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Impulsive Differential Equations with Fractional Derivatives

Ivan Matychyn
Glushkov Institute of Cybernetics
Department of Optimization of Controlled Processes
Kiev, 03187, Ukraine
matychyn@gmail.com

Viktoriia Onyshchenko
State University of Telecommunications
Department of Mathematics
Kiev, 03110, Ukraine
oviva@gmx.com

Abstract

Linear fractional differential equations whose right hand side contains additive Dirac distributions are investigated. Analytical solutions to these equations are obtained on the basis of the Laplace transform method.

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1 Introduction

Fractional calculus and fractional differential equations (FDEs) have become important in recent decades as mathematical models of processes that exhibit such properties as long-term memory and self-similarity. FDEs appear naturally in a number of fields such as rheology, seismology, biophysics, blood flow phenomena, aerodynamics, fluid flow in porous media, viscoelasticity, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. There exists an extensive literature

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on this topic including monographs [11, 15]. In particular, the problems of control of FDEs and fractional variational problems are addressed in the papers [4, 7, 14].

Impulsive differential equations have recently received considerable attention as mathematical models of processes where some parameters can change instantly in a jump-like manner.

The monographs [16, 19] are devoted to the impulse differential equations and related issues, i.e., stability, control etc. To describe the impulsive character of the processes special types of differential equations allowing discontinuous solutions [19] can be employed.

A common approach to modeling the impulsive behavior is the use of difference equations to describe the impulse impact. Thus, impulsive differential equations are inherently related to the difference equations. In many cases an impulsive differential equation can be reduced to a difference one. Dynamic equations on time scales [3] represent a state-of-the-art approach to this problem, which allows a unification of the theory of difference equations with that of differential equations.

Other approaches employed to deal with the impulsive behavior are based on the technique of generalized functions (distributions) such as Dirac delta function. All these approaches have their advantages and drawbacks.

Both FDEs and impulsive differential equations have drawn intense attention from researchers in the last decades due to the numerous applications. The idea that combining these two classes of differential equations may yield an interesting and promising object of investigation, viz., impulsive FDEs, prompted numerous papers. For the recent developments in theory and applications of impulsive FDEs, we refer the reader to the papers [2, 8, 13, 17, 18, 20] and the references therein. The recent surge in development of the theory of impulsive FDEs has motivated the present work.

Here we investigate linear fractional differential equations whose right hand side contains additive Dirac distributions. Analytical solutions to these equations are obtained on the basis of the Laplace transform method.

2 Preliminary Results

In [5] the Mittag–Leffler generalized matrix function was introduced:

$$E_{\rho,\mu}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(k\rho + \mu)}, \tag{2.1}$$

where $\rho > 0$, $\mu \in \mathbb{C}$, and B is an arbitrary square matrix of order n. It should be noted that $E_{\rho,\mu}(B)$ generalizes the matrix exponential as

$$E_{1,1}(B) = e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}. \tag{2.2}$$

The Mittag–Leffler generalized matrix function plays important role in study of the linear systems of fractional order. Denote by I the identity matrix of order n. The following lemma [4] allows us to find the Laplace transforms of expressions involving the Mittag–Leffler matrix function.

Lemma 2.1. *Let $\alpha > 0$, $\beta > 0$, and let A be an arbitrary square matrix of order m. Then the following formula is true:*

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(At^{\alpha}); s\right\} = s^{\alpha-\beta}(s^{\alpha}I - A)^{-1}.$$

In what follows we also use the following properties of the Laplace transform.

- Time shifting property:

$$\mathcal{L}\{f(x-a)\chi(x-a); s\} = e^{-as}F(s), \tag{2.3}$$

  where $\chi(x)$ is the Heaviside unit step function and $F(s) = \mathcal{L}\{f(x); s\}$.

- Convolution property:

$$\mathcal{L}\{f(x) * g(x); s\} = F(s)G(s), \tag{2.4}$$

  where $f(x) * g(x) = \displaystyle\int_0^x f(t)g(x-t)dt$ is a convolution of the functions f, g and $G(s) = \mathcal{L}\{g(x); s\}$.

Now let us recall that the Riemann–Liouville fractional integral and derivative of fractional order are defined as follows:

$$J^{\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)dt, \quad 0 < \nu < 1,$$
$$D^{\alpha}f(x) = \frac{d^k}{dx^k}J^{k-\alpha}f(x), \quad k-1 < \alpha < k.$$

The following formulas for their Laplace transforms hold true:

$$\mathcal{L}\{J^{\nu}f(x); s\} = s^{-\nu}F(s), \quad 0 < \nu < 1,$$
$$\mathcal{L}\{D^{\alpha}f(x); s\} = s^{\alpha}F(s) - \sum_{i=0}^{k-1} D^{\alpha-i-1}f(x)\big|_{x=0+}, \quad k-1 < \alpha < k. \tag{2.5}$$

The derivative of negative order appearing in (2.5) should be interpreted as integral, i.e., $D^{\alpha-k}f(x)\big|_{x=0+} = J^{k-\alpha}f(x)\big|_{x=0+}$.

3 Impulsive FDEs

In this section, we give the main results on the analytical solutions to the linear systems of FDEs involving impulses represented in terms of the Dirac delta functions.

3.1 State-independent Impulses

Let $\mathbb{R}_+ = [0, \infty)$ be the positive semi-axis.

Suppose $IMP = \{x_i\}_{i \in I}$ is a sequence of points from $\mathbb{R}_+$ (finite or infinite) such that any finite interval contains a finite number of its elements indexed by the set $I = \{1, 2, \ldots\}$, $I \subset \mathbb{N}$. Denote $IMP(x) = \{x_i \in IMP : \ x_i < x\}$, $I(x) = \{i \in I : \ x_i \in IMP(x)\}$, and $\nu(x) = \text{card } I(x)$. The identification $x \equiv x^-$ is assumed hereafter for the sake of brevity. We will also denote $x_0 = 0$.

Consider the fractional differential equation involving impulses

$$D^\alpha y(x) = Ay(x) + Bu(x) + \sum_{i \in I} C_i \delta(x - x_i), \tag{3.1}$$

where $0 < \alpha < 1$, $y : \mathbb{R}_+ \to \mathbb{R}^n$, $u : \mathbb{R}_+ \to \mathbb{R}^m$, A, B are constant matrices of size $n \times n$ and $n \times m$ respectively, $0 < \alpha < 1$, and $C_i \in \mathbb{R}^n$ are fixed constant vectors, under the initial condition

$$J^{1-\alpha} y(x)\big|_{x=0+} = y_0. \tag{3.2}$$

Remark 3.1. Although $y(x)$ is not defined at the impulses' instants x_i, in the sequel we implicitly assume solutions of (3.1) to be left-continuous by formally defining the Heaviside unit step function as left-continuous:

$$\chi(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases} \tag{3.3}$$

Lemma 3.2. *The initial value problem* (3.1)*,* (3.2) *has a unique solution given by the formula*

$$\begin{aligned} y(x) =& x^{\alpha-1} E_{\alpha,\alpha}(Ax^\alpha) y_0 \\ &+ \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(A(x-t)^\alpha) Bu(t) dt \\ &+ \sum_{i \in I} \chi(x - x_i)(x - x_i)^{\alpha-1} E_{\alpha,\alpha}(A(x - x_i)^\alpha) C_i \\ =& x^{\alpha-1} E_{\alpha,\alpha}(Ax^\alpha) y_0 \\ &+ \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(A(x-t)^\alpha) Bu(t) dt \\ &+ \sum_{i \in I(x)} (x - x_i)^{\alpha-1} E_{\alpha,\alpha}(A(x - x_i)^\alpha) C_i \end{aligned} \tag{3.4}$$

provided that the integral on the right-hand side converges.

Proof. Let us apply the Laplace transform to the both sides of (3.1). In view of (2.5), (3.2) we obtain

$$s^\alpha Y(s) - y_0 = AY(s) + BU(s) + \sum_{i \in I} e^{-x_i s} C_i$$

or

$$Y(s) = (s^\alpha I - A)^{-1} y_0 + (s^\alpha I - A)^{-1} BU(s) + \sum_{i\in I} e^{-x_i s}(s^\alpha I - A)^{-1} C_i.$$

In virtue of (2.3), (2.4), and Lemma 2.1 this yields the formula (3.4) for the solution of (3.1), (3.2). The uniqueness of (3.4) follows from the uniqueness of the Laplace transform. □

Remark 3.3. The solution to the IVP (3.1), (3.2) given by (3.4) becomes infinite at x_i^+, $i \in I$ and is left-continuous.

Remark 3.4. If $\alpha = 1$, then the IVP (3.1), (3.2) takes on the form

$$\begin{aligned}\dot{y}(x) &= Ay(x) + Bu(x) + \sum_{i\in I} C_i \delta(x - x_i),\\ y(0+) &= y_0.\end{aligned}$$

In view of (2.2), the solution (3.4) becomes

$$y(x) = e^{Ax} y_0 + \int_0^x e^{A(x-t)} Bu(t)dt + \sum_{i\in I(x)} e^{A(x-x_i)} C_i,$$

which is consistent with known formulas for the solutions of linear impulsive systems with constant coefficients.

3.2 State-dependent Impulses

To incorporate possible state-dependence of impulses, let us modify equation (3.1) as follows:

$$D^\alpha y(x) = Ay(x) + Bu(x) + \sum_{i\in I} c_i y(x)\delta(x - x_i), \tag{3.5}$$

where c_i are fixed constants. The latter equation implies that at time instances x_i the system is affected by the impulses $c_i y(x_i)\delta(0)$, while on the intervals (x_i, x_{i+1}) its evolution is governed by the equation $D^\alpha y(x) = Ay(x) + Bu(x)$.

Lemma 3.5. *On each interval* $(x_{\nu(x)-1}, x_{\nu(x)}]$ *the equation* (3.5) *equipped with the ini-*

tial condition (3.2) *has a unique solution given by the formula*

$$\begin{aligned} y(x) =& x^{\alpha-1}E_{\alpha,\alpha}(Ax^\alpha)y_0 \\ &+\int_0^x (x-t)^{\alpha-1}E_{\alpha,\alpha}(A(x-t)^\alpha)Bu(t)dt \\ &+\sum_{i=1}^{\nu(x)} c_i\chi(x-x_i)(x-x_i)^{\alpha-1}E_{\alpha,\alpha}(A(x-x_i)^\alpha)y(x_i) \\ =& x^{\alpha-1}E_{\alpha,\alpha}(Ax^\alpha)y_0 \\ &+\int_0^x (x-t)^{\alpha-1}E_{\alpha,\alpha}(A(x-t)^\alpha)Bu(t)dt \\ &+\sum_{i=1}^{\nu(x)} c_i(x-x_i)^{\alpha-1}E_{\alpha,\alpha}(A(x-x_i)^\alpha)y(x_i) \end{aligned} \tag{3.6}$$

provided that the integral on the right-hand side converges.

Proof. Applying the Laplace transform to the both sides of (3.5) and taking into account (2.5), (3.2) we obtain

$$s^\alpha Y(s) - y_0 = AY(s) + BU(s) + \sum_{i=1}^{\nu(x)} c_i e^{-x_i s} y(x_i)$$

or

$$Y(s) = (s^\alpha I - A)^{-1}y_0 + (s^\alpha I - A)^{-1}BU(s) + \sum_{i=1}^{\nu(x)} c_i e^{-x_i s}(s^\alpha I - A)^{-1}y(x_i).$$

The latter expression yields, in view of (2.3), (2.4), and Lemma 2.1, the formula (3.6) for the solution of (3.5), (3.2). The uniqueness of (3.6) follows from the uniqueness of the Laplace transform. □

4 Example

From the practical point of view, of considerable interest is the equation of the form

$$az'' + bD^\nu z + cz = u, \quad 0 < \nu < 2,$$

describing oscillations with fractional damping. Equations of such kind arise in describing vibrations of a plane wing in supersonic gas flow [12] resulting in the flutter-type phenomena, nano-dimensional sensors [10], etc. At $\nu = 3/2$ this equation is called the Bagley–Torvik equation and describes the motion of a rigid plate in a Newtonian fluid [1].

Consider the Bagley–Torvik equation involving additive delta function on the right-hand side

$$az''(x) + bD^{3/2}z(x) + cz(x) = u(x) + d\delta(x - x_1), \tag{4.1}$$

where a, b, c, and d are some constants, under equilibrium initial state conditions

$$z(0) = 0, \quad z'(0) = 0. \tag{4.2}$$

Under the assumption (4.2) the Riemann–Liouville derivative coincides with the regularized Caputo derivative, defined by the expression

$$D^{(\alpha)}f(x) = J^{k-\alpha}\frac{d^k}{dx^k}f(x), \quad k - 1 < \alpha < k.$$

Denote $y_1(x) = z(x)$, $y_2(x) = D^{1/2}z(x)$, $y_3(x) = z'(x)$, $y_4(x) = D^{3/2}z(x)$. Then [9], $D^{1/2}y_1 = D^{(1/2)}y_1 = y_2$, $D^{1/2}y_2 = D^{(1/2)}y_2 = y_3$, $D^{1/2}y_3 = D^{(1/2)}y_3 = y_4$, $D^{1/2}y_4 = D^{(1/2)}y_4 = z''$. Therefore, the equation (4.1) under the initial conditions (4.2) is equivalent to the system

$$D^{1/2}y = Ay + Bu + C\delta(x - x_1), \tag{4.3}$$

where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c/a & 0 & 0 & -b/a \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/a \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d/a \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$
under zero initial conditions

$$y(0) = y^0 = \operatorname{col}(0, 0, 0, 0). \tag{4.4}$$

By virtue of (3.4), the solution to this system is given by the formula

$$y(x) = \int_0^x \frac{1}{\sqrt{x - t}} E_{\frac{1}{2},\frac{1}{2}}\left(A\sqrt{x - t}\right) Bu(t)dt + \frac{\chi(x - x_1)}{\sqrt{(x - x_1)}} E_{\frac{1}{2},\frac{1}{2}}(A\sqrt{x - x_1})C. \tag{4.5}$$

In order to evaluate Mittag–Leffler matrix function $E_{\frac{1}{2};\frac{1}{2}}\left(A\sqrt{x}\right)$, one can apply technique of the Lagrange–Sylvester interpolation polynomials. In such a way the function can be expressed in terms of scalar generalized Mittag–Leffler function [6].

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