2), (1.3) and (1.4) have similar properties that will be discussed in the sequel.

2 Basic Definitions and Theorems

In this section we recall some definitions and known results from [5] and [3] that will be useful in the investigation of the behaviour of solutions of difference equations (1.2), (1.3) and (1.4).

Let I be some interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function.

Definition 2.1. A point $\bar{x} \in I$ is called an equilibrium point of equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{2.1}$$

if $\bar{x} = f(\bar{x}, \bar{x})$. That is, $x_n = \bar{x}$ is a solution of (2.1) for all $n \geq 0$.

Definition 2.2. A solution $\{x_n\}$ of (2.1) is said to be periodic with period p if

$$x_{n+p} = x_n \quad \text{for} \quad \text{all} \quad n \geq -1. \tag{2.2}$$

Definition 2.3. A solution $\{x_n\}$ of (2.1) is said to be periodic with prime period p, or a p-cycle if it is periodic with period p and p is the least positive integer for which (2.2) holds.

Definition 2.4. A solution $\{x_n\}$ of (2.1) is said to be eventually periodic with prime period p if exists such integer $N \geq -1$ that sequence $\{x_n\}_{n=N}^{\infty}$ is periodic with period p, that is, $x_{n+p} = x_n$ for all $n \geq N$.

Let $p = \dfrac{\partial f}{\partial u}(\bar{x}, \bar{x})$ and $q = \dfrac{\partial f}{\partial v}(\bar{x}, \bar{x})$ denote the partial derivatives of $f(u, v)$ evaluated at the equilibrium $\bar{x}$ of (2.1).

Theorem 2.5 (See [5, Theorem 1.1.1 (Linearized Stability)])**.** *The following statements are true:*

1. *If both roots of the quadratic equation*

$$\lambda^2 - p\lambda - q = 0 \tag{2.3}$$

lie in the open unit disk $|\lambda| < 1$*, then the equilibrium* $\bar{x}$ *of* (2.1) *is locally asymptotically stable.*

2. *If at least one of the roots of* (2.3) *has absolute value greater than one, then the equilibrium* $\bar{x}$ *of* (2.1) *is unstable.*

3. *A necessary and sufficient condition for a root of* (2.3) *to have absolute value equal to one is*

$$|p| = |1 - q|$$

or

$$q = -1 \quad \text{and} \quad |p| \leq 2.$$

In this case the equilibrium $\bar{x}$ *is called a nonhyperbolic point.*

Theorem 2.6 (See [3, Theorem 1.6.6])**.** *Let* I *be a set of real numbers and let* $f : I \times I \to I$ *be a function* $f(u, v)$ *that decreases in* u *and increases in* v*. Then for every solution* $\{x_n\}$ *of the* (2.1)*, the subsequences* x_{2n} *and* x_{2n+1} *do exactly one of the following:*

1. *They are both monotonically increasing.*

2. *They are both monotonically decreasing.*

3. *Eventually, one of them is monotonically increasing and the other is monotonically decreasing.*

Let us consider a special case of (1.1) in the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n}, \tag{2.4}$$

where $\alpha, \beta, \gamma, A, B \in [0, \infty)$ and $C = 0$ with $\alpha + \beta + \gamma, A + B \in (0, \infty)$. The next theorem gives necessary and sufficient conditions for (2.4) to have a prime period-two solution and the formulas for this solution.

Theorem 2.7 (See [5, Theorem 2.7.1])**.** *Assume* $B > 0$*. Then the following statements hold:*

1. *Equation* (2.4) *has a prime period-two solution if and only if* $\gamma = \beta + A$ *holds.*

2. *When* $\gamma = \beta + A$ *holds, all period-two solutions*

$$\ldots, \phi, \psi, \phi, \psi, \ldots$$

of (2.4) *are given by the formulas:*

$$\phi > \frac{\beta}{B}, \quad \phi \neq \frac{\beta + \sqrt{\beta^2 + \alpha\beta}}{B}, \quad \text{and} \quad \psi = \frac{\alpha + \beta\phi}{-\beta + B\phi}. \tag{2.5}$$

3. *When $\gamma = \beta + A$ holds every solution of* (2.4) *converges to a period-two solution.*

The importance of period two solutions is well known and has a great influence on the behavior of solutions of difference equations. The cases when solutions with period two exist are as follows:

1. Positive period two solutions on a hyperbola.
2. Positive period two solutions on a line (when all solutions are bounded for all values of the parameters).
3. Positive unique period two solutions.
4. Period two solutions when one of the initial conditions is 0.

3 Results

An important tool to investigate the behaviour of difference equations is the linearized stability theorem (see Theorem 2.5). First of all, using equation $\bar{x} = f(\bar{x}, \bar{x})$ we determine the following equilibrium points: $\bar{x} = 2$ for (1.2); $\bar{x} = 1$ for (1.3); $\bar{x} = 0$ and $\bar{x} = 2$ for (1.4). Secondly, we obtain characteristic equations about the equilibrium points and find the roots of these quadratic equations:

1. for (1.2), about the equilibrium point $\bar{x} = 2$ we obtain $\lambda_1 = -1$ and $\lambda_2 = \dfrac{1}{2}$;
2. for (1.3), about the equilibrium point $\bar{x} = 1$ we obtain $\lambda_1 = -1$ and $\lambda_2 = \dfrac{1}{2}$;
3. for (1.4),
   (a) about the equilibrium point $\bar{x} = 0$ we have $\lambda_1 = -1$ and $\lambda_2 = 2$;
   (b) about the equilibrium point $\bar{x} = 2$ we have $\lambda_1 = -1$ and $\lambda_2 = \dfrac{2}{3}$.

By Theorem 2.5, we deduce that the equilibrium point $\bar{x} = 0$ is unstable; in all other cases it is not known whether the equilibrium points are stable or not.

Since all the characteristic equations have a root equal to -1, we obtain the following general theorem.

Theorem 3.1. *When $C = 0$ and $\gamma = \beta + A$ hold in* (1.1)*, then roots of the characteristic equation of the linearized equation $\lambda^2 - p\lambda - q = 0$ are given by the formulas*

$$\lambda_1 = -1, \quad \lambda_2 = 1 + \frac{\beta A - \alpha B - B\gamma\bar{x}}{(A + B\bar{x})^2} \in (0, 1),$$

where $\bar{x}$ is a positive equilibrium point of equation (2.4).

Proof. Since $\gamma = \beta + A$, we obtain that the equilibrium equation of (2.4) is in the form

$$B\bar{x}^2 - 2\beta\bar{x} - \alpha = 0. \tag{3.1}$$

The only positive equilibrium of equation (2.4) is $\bar{x} = \dfrac{\beta + \sqrt{\beta^2 + B\alpha}}{B}$ and the linearized equation associated with equation (2.4) about $\bar{x}$ is

$$y_{n+1} = \frac{\beta A - B\alpha - B\gamma\bar{x}}{(A + B\bar{x})^2} y_n + \frac{\gamma}{A + B\bar{x}} y_{n-1}, \quad n = 0, 1, \ldots. \tag{3.2}$$

The characteristic equation of (3.2) is

$$\lambda^2 - \frac{\beta A - B\alpha - B\gamma\bar{x}}{(A + B\bar{x})^2}\lambda - \frac{\gamma}{A + B\bar{x}} = 0. \tag{3.3}$$

Since it is difficult to calculate the roots of quadratic equation (3.3), we check whether equation (3.3) has a root $\lambda = -1$:

$$\begin{aligned}
0 &= 1 + \frac{\beta A - B\alpha - B\gamma\bar{x}}{(A + B\bar{x})^2} - \frac{\gamma}{A + B\bar{x}} \iff \\
0 &= (A + B\bar{x})^2 + (\beta A - B\alpha - B\gamma\bar{x}) - \gamma A - B\gamma\bar{x} \iff \\
0 &= A^2 + 2AB\bar{x} + B^2\bar{x}^2 + \beta A - B\alpha - 2B(A + \beta)\bar{x} - (A + \beta) \iff \\
0 &= B^2\bar{x}^2 - 2B\beta\bar{x} - B\alpha \iff \\
0 &= B\bar{x}^2 - 2\beta\bar{x} - \alpha.
\end{aligned}$$

The last equality is correct because it is the equilibrium equation (3.1). Hence $\lambda_1 = -1$ is a root of equation (3.3). Since one root of the quadratic equation is negative and the multiplication of both roots is $-\dfrac{\gamma}{A + B\bar{x}} < 0$, using Vieta's theorem we deduce that $\lambda_2 = \dfrac{\gamma}{A + B\bar{x}} > 0$ and $\lambda_1 + \lambda_2 = \dfrac{\beta A - B\alpha - B\gamma\bar{x}}{(A + B\bar{x})^2}$. Hence

$$\lambda_2 = 1 + \frac{\beta A - \alpha B - B\gamma\bar{x}}{(A + B\bar{x})^2}.$$

Now we show that the numerator $\beta A - \alpha B - B\gamma\bar{x}$ is negative:

$$\begin{aligned}
\beta A - \alpha B - B\gamma\bar{x} &= \beta A - \alpha B - B(\beta + A)\frac{\beta + \sqrt{\beta^2 + B\alpha}}{B} \\
&= \beta A - \alpha B - (\beta + A)(\beta + \sqrt{\beta^2 + B\alpha}) \\
&= -\alpha B - \beta^2 - (\beta + A)\sqrt{\beta^2 + B\alpha} < 0.
\end{aligned}$$

This completes the proof that $0 < \lambda_2 < 1$. □

Open Problem. Show whether or not some connection between (1.1) having a characteristic root of $\lambda = -1$ and (1.1) having period-two solutions.

Since $B > 0$, we can use Theorem (2.7) to obtain the following period-two solutions:

1. for (1.2) and (1.4) the period-two solution is
$$\ldots, \phi, \psi = \frac{\phi}{\phi - 1}, \phi, \psi = \frac{\phi}{\phi - 1}, \ldots;$$
2. for (1.3) the period-two solution is
$$\ldots, \phi, \psi = \frac{1}{\phi}, \phi, \psi = \frac{1}{\phi}, \ldots.$$

Observe that if $x_{-1} = \phi$ and $x_0 = \dfrac{1}{\phi}$, then the solution of (1.3) is periodic with period two:
$$\ldots \phi, \frac{1}{\phi}, \phi, \frac{1}{\phi}, \ldots.$$
However, we note the following.

Theorem 3.2. *There are no solutions of* (1.3) *that are eventually periodic with period two with the form*
$$\ldots \phi, \frac{1}{\phi}, \phi, \frac{1}{\phi}, \ldots.$$

Proof. Suppose to the contrary that there exists k such that $x_{2k} = \alpha \neq \phi$ and $x_{2k+1} = \beta \neq \dfrac{1}{\phi}$ that $x_{2k+2} = \phi$ and $x_{2k+3} = \dfrac{1}{\phi}$. Now we express α and β in terms of ϕ:
$$x_{2k+2} = \frac{1 + \alpha}{1 + \beta} = \phi, \quad x_{2k+3} = \frac{1 + \beta}{1 + \phi} = \frac{1}{\phi}. \tag{3.4}$$
From (3.4), we obtain that $\alpha = \phi$ and $\beta = \dfrac{1}{\phi}$ which leads to a contradiction with the assumption. Thus there are no initial conditions such that the solution of (1.3) is eventually periodic with period two. □

Remark 3.3. Similarly, one can show that there are no initial conditions such that the solution of (1.2) or (1.4) is eventually periodic with period two.

For (1.2) and (1.3) the function $f(x_n, x_{n-1})$ is decreasing after the argument x_n and increasing after the argument x_{n-1}. The function $f(x_n, x_{n-1})$ is not decreasing in the argument x_n for equation (1.4) in general. However, if $x_{-1} > 1$ and $x_0 > 1$, one has $x_n > 1$ for all $n \geq -1$ in the equation (1.4), and the function $f(x_n, x_{n-1})$ is decreasing in the argument x_n. Using Theorem 2.6 we obtain the following theorems and prove that the statements mentioned in Theorem 2.6 are fulfilled for the equation (1.4) for all positive values of the initial conditions.

Theorem 3.4. *The following statements are true.*

1. *If* $x_{-1} \cdot x_0 > 1$*, then both subsequences* x_{2n} *and* x_{2n+1} *of the solution of* (1.3) *are monotonically decreasing.*
2. *If* $x_{-1} \cdot x_0 < 1$*, then both subsequences* x_{2n} *and* x_{2n+1} *of the solution of* (1.3) *are monotonically increasing.*
3. *If* $x_{-1} \cdot x_0 = 1$*, then the solution of* (1.3) *is* $\dots, x_{-1}, \frac{1}{x_{-1}}, x_{-1}, \frac{1}{x_{-1}}, \dots$.

Proof. We give the prove of Statement 1; the other proofs are similar. Suppose $x_{-1} \cdot x_0 > 1$. Since initial conditions x_{-1} and x_0 are positive numbers, the inequalities $x_{-1} > \frac{1}{x_0}$ and $x_0 > \frac{1}{x_{-1}}$ hold. Using the given recurrence relation (1.3) we calculate x_1:

$$x_1 = \frac{1+x_{-1}}{1+x_0} < \frac{1+x_{-1}}{1+\frac{1}{x_{-1}}} = \frac{(1+x_{-1}) \cdot x_{-1}}{1+x_{-1}} = x_{-1}.$$

Similarly we calculate x_2 in terms of x_{-1} and x_0:

$$x_2 = \frac{1+x_0}{1+x_1} = \frac{1+x_0}{1+\frac{1+x_{-1}}{1+x_0}} = \frac{(1+x_0)^2}{x_0+2+x_{-1}} < \frac{(1+x_0)^2}{x_0+2+\frac{1}{x_0}} = \frac{x_0 \cdot (1+x_0)^2}{x_0^2+2x_0+1} = x_0.$$

Next, we verify that $x_1 \cdot x_2 > 1$:

$$\begin{aligned} x_1 \cdot x_2 &= \frac{1+x_{-1}}{1+x_0} \cdot \frac{(1+x_0)^2}{x_0+2+x_{-1}} = \frac{(1+x_{-1})(1+x_0)}{x_0+2+x_{-1}} \\ &= \frac{1+x_{-1}+x_0+x_{-1} \cdot x_0}{x_0+2+x_{-1}} > \frac{1+x_{-1}+x_0+1}{x_0+2+x_{-1}} = 1. \end{aligned}$$

It follows, by induction, that

$$\begin{aligned} x_{-1} > x_1 > x_3 > \dots > x_{2n+1} > \dots \\ x_0 > x_2 > x_4 > \dots > x_{2n} > \dots, \end{aligned}$$

that is, both subsequences x_{2n} and x_{2n+1} of the solution of (1.3) are monotonically decreasing. □

The behavior of solutions of (1.3) when $x_{-1} \cdot x_0 > 1$ are represented graphically in Figure 3.1, where on the horizontal axis we mark values of the odd terms of solution sequence and on the vertical axis – values of the even terms. Thus for every pair of initial conditions x_{-1} and x_0 we obtain the following sequence of points forming the trajectory of the resulting solution:

$$(x_{-1}; x_0),\ (x_1; x_2),\ (x_3; x_4), \dots. \tag{3.5}$$

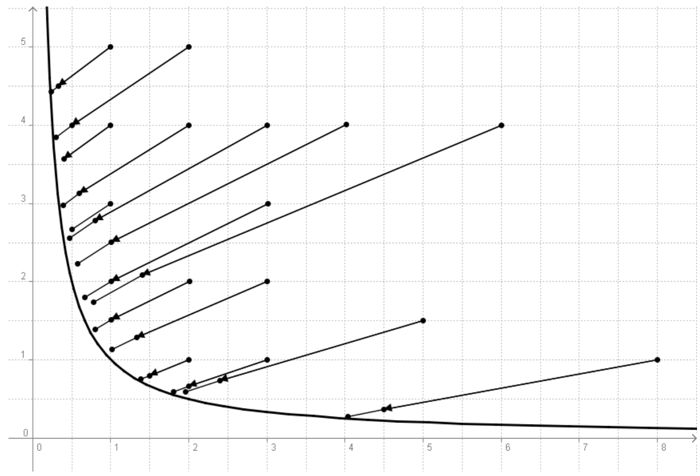


Figure 3.1: Behavior of solution subsequences of (1.3) when $x_{-1} \cdot x_0 > 1$.

Each trajectory is depicted in Figure 3.1 as a sequence of arrows connecting the points in (3.5). The bold line in Figure 3.1 is the curve $\psi = \dfrac{1}{\phi}$ and represents all period-two solutions. Since $x_{-1} \cdot x_0 > 1$, then by Theorem 3.4, the trajectories begin above the bold curve and are decreasing towards it.

Theorem 3.5. *The following statements are true.*

1. *If* $x_{-1} \cdot x_0 > x_{-1} + x_0$, *then both subsequences* x_{2n} *and* x_{2n+1} *of the solution of* (1.2) *and* (1.4) *are monotonically decreasing.*

2. *If* $x_{-1} \cdot x_0 < x_{-1} + x_0$, *then both subsequences* x_{2n} *and* x_{2n+1} *of the solution of* (1.2) *and* (1.4) *are monotonically increasing.*

3. *If* $x_{-1} \cdot x_0 = x_{-1} + x_0$, *then the solution of* (1.2) *and* (1.4) *is*
$$\dots, x_{-1}, \frac{x_{-1}}{x_{-1}-1}, x_{-1}, \frac{x_{-1}}{x_{-1}-1}, \dots.$$

Proof. We prove Statement 2 for equation (1.4); the other proofs are similar. Suppose $x_{-1} \cdot x_0 < x_{-1} + x_0$. Using the given recurrence relation (1.4) we calculate x_1:

$$x_1 = \frac{x_0 + 2x_{-1}}{1 + x_0} = \frac{x_0 + x_{-1} + x{-1}}{1 + x_0} > \frac{x_{-1} \cdot x_0 + x_{-1}}{1 + x_0} = \frac{x_{-1}(1 + x_0)}{1 + x_0} = x_{-1}.$$

Similarly we calculate x_2 in terms of x_{-1} and x_0:

$$\begin{aligned} x_2 &= \frac{x_1 + 2x_0}{1 + x_1} = \frac{x_0 + 2x_{-1} + 2x_0 + 2x_0^2}{1 + 2x_0 + 2x_{-1}} > \frac{x_0 + 2x_{-1} \cdot x_0 + 2x_0^2}{1 + 2x_0 + 2x_{-1}} \\ &= \frac{x_0(1 + 2x_{-1} + 2x_0)}{1 + 2x_0 + 2x_{-1}} = x_0. \end{aligned}$$

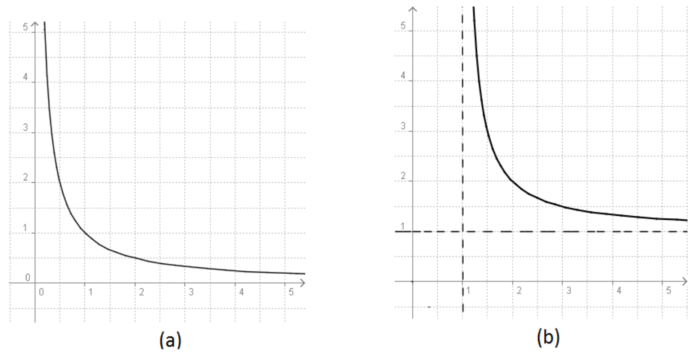


Figure 3.2: Graphs of functions $\psi = \dfrac{\phi}{\phi - 1}$ and $\psi = \dfrac{1}{\phi}$

Next, we verify that $x_1 \cdot x_2 < x_1 + x_2$:

$$\begin{aligned}
\frac{x_0 + 2x_{-1}}{1 + x_0} \cdot \frac{x_1 + 2x_0}{1 + x_1} < \frac{x_0 + 2x_{-1}}{1 + x_0} + \frac{x_1 + 2x_0}{1 + x_1} &\iff \\
(x_0 + 2x_{-1})(x_1 + 2x_0) < (x_0 + 2x_{-1})(1 + x_1) + (x_1 + 2x_0)(1 + x_0) &\iff \\
4x_0x_{-1} < 3x_0 + 2x_{-1} + x_1(1 + x_0) &\iff \\
4x_0x_{-1} < 3x_0 + 2x_{-1} + \frac{x_0 + 2x_{-1}}{1 + x_0}(1 + x_0) &\iff \\
4x_0x_{-1} < 3x_0 + 2x_{-1} + x_0 + 2x_{-1} &\iff \\
x_0x_{-1} < x_0 + x_{-1}&.
\end{aligned}$$

It follows by induction that

$$\begin{aligned}
x_{-1} < x_1 < x_3 < \cdots < x_{2n+1} < \ldots, \\
x_0 < x_2 < x_4 < \cdots < x_{2n} < \ldots,
\end{aligned}$$

that is, both subsequences x_{2n} and x_{2n+1} of the solution of (1.4) are monotonically increasing. The proof for equation (1.2) is similar. □

Remark 3.6. To obtain a period two solution for equations (1.2) and (1.4) it is necessary that $x_{-1} > 1$ and $x_0 > 1$.

Corollary 3.7. *If a point* $(x_{-1}; x_0)$ *is above (below) the graph of the function* $\psi = \dfrac{\phi}{\phi - 1}$ *(see Figure 3.2(a)), then point sequences* (3.5) *are decreasing (increasing) for equations* (1.2) *and* (1.4).

Corollary 3.8. *If a point* $(x_{-1}; x_0)$ *is above (below) the graph of the function* $\psi = \dfrac{1}{\phi}$ *(see Figure 3.2(b)), then point sequences* (3.5) *are decreasing (increasing) for equation* (1.3).

4 Some Numerical Examples and Hypothesis

Now we consider the difference equation (1.3) and the case when $x_{-1} = x_0 = a$. Then

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= \frac{1+a}{2} \\
x_3 &= \frac{4}{a+3} \\
x_4 &= \frac{a^2+6a+9}{2a+14} \\
x_5 &= \frac{2a^2+28a+98}{a^3+11a^2+47a+69} \\
x_6 &= \frac{(a^2+8a+23)(a^3+11a^2+47a+69)}{(2a+14)(a^3+13a^2+75a+167)} = \frac{a^5+P_4(a)}{2a^4+P_3(a)} \\
x_7 &= \frac{2a^7+P_6(a)}{a^8+P_7(a)} \\
x_8 &= \frac{a^{13}+P_{12}(a)}{2a^{12}+P_{11}(a)} \\
x_9 &= \frac{2a^{20}+P_{19}(a)}{a^{21}+P_{20}(a)} \\
\ldots \\
x_{2n-1} &= \frac{2a^k+P_{k-1}(a)}{a^{k+1}+P_k(a)} \\
x_{2n} &= \frac{a^{m+1}+P_{m-1}(a)}{2a^m+P_m(a)}.
\end{aligned}
$$

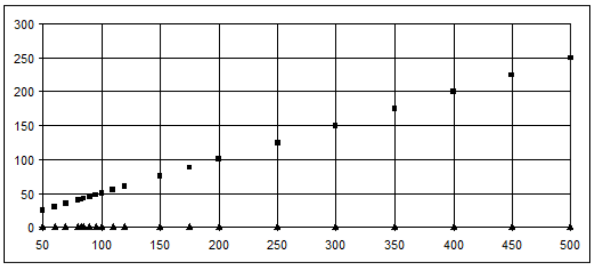
We denote a polynomial of order n as $P_n(a)$. We can observe that, for sufficiently large values of a the solution of (1.3) converges to a period-two solution that is close $\left\{\frac{a}{2}, \frac{2}{a}\right\}$. Numerical calculations show that for sufficiently large values of a, the solution of (1.3) appears to converge to a period-two solution $\left\{\frac{a-1}{2}, \frac{2}{a-1}\right\}$ (as it is seen in Figure 4.1 for large values of a points lie on a straight line).

Similar relationships can be obtained for the following:

1. Equation (1.2)

$$x_{2n-1} = \frac{a^k+P_{k-1}(a)}{a^k+P'_{k-1}(a)}, \quad x_{2n} = \frac{a^m+P_{m-1}(a)}{2a^{m-1}+P_{m-2}(a)}.$$

   Numerical calculations show that, for sufficiently large values of a, the solution of (1.2) appears to converge to a period-two solution $\left\{1, \frac{a}{2}\right\}$ (see Figure 4.2).



a	x_{2n-1}	x_{2n}
60	0,03379	29,59070
70	0,02892	34,57877
100	0,02018	49,55649
150	0,01342	74,53840
175	0,01149	87,03310
200	0,01005	99,52909
250	0,00803	124,52341
500	0,00401	249,51185
1000	0,00200	499,50596
5000	0,00040	2499,50120
10000	0,00020	4999,50060

Figure 4.1: Numerical results for equation (1.3) where initial values are $x_{-1} = x_0 = a$ and n is large

a	x_{2n-1}	x_{2n}
100	1,0204	50,0570
150	1,0135	75,0386
250	1,0081	125,0235
300	1,0067	150,0197
500	1,0040	250,0119
600	1,0033	300,0099
800	1,0025	400,0075
1000	1,0020	500,0060
3000	1,0007	1500,0020
5000	1,0004	2500,0012

a	x_{2n-1}	x_{2n}
100	1,0207	49,2583
200	1,0102	99,1949
250	1,0081	124,1815
500	1,0040	249,1538
1000	1,0020	499,1396
2000	1,0010	999,1323
3000	1,0007	1499,1299
5000	1,0004	2499,1279
10000	1,0002	4999,1265
100000	1,0000	49999,1251

Figure 4.2: Numerical results for equations (1.2) and (1.4) where initial values are $x_{-1} = x_0 = a$ and n is large

2. Equation (1.4)

$$x_{2n-1} = \frac{a^k + P_{k-1}(a)}{a^k + P'_{k-1}(a)}, \qquad x_{2n} = \frac{a^m + P_{m-1}(a)}{2a^{m-1} + P_{m-2}(a)}.$$

The calculations show that for sufficiently large values of a the solution of (1.4) converges to a period-two solution $\left\{1, \ \frac{a}{2} - 0.88\right\}$ (see Figure 4.2).

If $x_0 >> x_{-1}$, then using numerical calculations (see Figure 4.3) we can hypothesize that the solutions of (1.3) converge to the period-two solution $x_0 - x_{-1}$ and $\dfrac{1}{x_0 - x_{-1}}$.

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x_{-1} \ x_0	0.1	0.7	1	4	5	50	100
100	0,0100	0,0101	0,0101	0,0104	0,0105	0,0150	0,0202
	99,9110	99,3214	99,0292	96,1972	95,2881	66,6324	49,5565
200	0,0050	0,0050	0,0050	0,0051	0,0051	0,0062	0,0075
	199,9055	199,3108	199,0148	196,1017	195,1493	160,0759	133,2886
500	0,0020	0,0020	0,0020	0,0020	0,0020	0,0022	0,0024
	499,9022	499,3044	499,0060	496,0415	495,0611	454,6126	416,7466
1000	0,0010	0,0010	0,0010	0,0010	0,0010	0,0010	0,0011
	999,9011	999,3022	999,0030	996,0209	995,0308	952,4226	909,1571
10000	0,0001	0,0001	0,0001	0,0001	0,0001	0,0001	0,0001
	9999,9001	9999,3002	9999,0003	9996,0021	9995,0031	9950,2538	9900,9998

Figure 4.3: Numerical calculations for (1.3)

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