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Oscillation of Nonlinear Neutral Type Dynamic Equations

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Abstract

This paper is intended to study the oscillatory behaviour of solutions of the higher order nonlinear neutral type functional dynamic equation with oscillating coefficients of the following form:

$$[y(t)+p(t)y(\tau(t))]^{\Delta^n}+\sum_{i=1}^{m}q_i(t)f_i(y(\phi_i(t)))=s(t)$$

where $n\geq 2$. We obtain sufficient conditions for oscillatory behaviour of its solutions. Our results complement the oscillation results for neutral dynamic equations and improve some oscillation results for neutral differential/difference equations.

AMS Subject Classifications: 34N05.
Keywords: Time scale, nonlinear neutral dynamic equation, oscillating coefficient.

1 Introduction

In this paper we consider the higher order nonlinear dynamic equation of the form

$$[y(t)+p(t)y(\tau(t))]^{\Delta^n}+\sum_{i=1}^{m}q_i(t)f_i(y(\phi_i(t)))=s(t) \tag{1.1}$$

where $n\geq 2$, $p(t)$, $q_i(t)\in C_{\mathrm{rd}}[t_0,\infty)_{\mathbb{T}}$ for $i=1,2,\ldots,m$; $p(t)$ and $s(t)$ are oscillating functions $(p(t):\mathbb{T}\to\mathbb{R})$, $q_i(t)$ are positive real valued for $i=1,2,\ldots,m$; $\phi_i(t)\in C_{\mathrm{rd}}[t_0,\infty)_{\mathbb{T}}$, $\phi_i^{\Delta}(t)>0$, the variable delays τ, $\phi_i:[t_0,\infty)_{\mathbb{T}}\to\mathbb{T}$ with $\tau(t)$, $\phi_i(t)<$

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t for all $t \in [t_0, \infty)_{\mathbb{T}}$, $\phi_i(t) \to \infty$ as $t \to \infty$ for $i = 1, 2, \ldots, m$; $\tau(t) \to \infty$ as $t \to \infty$; $f_i(u) \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing functions, $uf_i(u) > 0$ for $u \neq 0$ and $i = 1, 2, \ldots, m$.

A dynamic equation is said to be a delay dynamic equation if $\tau(t) < t$. Equation (1.1) is called neutral dynamic equation if the highest order differential operator is applied both to the unknown function and to its composition with a delay function. A solution $y(t)$ to equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

In the literature, there are a few papers devoted to the study of delay difference/ differential equations with an oscillating coefficients in the neutral part of the equation. The problem of obtaining sufficient conditions for oscillatory behaviour of the solutions has been studied by a number of authors, see [8, 10] and references given there. The readers are referred to [2] for the oscillation theory of higher order neutral difference equations and to [1] for the fundamental studies on the oscillation theory.

Our work is inspired by [6] and [7], where the authors study the oscillatory behaviour of solutions of higher order nonlinear neutral type differential and difference equations with oscillating coefficients, respectively. We note that equation (1.1) involves some different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. This paper is intended to study the oscillatory behaviour of solutions of equation (1.1).

For the sake of convenience, the function $z(t)$ is defined by

$$z(t) = y(t) + p(t)y(\tau(t)) - r(t), \tag{1.2}$$

where $r(t) \in C_{\mathrm{rd}}[t_0, \infty)_{\mathbb{T}}$ is n times Δ-differentiable. The function $r(t)$ is an oscillating function with the property $r^{\Delta^n}(t) = s(t)$.

2 Basic Definitions and Some Auxiliary Lemmas

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $\sigma(t) = t$, then t is called *right-dense*, and if $\rho(t) = t$, then t is called *left-dense*. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

We introduce the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.1 (See [3]). The function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Theorem 2.2 (See [3]). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^\Delta(t)$ and $w^{\widetilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^\kappa$, then*

$$(w \circ \nu)^\Delta = \left(w^{\widetilde{\Delta}} \circ \nu\right) \nu^\Delta,$$

where we denote the derivative on $\widetilde{\mathbb{T}}$ by $\widetilde{\Delta}$.

Definition 2.3 (See [3]). Let $f : \mathbb{T} \to \mathbb{R}$ be a function. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$, then F is said to be an antiderivative of f. We define the Cauchy integral by

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

Theorem 2.4 (See [4]). *Let u and v be continuous functions on $[a, b]$ that are Δ-differentiable on $[a, b)$. If u^Δ and v^Δ are integrable from a to b, then*

$$\int_a^b u^\Delta(t)v(t)\Delta t + \int_a^b u^\sigma(t)v^\Delta(t)\Delta t = u(b)v(b) - u(a)v(a).$$

Let $\widetilde{\mathbb{T}} = \mathbb{T} \cup \{\sup \mathbb{T}\} \cup \{\inf \mathbb{T}\}$. If $\infty \in \widetilde{\mathbb{T}}$, we call ∞ left-dense, and $-\infty$ is called right-dense provided $-\infty \in \widetilde{\mathbb{T}}$. For any left-dense $t_0 \in \widetilde{\mathbb{T}}$ and any $\varepsilon > 0$, the set

$$L_\varepsilon(t_0) = \{t \in \mathbb{T} : 0 < t_0 - t < \varepsilon\}$$

is nonempty, and so is $L_\varepsilon(\infty) = \left\{t \in \mathbb{T} : t > \frac{1}{\varepsilon}\right\}$ if $\infty \in \widetilde{\mathbb{T}}$.

Lemma 2.5 (See [5]). *Let $n \in \mathbb{N}$ and f be n-times differentiable on $\mathbb{T}$. Assume $\infty \in \widetilde{\mathbb{T}}$. Suppose there exists $\varepsilon > 0$ such that*

$$f(t) > 0, \operatorname{sgn}\left(f^{\Delta^n}(t)\right) \equiv s \in \{-1, +1\} \text{ for all } t \in L_\varepsilon(\infty),$$

and $f^{\Delta^n}(t) \neq 0$ on $L_\delta(\infty)$ for any $\delta > 0$. Then there exists $\upsilon \in [0, n] \cap \mathbb{N}_0$ such that $n + \upsilon$ is even for $s = 1$ and odd for $s = -1$ with

$$\begin{cases} (-1)^{\upsilon+j} f^{\Delta^j}(t) > 0 \text{ for all } t \in L_\varepsilon(\infty), j \in [\upsilon, n-1] \cap \mathbb{N}_0 \\ f^{\Delta^j}(t) > 0 \text{ for all } t \in L_{\delta_j}(\infty) \text{ (with } \delta_j \in (0, \varepsilon)), j \in [1, \upsilon - 1] \cap \mathbb{N}_0. \end{cases}$$

Lemma 2.6 (See [5]). *Let f be n-times differentiable on $\mathbb{T}^{\kappa^n}$, $t \in \mathbb{T}$, and $\alpha \in \mathbb{T}^{\kappa^n}$. Then with the functions h_k defined as $h_n(t,s) = (-1)^n g_n(s,t)$,*

$$h_0(r,s) \equiv 1 \text{ and } h_{k+1}(r,s) = \int_s^r h_k(\tau,s)\,\Delta s \text{ for } k \in \mathbb{N}_0,$$

we have

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha) f^{\Delta^k}(\alpha) + \int_\alpha^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\,\Delta\tau.$$

Lemma 2.7 (See [5]). *Let f be n-times differentiable on $\mathbb{T}^{\kappa^n}$ and $m \in \mathbb{N}$ with $m < n$. Then we have for all $\alpha \in \mathbb{T}^{\kappa^{n-1+m}}$ and $t \in \mathbb{T}^{\kappa^m}$*

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} h_k(t,\alpha) f^{\Delta^{k+m}}(\alpha) + \int_\alpha^{\rho^{n-m-1}(t)} h_{n-m-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\,\Delta\tau.$$

Lemma 2.8 (See [5]). *Suppose f is n-times differentiable and $g_k, 0 \le k \le n-1$, are differentiable at $t \in \mathbb{T}^{\kappa^n}$ with*

$$g_{k+1}^{\Delta}(t) = g_k(\sigma(t)) \text{ for all } 0 \le k \le n-2.$$

Then we have

$$\left[\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k} g_k\right]^{\Delta} = f g_0^{\Delta} + (-1)^{n-1} f^{\Delta^n} g_{n-1}^{\sigma}.$$

3 Main Results

We will need the following lemma in order to prove our results.

Lemma 3.1. *Let f be n-times differentiable on $\mathbb{T}^{\kappa^n}$. If $f^{\Delta} > 0$, then for every λ, $0 < \lambda < 1$, we have*

$$f(t) \ge \lambda(-1)^{n-1} g_{n-1}(\sigma(T^*),t) f^{\Delta^{n-1}}(t). \tag{3.1}$$

Proof. Let υ, $0 \le \upsilon \le n-1$, be the integer assigned to the function f as in Lemma 2.5. Because of $f^{\Delta} > 0$, we always have $\upsilon > 0$. Furthermore, let $T^* \ge T$ be assigned to f by Lemma 2.5. Then, using the Taylor formula (Lemma 2.6) on time scales, for every $\rho^{n-1}(t) \ge T^*$ we obtain

$$f(t) \ge \int_{T^*}^{\rho^{n-1}(t)} (-1)^{n-1} g_{n-1}(\sigma(\tau),t) f^{\Delta^n}(\tau)\,\Delta\tau. \tag{3.2}$$

By Theorem 2.4 and (3.2) we have

$$f(t) \geq (-1)^{n-1} g_{n-1}(\sigma(t), t) f^{\Delta^{n-1}}(t) - \int_{T^*}^{\rho^{n-1}(t)} (-1)^{n-1} g_{n-1}(\sigma(\tau), t) f^{\Delta^{n-1}}(\tau) \Delta\tau.$$

Since f is n-times differentiable on $\mathbb{T}^{\kappa^n}$ and $m \in \mathbb{N}$ with $m < n$, we have, by Lemma 2.7 with n and f substituted by $n-m$ and f^{Δ^m}, respectively:

$$f^{\Delta^m}(t) \geq \int_{T^*}^{\rho^{n-m-1}(t)} (-1)^{n-m-1} g_{n-m-1}(\sigma(\tau), t) f^{\Delta^n}(\tau) \Delta\tau.$$

Also for every $\rho^{n-1}(t)$, s with $\rho^{n-1}(t) \geq T^*$ and $T^* \leq s \leq t$ we have

$$f^{\Delta^m}(s) \geq (-1)^{n-m-1} g_{n-m-1}(\sigma(T^*), t) f^{\Delta^n}(t).$$

This is obvious for $m = n-1$ and, when $m < n-1$, it can be derived by applying the Taylor formula. Thus for all $t \geq T^*$ we get

$$f(t) \geq (-1)^{n-1} g_{n-1}(\sigma(T^*), t) f^{\Delta^{n-1}}(t)$$

and therefore the proof can be immediately completed. □

Lemma 3.1 is an extension of results presented in [1, 1.8.14] and [9, Lemma 2]. Indeed, for $\mathbb{T} = \mathbb{Z}$, we have $\rho(t) = t-1$, $\sigma(t) = t+1$ and

$$g_{n-1}(\sigma(T^*), t) = \frac{(t - T^* - 1)^{(n-1)}}{(n-1)!}.$$

Hence, we get the inequality in [1]

$$u(t) \geq \frac{1}{(n-1)!} (n - n_1)^{(n-1)} \Delta^{n-1} u\left(2^{n-m-1} n\right).$$

In the case $\mathbb{T} = \mathbb{R}$, we have $\rho(t) = \sigma(t) = t$ and

$$g_{n-1}(\sigma(T^*), t) = \frac{(t - T^*)^{(n-1)}}{(n-1)!}.$$

Hence, we get the inequality in [9]

$$u(t) \geq \frac{\vartheta}{(n-1)!} (t)^{n-1} u^{n-1}(t).$$

Furthermore there might be other time scales that we cannot appreciate at this time due to our current lack of "real-world" examples.

Theorem 3.2. *Assume that n is odd and*

(C1) $\lim_{t\to\infty} p(t) = 0$ *and* $\lim_{t\to\infty} r(t) = 0$*;*

(C2) $\int_{t_0}^{\infty} s^{n-1} \sum_{i=1}^{m} q_i(s)\, \Delta s = \infty.$

Then, every bounded solution to equation (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Assume that equation (1.1) has a bounded nonoscillatory solution $y(t)$. Without loss of generality, assume that $y(t)$ is eventually positive (the proof is similar when $y(t)$ is eventually negative). That is, $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\phi_i(t)) > 0$ for $t \geq t_1 \geq t_0$ and $i = 1, 2, \ldots, m$. Assume further that $y(t)$ does not tend to zero as $t \to \infty$. By (1.1)–(1.2) we have

$$z^{\Delta^n}(t) = -\sum_{i=1}^{m} q_i(t)\, f_i(y(\phi_i(t))) < 0 \tag{3.3}$$

for $t \geq t_1$. It follows that $z^{\Delta^j}(t)$, $j \in [0, n-1] \cap \mathbb{N}_0$ is strictly monotone and eventually of constant sign. Since $p(t)$ and $r(t)$ are oscillating functions, there exists a $t_2 \geq t_1$ such that if $t \geq t_2$, then $z(t) > 0$ eventually. Since $y(t)$ is bounded, by virtue of (C1) and (1.2), there is a $t_3 \geq t_2$, such that $z(t)$ is also bounded for $t \geq t_3$. Because n is odd and $z(t)$ is bounded, by Lemma 2.5, when $\upsilon = 0$ (otherwise $z(t)$ is not bounded) there exists $t_4 \geq t_3$ such that for $t \geq t_4$ we have $(-1)^j z^{\Delta^j}(t) > 0$, $j \in [0, n-1] \cap \mathbb{N}_0$.

In particular, since $z^{\Delta}(t) < 0$ for $t \geq t_4$, $z(t)$ is decreasing. Since $z(t)$ is bounded, we write $\lim_{t\to\infty} z(t) = L$, $(-\infty < L < \infty)$. Assume that $0 \leq L < \infty$. Let $L > 0$. Then there exists a constant $c > 0$ and a $t_5 \geq t_4$ such that $z(t) > c > 0$ for $t \geq t_5$. Since $y(t)$ is bounded, $\lim_{t\to\infty} p(t) y(\tau(t)) = 0$ by (C1). Therefore, there exists a constant $c_1 > 0$ and a $t_6 \geq t_5$ such that $y(t) = z(t) - p(t) y(\tau(t)) + r(t) > c_1 > 0$ for $t \geq t_6$. So that we can find a t_7 with $t_7 \geq t_6$ such that $y(\phi_i(t)) > c_1 > 0$ for $t \geq t_7$. From (3.3) we have

$$z^{\Delta^n}(t) = -\sum_{i=1}^{m} q_i(t) f_i(c_1) < 0 \tag{3.4}$$

for $t \geq t_7$. By multiplying (3.4) by t^{n-1} and integrating it from t_7 to t, we obtain

$$F(t) - F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} q_i(s)\, s^{n-1} \Delta s, \tag{3.5}$$

where

$$F(t) = \sum_{i=1}^{n-1} (-1)^{i+1} \left(t^{n-1}\right)^{\Delta^i} z^{\Delta^{n-i}}\left(\sigma^i(t)\right)$$

and

$$\sigma^{i}(t) = \sigma\left(\sigma^{i-1}(t)\right).$$

Since $(-1)^{k} z^{\Delta^{k}}(t) > 0$ for $k = 0, 1, 2, \ldots, n-1$ and $t \geq t_4$, we have $F(t) > 0$ for $t \geq t_7$. From (3.5) we have

$$-F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} q_i(s)\, s^{n-1} \Delta s.$$

By (C2), we obtain

$$-F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} q_i(s)\, s^{n-1} \Delta s = -\infty$$

as $t \to \infty$. This is a contradiction. Hence, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is $\lim_{t\to\infty} z(t) = 0$. Since $y(t)$ is bounded, by (C1), we obtain

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} z(t) - \lim_{t\to\infty} p(t)y(t) + \lim_{t\to\infty} r(t) = 0.$$

Now let us consider the case $y(t) < 0$ for $t \geq t_1$. By (1.1)–(1.2), we have

$$z^{\Delta^{n}}(t) = -\sum_{i=1}^{m} q_i(t) f_i\left(y\left(\phi_i(t)\right)\right) > 0$$

for $t \geq t_1$. That is, $z^{\Delta^{n}} > 0$. It follows that $z^{\Delta^{j}}(t)$, $(j \in [0, n-1] \cap \mathbb{N}_0)$ is strictly monotone and eventually of constant sign. Since $p(t)$ and $r(t)$ are oscillating functions, there exists a $t_2 \geq t_1$ such that if $t \geq t_2$, then $z(t) < 0$ eventually. Since $y(t)$ is bounded, by (C1) and (1.2) there is $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Assume that $x(t) = -z(t)$. Then $x^{\Delta^{n}}(t) = -z^{\Delta^{n}}(t)$. Therefore, $x(t) > 0$ and $x^{\Delta^{n}}(t) < 0$ for $t \geq t_3$. Hence, we observe that $x(t)$ is bounded. Since n is odd, by Lemma 2.5, there exists $t_4 \geq t_3$ and $\upsilon = 0$ (otherwise $x(t)$ is not bounded) such that $(-1)^{j} x^{\Delta^{j}}(t) > 0$, $j \in [0, n-1] \cap \mathbb{N}_0$ and $t \geq t_4$. That is, $(-1)^{j} z^{\Delta^{j}}(t) < 0$, $j \in [0, n-1] \cap \mathbb{N}_0$ and $t \geq t_4$. In particular, for $t \geq t_4$ we have $z^{\Delta}(t) > 0$. Therefore, $z(t)$ is increasing. So, we can assume that $\lim_{t\to\infty} z(t) = L$, $(-\infty < L \leq 0)$. As in the proof of $y(t) > 0$, we may obtain that $L = 0$. As for the rest, it is similar to the case of $y(t) > 0$. That is, $\lim_{t\to\infty} y(t) = 0$. This contradicts our assumption. Hence, the proof is completed. □

Theorem 3.3. *Assume that n is even and (C1) holds. Moreover, the following conditions are satisfied:*

(C3) there is a function $\varphi(t)$ such that $\varphi(t) \in C_{\mathrm{rd}}[t_0,\infty)_{\mathbb{T}}$,

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\varphi(s)\sum_{i=1}^{m}q_i(s)\,\Delta s=\infty$$

and

$$\lim_{t\to\infty}\sup\int_{t_{10}}^{t}\frac{\left[\varphi^{\Delta}(s)\right]^2}{\varphi(s)\,g_{n-2}^{\sigma}\left(\sigma(\phi_i(s)),\phi_i(s)\right)}\Delta s<\infty$$

for $\varphi(t)$ and $i=1,2,\dots,m$.

Then, every bounded solution to equation (1.1) *is oscillatory.*

Proof. Suppose that equation (1.1) has a bounded nonoscillatory solution $y(t)$. Without loss of generality we can assume that $y(t)$ is eventually positive (the proof is similar when $y(t)$ is eventually negative). That is, $y(t)>0$, $y(\tau(t))>0$ and $y(\phi_i(t))>0$ for $t\ge t_1\ge t_0$. By (1.1)–(1.2), we have (3.3) for $t\ge t_1$. Then $z^{\Delta^n}(t)<0$. It follows that $z^{\Delta^j}(t)$, $j\in[0,n-1]\cap\mathbb{N}_0$ is strictly monotone and eventually of constant sign. Since $p(t)$ and $r(t)$ are oscillating functions, there exists $t_2\ge t_1$ such that for $t\ge t_2$, we have $z(t)>0$. Since $y(t)$ is bounded, by (C1) and (1.2), there is $t_3\ge t_2$, such that $z(t)$ is also bounded for $t\ge t_3$. Because n is even, by Lemma 2.5 when $\upsilon=1$ (otherwise $z(t)$ is not bounded), there exists $t_4\ge t_3$ such that for $t\ge t_4$ we have

$$(-1)^{j+1}z^{\Delta^j}(t)>0, j\in[0,n-1]\cap\mathbb{N}_0. \tag{3.6}$$

In particular, since $z^{\Delta}(t)>0$ for $t\ge t_4$, $z(t)$ is increasing. Since $y(t)$ is bounded, $\lim_{t\to\infty}p(t)y(\tau(t))=0$ by (C1). Then there exists $t_5\ge t_4$ and a positive integer δ such that, by (1.2),

$$y(t)=z(t)-p(t)y(\tau(t))+r(t)>\frac{1}{\delta}z(t)>0$$

for $t\ge t_5$. We may find a $t_6\ge t_5$ such that for $t\ge t_6$ and $i=1,2,\dots,m$.

$$y(\phi_i(t))>\frac{1}{\delta}z(\phi_i(t))>0. \tag{3.7}$$

From (3.3), (3.7) and the properties of f we have

$$\begin{aligned} z^{\Delta^n}(t) &\le -\sum_{i=1}^{m}q_i(t)f_i\left(\frac{1}{\delta}z(\phi_i(t))\right)\\ &= -\sum_{i=1}^{m}q_i(t)\frac{f_i\left(\frac{1}{\delta}z(\phi_i(t))\right)}{z(\phi_i(t))}z(\phi_i(t)) \end{aligned} \tag{3.8}$$

for $t \geq t_6$. Since $z(t) > 0$ is bounded and increasing, $\lim_{t\to\infty} z(t) = L$, $(0 < L < +\infty)$. By the continuity of f, we have

$$\lim_{t\to\infty} \frac{f_i\left(\frac{1}{\delta} z\left(\phi_i(t)\right)\right)}{z\left(\phi_i(t)\right)} = \frac{f_i\left(\frac{L}{\delta}\right)}{L} > 0.$$

Then there is $t_7 \geq t_6$ such that for $t \geq t_7$, $i = 1, 2, \ldots, m$, we have

$$\frac{f_i\left(\frac{1}{\delta} z\left(\phi_i(t)\right)\right)}{z\left(\phi_i(t)\right)} = \frac{f_i\left(\frac{L}{\delta}\right)}{2L} = \alpha > 0. \tag{3.9}$$

By (3.8)–(3.9),

$$z^{\Delta^n}(t) \leq -\alpha \sum_{i=1}^{m} q_i(t) z\left(\phi_i(t)\right), \text{ for } t \geq t_7. \tag{3.10}$$

Set

$$w(t) = \frac{z^{\Delta^{n-1}}(t)}{z\left(\frac{1}{\delta}\phi_i(t)\right)}. \tag{3.11}$$

We know from (3.6) that there is $t_8 \geq t_7$ such that, for sufficiently large $t \geq t_8$, $w(t) > 0$. Therefore, Δ-differentiating (3.11) we obtain

$$\begin{aligned}
w^{\Delta}(t) &= \frac{z^{\Delta^n}(t)}{z\left(\delta^{-1}\phi_i(t)\right)} - \frac{z^{\Delta^{n-1}}\left(\sigma(t)\right) z^{\Delta}\left(\delta^{-1}\phi_i(t)\right)\delta^{-1}\phi_i^{\Delta}(t)}{z\left(\delta^{-1}\phi_i(t)\right) z\left(\delta^{-1}\phi_i\left(\sigma(t)\right)\right)} \\
&= \frac{z^{\Delta^n}(t) z\left(\delta^{-1}\phi_i\left(\sigma(t)\right)\right) - z^{\Delta^{n-1}}\left(\sigma(t)\right) z^{\Delta}\left(\delta^{-1}\phi_i(t)\right)\delta^{-1}\phi_i^{\Delta}(t)}{z\left(\delta^{-1}\phi_i(t)\right) z\left(\delta^{-1}\phi_i\left(\sigma(t)\right)\right)} \\
&\leq \frac{z^{\Delta^n}(t)\, z\left(\delta^{-1}\phi_i\left(\sigma(t)\right)\right) - z^{\Delta^{n-1}}\left(\sigma(t)\right) z^{\Delta}\left(\delta^{-1}\phi_i(t)\right)\delta^{-1}\phi_i^{\Delta}(t)}{\left[z\left(\delta^{-1}\phi_i(t)\right)\right]^2} \\
&= \frac{z^{\Delta^n}(t)\, z\left(\delta^{-1}\phi_i\left(\sigma(t)\right)\right)}{\left[z\left(\delta^{-1}\phi_i(t)\right)\right]^2} - \frac{z^{\Delta^{n-1}}\left(\sigma(t)\right) z^{\Delta}\left(\delta^{-1}\phi_i(t)\right)\delta^{-1}\phi_i^{\Delta}(t)}{\left[z\left(\delta^{-1}\phi_i(t)\right)\right]^2} \\
&\leq \frac{z^{\Delta^n}(t)}{z\left(\delta^{-1}\phi_i(t)\right)} - \frac{1}{\delta}\frac{z^{\Delta^{n-1}}(t)\, z^{\Delta}\left(\delta^{-1}\phi_i(t)\right)\phi_i^{\Delta}(t)}{\left[z\left(\delta^{-1}\phi_i(t)\right)\right]^2} \\
&= \frac{z^{\Delta^n}(t)}{z\left(\delta^{-1}\phi_i(t)\right)} - \frac{1}{\delta} w(t) \frac{z^{\Delta}\left(\delta^{-1}\phi_i(t)\right)\phi_i^{\Delta}(t)}{z\left(\delta^{-1}\phi_i(t)\right)}.
\end{aligned} \tag{3.12}$$

By (3.6) there is $t \geq t_9$, such that $z^{\Delta}(t) > 0$ and $z^{\Delta^{n-1}}(t) > 0$ for an even n. Since $z(t) > 0$ is increasing, $z\left(\delta^{-1}\phi_i\left(\sigma(t)\right)\right) > z\left(\delta^{-1}\phi_i(t)\right)$ for $i = 1, 2, \ldots, m$. Therefore, by Lemma 3.1, we get

$$z\left(\delta^{-1}\phi_i(t)\right) \geq \lambda\, (-1)^{n-1} g_{n-1}\left(\sigma\left(\phi_i(t)\right), \phi_i(t)\right) z^{\Delta^{n-1}}\left(\phi_i(t)\right). \tag{3.13}$$

Then, Δ–differentiating (3.13), using Lemma 2.8 and

$$g_{n-1}^{\Delta}\left(\sigma(t), t\right) = g_{n-2}^{\sigma}(\sigma(t), t),$$

we get

$$\begin{aligned} \left[z\left(\delta^{-1}\phi_i(t)\right)\right]^{\Delta} &\geq \lambda\left(-1\right)^{n-2} g_{n-1}^{\Delta}\left(\sigma(\phi_i(t)),\phi_i(t)\right) z^{\Delta^{n-1}}\left(\phi_i(t)\right) \\ &\geq \lambda\left(-1\right)^{n-2} g_{n-2}^{\sigma}\left(\sigma(\phi_i(t)),\phi_i(t)\right) z^{\Delta^{n-1}}\left(\phi_i(t)\right). \end{aligned}$$

By Lemma 2.4, we have

$$z^{\Delta}\left(\delta^{-1}\phi_i(t)\right)\delta^{-1}\phi_i^{\Delta}(t) \geq \lambda\left(-1\right)^{n-2} g_{n-2}^{\sigma}\left(\sigma(\phi_i(t)),\phi_i(t)\right) z^{\Delta^{n-1}}\left(\phi_i(t)\right).$$

Since $\phi_i(t) \leq t$, we obtain

$$z^{\Delta}\left(\delta^{-1}\phi_i(t)\right) \geq \frac{\delta\lambda\left(-1\right)^{n-2} g_{n-2}^{\sigma}\left(\sigma(\phi_i(t)),\phi_i(t)\right) z^{\Delta^{n-1}}(t)}{\phi_i^{\Delta}(t)}. \tag{3.14}$$

Hence, by (3.10), (3.13) and (3.14), we conclude

$$\begin{aligned} w^{\Delta}(t) &\leq \frac{-\alpha\sum\limits_{i=1}^{m} q_i(t) z\left(\phi_i(t)\right)}{z\left(\delta^{-1}\phi_i(t)\right)} \\ &\quad -\frac{1}{\delta} w(t) \frac{\delta\lambda\left(-1\right)^{n-2} g_{n-2}^{\sigma}\left(\sigma(\phi_i(t)),\phi_i(t)\right) z^{\Delta^{n-1}}(t)}{\phi_i^{\Delta}(t)} \frac{\phi_i^{\Delta}(t)}{z\left(\delta^{-1}\phi_i(t)\right)} \\ &\leq -\alpha\sum_{i=1}^{m} q_i(t) - \lambda\left(-1\right)^{n-2} w^2(t) g_{n-2}^{\sigma}\left(\sigma(\phi_i(t)),\phi_i(t)\right) \end{aligned}$$

and then

$$\alpha\sum_{i=1}^{m} q_i(t) \leq -w^{\Delta}(t) - \lambda\left(-1\right)^{n-2} w^2\left(t\right) g_{n-2}^{\sigma}\left(\sigma(\phi_i(t)),\phi_i(t)\right) \tag{3.15}$$

for $t \geq t_{10}$. Multiplying (3.15) by $\varphi(t)$ and integrating it from t_{10} to t we obtain, by Theorem 2.4,

$$\begin{aligned} \alpha\int_{t_{10}}^{t} \varphi\left(s\right)\sum_{i=1}^{m} q_i\left(s\right)\Delta s &\leq -\int_{t_{10}}^{t} \varphi\left(s\right) w^{\Delta}\left(s\right)\Delta s \\ &\quad -\int_{t_{10}}^{t} \lambda\left(-1\right)^{n-2}\varphi\left(s\right) w^2\left(s\right) g_{n-2}^{\sigma}\left(\sigma(\phi_i(s)),\phi_i\left(s\right)\right)\Delta s \end{aligned}$$

$$\begin{aligned}
&\leq -\left[\varphi(t)w(t)-\varphi(t_{10})\,w(t_{10})-\int_{t_{10}}^{t}\varphi^{\Delta}(s)\,w^{\sigma}(t)\Delta s\right] \\
&\quad -\int_{t_{10}}^{t}\lambda\,(-1)^{n-2}\,\varphi(s)\,w^{2}(s)\,g_{n-2}^{\sigma}\left(\sigma(\phi_i(s)),\phi_i(s)\right)\Delta s \\
&\leq \varphi(t_{10})\,w(t_{10})+\int_{t_{10}}^{t}\varphi^{\Delta}(s)\,w^{\sigma}(t)\Delta s \\
&\quad -\lambda\int_{t_{10}}^{t}\varphi(s)\,w^{2}(s)\,g_{n-2}^{\sigma}\left(\sigma(\phi_i(s)),\phi_i(s)\right)\Delta s \\
&\leq \varphi(t_{10})\,w(t_{10})-\lambda\int_{t_{10}}^{t}\varphi(s)\,g_{n-2}^{\sigma}\left(\sigma(\phi_i(s)),\phi_i(s)\right) \\
&\quad \times\left[w(s)-\frac{\varphi^{\Delta}(s)}{2\lambda\varphi(s)\,g_{n-2}^{\sigma}\left(\sigma(\phi_i(s)),\phi_i(s)\right)}\right]^{2}\Delta s \\
&\quad +\int_{t_{10}}^{t}\frac{\left[\varphi^{\Delta}(s)\right]^{2}}{4\lambda\varphi(s)\,g_{n-2}^{\sigma}\left(\sigma(\phi_i(s)),\phi_i(s)\right)}\Delta s \\
&\leq \varphi(t_{10})\,w(t_{10})+\int_{t_{10}}^{t}\frac{\left[\varphi^{\Delta}(s)\right]^{2}}{4\lambda\varphi(s)\,g_{n-2}^{\sigma}\left(\sigma(\phi_i(s)),\phi_i(s)\right)}\Delta s.
\end{aligned}$$

Therefore, by (C3), we conclude

$$\begin{aligned}
\infty &= \alpha\limsup_{t\to\infty}\int_{t_{10}}^{t}\varphi(s)\sum_{i=1}^{m}q_i(s)\,\Delta s \\
&\leq \varphi(t_{10})\,w(t_{10})+\frac{1}{4\lambda}\limsup_{t\to\infty}\int_{t_{10}}^{t}\frac{\left[\varphi^{\Delta}(s)\right]^{2}}{\varphi(s)\,g_{n-2}^{\sigma}\left(\sigma(\phi_i(s)),\phi_i(s)\right)}\Delta s \\
&< \infty.
\end{aligned}$$

This is a contradiction. Now let us consider the case $y(t) < 0$ for $t \geq t_1$. By (1.1)–(1.2), we have

$$z^{\Delta^n}(t) = -\sum_{i=1}^{m} q_i(t) f_i\left(y\left(\phi_i(t)\right)\right) > 0$$

for $t \geq t_1$. That is, $z^{\Delta^n} > 0$. It follows that $z^{\Delta^j}(t)$, $(j \in [0, n-1] \cap \mathbb{N}_0)$ is strictly monotone and eventually of constant sign. Since $p(t)$ is an oscillatory function, there exists $t_2 \geq t_1$ such that $z(t) < 0$ for $t \geq t_2$. Since $y(t)$ is bounded, by (C1) and (1.2), there is $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Assume that $x(t) = -z(t)$. Then $x^{\Delta^n}(t) = -z^{\Delta^n}(t)$. Therefore, $x(t) > 0$ and $x^{\Delta^n}(t) < 0$ for $t \geq t_3$. Hence, we observe that $x(t)$ is bounded. Since n is odd, by Lemma 2.5, there exists $t_4 \geq t_3$ and $v = 1$ (otherwise $x(t)$ is not bounded) such that $(-1)^k x^{\Delta^k}(t) >$, $k \in [0, n-1] \cap \mathbb{N}_0$ and $t \geq t_4$. That is, $(-1)^k z^{\Delta^k}(t) < 0$, $k \in [0, n-1] \cap \mathbb{N}_0$ and $t \geq t_4$. In particular, for $t \geq t_4$ we have $z^{\Delta}(t) > 0$. Therefore, $z(t)$ is increasing. For the rest of the proof, we can proceed similarly to the case of $y(t) > 0$. Hence, the proof is completed. □

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