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Positive Solutions to a Second Order Dynamic Equation

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Abstract

In this paper, we establish some criteria for the existence of multiple positive solutions for certain two point boundary value problems for the dynamic equation

$$-u^{\Delta\Delta} = f(t, u^\sigma)$$

on a time scale $\mathbb{T}$.

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1 Introduction

Let $\mathbb{T}$ be a time scale, and suppose that $[a, \sigma^2(b)] \subseteq \mathbb{T}$. In this paper we consider the dynamic equation

$$-u^{\Delta\Delta} = f(t, u^\sigma) \qquad a \le t \le b \tag{1.1}$$

and the boundary conditions

$$\begin{cases} \alpha u(a) - \beta u^\Delta(a) &= 0 \\ \gamma u(\sigma^2(b)) + \delta u^\Delta(\sigma(b)) &= 0. \end{cases} \tag{1.2}$$

Here, f is continuous and $f(t, u) \ge 0$ for $t \in [a, b]$ and $u \ge 0$, α, β, γ, $\delta \ge 0$ and $\rho := \gamma\beta(\sigma^2(b) - a) + \alpha\delta + \alpha\gamma > 0$. This problem has been studied in [3–10] and the references therein. Results of positive solutions to the differential equation ($\mathbb{T} = \mathbb{R}$)

$$-u'' = f(t, u) \qquad a \le t \le b$$

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are well-known and presented in the references above. Our goal here is to extend the results to a general time scale $\mathbb{T}$, by defining conditions on f under which (1.1)–(1.2) has positive solutions.

For $u > 0$, define

$$m(u) := \min_{t\in[a,b]} \frac{f(t,u)}{u},$$
$$M(u) := \max_{t\in[a,b]} \frac{f(t,u)}{u}.$$

Let $m(0)$, $M(0)$, $m(\infty)$, $M(\infty)$ denote limits of $m(u)$ and $M(u)$ as $u \to 0^+$ or $u \to \infty$ provided these limits exist in the extended reals. The conditions $m(0) = \infty = m(\infty)$ correspond to the assumption that f is sublinear at $u = 0$ and superlinear at $u = \infty$ (f is sub-superlinear). $M(0) = 0 = M(\infty)$ corresponds to the assumption that f is superlinear at $u = 0$ and sublinear at $u = \infty$ (f is super-sublinear). If $M(0) = 0$, $m(\infty) = \infty$, f is superlinear at both $u = 0$ and $u = \infty$. If $m(0) = \infty$, $M(\infty) = 0$, f is sublinear at both $u = 0$ and $u = \infty$.

Let $G(t,s)$ denote the Green's function to $-u^{\Delta\Delta} = 0$ subject to boundary conditions (1.2). Then

$$G(t,s) = \frac{1}{c}\begin{cases} \phi(t)\psi^\sigma(s) & t \le s \\ \phi^\sigma(s)\psi(t) & \sigma(s) \le t \end{cases} \tag{1.3}$$

where c is the constant $c = -W(\phi,\psi)(t) > 0$, and where ϕ and ψ are the solutions of

$$-\phi^{\Delta\Delta} = 0, \qquad \phi(a) = \beta, \quad \phi^\Delta(a) = \alpha \tag{1.4}$$

$$-\psi^{\Delta\Delta} = 0, \qquad \psi(\sigma^2(b)) = \delta, \quad \psi^\Delta(\sigma(b)) = -\gamma \tag{1.5}$$

respectively.

Lemma 1.1. $0 \le G(t,s) \le G(\sigma(s),s)$ *for all* $a \le t \le \sigma^2(b)$, $a \le s \le b$.

Proof. From equations (1.4) and (1.5), we see that $\phi(t) = \alpha(t-a)+\beta$ and $\psi(t) = -\gamma(t-\sigma^2(b))+\delta$. Then clearly, $\phi(t) \ge 0$ and $\psi(t) \ge 0$ for all $t \in (a,\sigma^2(b))$. Thus, $G(t,s) \ge 0$. Moreover, $G(t,s)$ is increasing (in t) if $t \le s$ and decreasing if $t \ge \sigma(s)$). By [2, Theorem 4.73], we have $G(t,\sigma(s)) = G(\sigma(s),t)$. Thus, the maximum point occurs at $t = s$, giving the result. □

It is well known that the BVP (1.1)–(1.2) has a solution if and only if u solves the integral equation

$$u(t) = \int_a^{\sigma(b)} G(t,s) f(s,u^\sigma(s))\Delta s := (Fu)(t).$$

Let $\xi := \min\left\{\frac{\gamma+4\delta}{4(\gamma+\delta)}, \frac{\alpha+4\beta}{4(\alpha+\beta)}\right\}$. Choose $t_1, t_2 \in \mathbb{T}$ such that

$$\frac{G(t,s)}{G(\sigma(s),s)} \ge \xi \tag{1.6}$$

for $t \in [t_1, \sigma(t_2)] := I$. Note that it is always possible to pick such an interval: for $s \in [a, b]$, if $\sigma(s)$ is either right-scattered or left-scattered, by continuity of G, there are $t_1 \neq t_2$ satisfying (1.6). If $\sigma(s)$ is isolated, it may be possible that $t_1 = t_2 = \sigma(s)$ is the only t-value satisfying (1.6). In this case, $\sigma^2(s) > \sigma(s)$, and we take $I = [\sigma(s), \sigma^2(s)]$. We define a cone $K \subseteq C[a, \sigma^2(b)]$ by

$$K := \left\{ u \in C[a, \sigma^2(b)] : u(t) \geq 0, \text{ and } \min_{t \in I} u^\sigma(t) \geq \xi \|u\| \right\}.$$

Lemma 1.2. $F(K) \subseteq K$.

Proof. Note that as $G(t, s) \geq 0$ for all $t \in [a, \sigma^2(b)]$ and $s \in [a, b]$, and $f(t, u) \geq 0$ for $u \geq 0$, we get $(Fu)(t) \geq 0$ for all $t \in [a, b]$. Let $t \in I$ and $u \in K$. Then

$$\begin{aligned}
(Fu)^\sigma(t) &= \int_a^{\sigma(b)} G(\sigma(t), s) f(s, u^\sigma(s)) \Delta s \\
&\geq \xi \int_a^{\sigma(b)} G(\sigma(s), s) f(s, u^\sigma(s)) \Delta s \\
&= \xi \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s) f(s, u^\sigma(s)) \Delta s \\
&= \xi \|Fu\|.
\end{aligned}$$

□

Fix $\tau_0 \in I$ and let

$$\eta = \left(\int_a^{\sigma(b)} G(\sigma(s), s) \Delta s \right)^{-1}$$

and

$$\theta = \left(\int_I G(\tau_0, s) \Delta s \right)^{-1}.$$

We will define two conditions on f (appearing in [3] and [9]):

(C_1) There is a $p > 0$ such that $0 \leq u \leq p$ and $a \leq t \leq b$ implies $f(t, u) \leq \eta p$.

(C_2) There is a $p > 0$ such that $\xi p \leq u \leq p$ and $t \in I$ implies $f(t, u) \geq \theta p$.

We now state the following result due to Krasnoselskii (see Guo and Lakshmikantham [11, Theorem 2.3.4]).

Lemma 1.3. *Let E be a Banach space, $K \subseteq E$ a cone and assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$ *or*

(ii) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Main Results

The first two theorems occurred in the continuous case in [3]. Here, they are presented for a general time scale $\mathbb{T}$.

Theorem 2.1 (f sub-superlinear)**.** *Assume* $m(0) = \infty = m(\infty)$ *and that there exist* $0 < p_1 < q < p_2$ *with* (C_1) *holding for* $p = p_1$ *and* $p = p_2$ *and* (C_2) *holds for* $p = q$. *Then the BVP* (1.1)–(1.2) *has at least four positive solutions satisfying*

$$0 < \|u_1\| < p_1 < \|u_2\| < q < \|u_3\| < p_2 < \|u_4\|.$$

Proof. Assume $m(0) = \infty$. Let $B > \dfrac{\theta}{\xi}$. Then there is $0 < r < p_1$ such that for $0 \leq u \leq r$, we have $f(t, u) > Bu$ for all $t \in [a, \sigma^2(b)]$. Define $K_r = \{u \in K : \|u\| \leq r\}$. Let $u \in \partial K_r$. Then

$$\begin{aligned}(Fu)(\tau_0) &= \int_a^{\sigma(b)} G(\tau_0, s) f(s, u^\sigma(s))\Delta s \\ &> B \int_I G(\tau_0, s) u^\sigma(s)\Delta s \\ &\geq \xi B\|u\| \int_I G(\tau_0, s)\Delta s \\ &> \|u\|.\end{aligned}$$

Thus, for $u \in \partial K_r$, we have $\|Fu\| > \|u\|$.

Similarly, since $m(\infty) = \infty$, there is $0 < p_2 < R_1$ such that $f(t, u) > Bu$ for $u \geq R_1$. Choose $R > \dfrac{R_1}{\xi}$. Then for $u \in \partial K_R$, we have

$$\begin{aligned}(Fu)(\tau_0) &= \int_a^{\sigma(b)} G(\tau_0, s) f(s, u^\sigma(s))\Delta s \\ &> B \int_I G(\tau_0, s) u^\sigma(s)\Delta s \\ &\geq \xi B\|u\| \int_I G(\tau_0, s)\Delta s \\ &> \|u\|.\end{aligned}$$

Now assume $u \in \partial K_{p_1}$. Then

$$\begin{aligned}\|Fu\| &= \max_{a\le t\le b}\int_a^{\sigma(b)} G(t,s)f(s,u^\sigma(s))\Delta s\\ &\le \int_a^{\sigma(b)} G(\sigma(s),s)f(s,u^\sigma(s))\Delta s\\ &\le \eta p_1\int_a^{\sigma(b)} G(\sigma(s),s)\Delta s\\ &\le p_1 = \|u\|.\end{aligned}$$

Thus, for $u \in \partial K_{p_1}$, we have $\|Fu\| \le \|u\|$. Similarly, for $u \in \partial K_{p_2}$, we have

$$\begin{aligned}\|Fu\| &= \max_{a\le t\le b}\int_a^{\sigma(b)} G(t,s)f(s,u^\sigma(s))\Delta s\\ &\le \int_a^{\sigma(b)} G(\sigma(s),s)f(s,u^\sigma(s))\Delta s\\ &\le \eta p_2\int_a^{\sigma(b)} G(\sigma(s),s)\Delta s\\ &\le p_2 = \|u\|.\end{aligned}$$

Assume now that $u \in \partial K_q$. Then

$$\begin{aligned}(Fu)(\tau_0) &= \int_a^{\sigma(b)} G(\tau_0,s)f(s,u^\sigma(s))\Delta s\\ &\ge \theta q\int_I G(\tau_0,s)\Delta s\\ &= q = \|u\|.\end{aligned}$$

Thus, we have for $u \in \partial K_q$, we have $\|Fu\| \ge \|u\|$. Therefore, the conclusion of the theorem holds. □

Next, we state the theorem for the super-sublinear case, and the proof is omitted as it is similar to the proof of Theorem 2.1.

Theorem 2.2 (f super-sublinear)**.** *Assume $M(0) = 0 = M(\infty)$ and that there exist $0 < p_1 < q < p_2$ with (C_2) holding for $p = p_1$ and $p = p_2$ and (C_1) holds for $p = q$. Then the BVP* (1.1)–(1.2) *has at least four positive solutions satisfying*

$$0 < \|u_1\| < p_1 < \|u_2\| < q < \|u_3\| < p_2 < \|u_4\|.$$

This result can be extended to obtain any number of finite or infinite positive solutions.

Corollary 2.3 (f sub-superlinear)**.** *Assume $m(0) = \infty = m(\infty)$ and that there exists a sequence of numbers $0 < p_1 < q_1 < p_2 < q_2 < \ldots$ with (C_1) holding for $p = p_k$ for each k and (C_2) holds for $p = q_k$ for each k. Then the BVP* (1.1)–(1.2) *has positive solutions satisfying*

$$0 < \|u_1\| < p_1 < \|u_2\| < q_1 < \|u_3\| < p_2 < \|u_4\| < q_2 < \|u_5\| < \ldots.$$

Corollary 2.4 (f super-sublinear)**.** *Assume $M(0) = 0 = M(\infty)$ and that there exists a sequence of numbers $0 < p_1 < q_1 < p_2 < q_2 < \ldots$ with (C_2) holding for $p = p_k$ for each k and (C_1) holds for $p = q_k$ for each k. Then*

$$0 < \|u_1\| < p_1 < \|u_2\| < q_1 < \|u_3\| < p_2 < \|u_4\| < q_2 < \|u_5\| < \ldots.$$

The proofs of these corollaries follow the same method as the proof of Theorem 2.1. Next, we give results for the sublinear and superlinear cases.

Theorem 2.5 (f sublinear)**.** *Assume $m(0) = \infty$ and $M(\infty) = 0$ and that there exist $0 < p < q$ with (C_1) holding for p and (C_2) holding for q. Then the BVP* (1.1)–(1.2) *has at least three positive solutions satisfying*

$$0 < \|u_1\| < p < \|u_2\| < q < \|u_3\|.$$

Theorem 2.6 (f superlinear)**.** *Assume $M(0) = 0$ and $m(\infty) = \infty$ and that there exist $0 < p < q$ with (C_2) holding for p and (C_1) holding for q. Then the BVP* (1.1)–(1.2) *has at least three positive solutions satisfying*

$$0 < \|u_1\| < p < \|u_2\| < q < \|u_3\|.$$

Again, the proofs of these two theorems follow the same method as the proof of Theorem 2.1.

We can use a fixed point theorem due to Leggett and Williams (see [12]) to find multiple positive solutions.

Definition 2.7. A continuous map $\alpha : K \to [0, \infty)$ is a concave positive functional if

$$\alpha(\lambda x + (1 - \lambda)y) \geq \lambda\alpha(x) + (1 - \lambda)\alpha(y)$$

for $0 \leq \lambda \leq 1$ and $x, y \in K$.

Lemma 2.8 (See [12])**.** *Assume $F : K_c \to K_c$ is completely continuous and there exists a concave positive functional $\alpha(u)$ such that $\alpha(u) \leq \|u\|$ for $u \in K_c$. Also, assume $0 < d < a < b \leq c$ such that*

(i) $a \leq \alpha(u)$ and $|u| \leq b$ implies $\alpha(Fu) > a$, where we assume $\{x \in K : a < \alpha(x),\ |x| \leq b\} \neq \emptyset$.

(ii) $a \le \alpha(u)$, $|u| \le c$ *and* $\|Fu\| > b$ *implies* $\alpha(Fu) > a$.

(iii) $u \in K_d$ *implies* $\|Fu\| < d$.

Then F has at least two positive fixed points.

Theorem 2.9. *Assume $M(0) = 0 = M(\infty)$ and there is $a > 0$ such that $f(t,u) \ge \frac{a}{\omega}$ for $a \le u \le \frac{a}{\xi}$ where*

$$\omega = \min_{t \in I} \int_I G(\sigma(t), s)\Delta s.$$

Then (1.1)–(1.2) *has at least two positive solutions.*

Proof. $M(0) = 0 = M(\infty)$ implies that there exist r and R such that $0 < r < a < \frac{a}{\xi} < R$ and $\|Fu\| < \|u\|$ for $u \in \partial K_r$ and also for $u \in \partial K_R$. Let $d = r$, $b = \frac{a}{\xi}$, and $c = R$ in the above lemma. Define $\alpha(u) := \min_{t \in I} u^\sigma(t)$. We need to check that all the conditions of the lemma are satisfied.
(i) If $a \le \alpha(u)$ and $|u| \le b$, then $a \le u^\sigma(t) \le b$ for all $t \in I$, and so

$$\begin{aligned}
\alpha(Fu) &= \min_{t \in I} \int_a^{\sigma(b)} G(\sigma(t), s) f(s, u^\sigma(s))\Delta s \\
&> \min_{t \in I} \int_I G(\sigma(t), s) a\omega^{-1}\Delta s \\
&= a
\end{aligned}$$

(ii) If $a \le \alpha(u)$, $|u| \le c$ and $\|Fu\| > b$, then since $G(\sigma(t), s) \ge \xi G(\sigma(s), s) \ge \xi G(\phi, s)$ for $t \in I$ and for any $\phi, s \in [a, \sigma^2(b)]$, we have that $\alpha(Fu) \ge \xi\|Fu\| > \xi b = a$.
(iii) $M(0) = 0$, so given $B > 0$, there is $r > 0$ such that for $0 < u < r$ we have $f(t,u) \le Bu$ for all $t \in [a, \sigma^2(b)]$. Let $B < \left(\int_a^{\sigma(b)} G(\sigma(s), s)\Delta s\right)^{-1}$. Then, if $u \in K_d$ we have

$$\begin{aligned}
\|Fu\| &= \int_a^{\sigma(b)} G(\sigma(s), s) f(s, u^\sigma(s))\Delta s \\
&\le B\|u\| \int_a^{\sigma(b)} G(\sigma(s), s)\Delta s \\
&< \|u\|
\end{aligned}$$

□

Next, we do not assume that f is continuous in u, but we shall assume f is monotone. The following lemma is due to Amann [1].

Lemma 2.10. *Let (X, K) be an ordered Banach space with a normal cone K. Assume for $u, v \in K$ with $u < v$, $F : [u, v] \to X$ is compact and increasing and satisfies*

$$u \leq Fu, \qquad Fv \leq v,$$

then F has maximal and minimal fixed points in $[u, v]$.

Theorem 2.11. *Assume $m(0) = \infty$ and $M(\infty) = 0$ and that $f(t, u)$ is increasing in u. Then the BVP* (1.1)–(1.2) *has at least one positive solution.*

Proof. By the previous lemma, we need to find $u \leq v$ which satisfy $u \leq Fu$ and $Fv \leq v$. By sublinearity of f, for any $\varepsilon > 0$, there is $N > 0$ such that $f(t, u) \leq N + \varepsilon u$ for $u \geq 0$ and $t \in [a, \sigma^2(b)]$. Hence, for $u \in K$,

$$(Fu)(t) \leq \int_a^{\sigma(b)} G(t, s)(N + \varepsilon u^\sigma(s))\Delta s.$$

Thus, if we take $v = R$ for R sufficiently large, we have $Fv \leq v$.

Now, let $\alpha_\varepsilon(t) := \varepsilon \int_a^{\sigma(b)} G(t, s)\Delta s$. By sublinearity, for any $N > 0$ and sufficiently small $\varepsilon > 0$, we have

$$f(t, \alpha_\varepsilon(t)) \geq N\alpha_\varepsilon(t).$$

Let $\delta_0 := \min_{t \in I} \int_a^{\sigma(b)} G(t, s)\Delta s > 0$. Since $e(t) := \int_a^{\sigma(b)} G(t, s)\Delta s > 0$ for $t \in [a, \sigma^2(b)]$ and $e^\Delta(a) > 0$ if $e(a) = 0$ and $e^\Delta(\sigma(b)) < 0$ if $e(\sigma^2(b)) = 0$, one can show that there exists $\delta_1 > 0$ such that

$$\int_I G(t, s)\Delta s \geq \delta_1 \int_a^{\sigma(b)} G(t, s)\Delta s.$$

Thus, we have

$$\begin{aligned}
(F(\alpha_\varepsilon))(t) &\geq N \int_a^{\sigma(b)} G(t, s)\alpha_\varepsilon^\sigma(s)\Delta s \\
&\geq N\,\delta_0\,\varepsilon \int_I G(t, s)\Delta s \\
&\geq N\,\delta_0\,\delta_1\,\varepsilon \int_a^{\sigma(b)} G(t, s)\Delta s \\
&= N\,\delta_0\,\delta_1\,\alpha_\varepsilon(t).
\end{aligned}$$

Noting that δ_0 and δ_1 are independent of N, choose N large enough so that $N\delta_0\delta_1 > 1$. Then let $u = \alpha_\varepsilon(t)$. Therefore, we have $u \leq Fu$ and $u \leq v$. □

3 Example

Example 3.1. We give an example of a dynamic equation that has at least 4 positive solutions using Theorem 2.1. Let $\mathbb{T} = q^{\mathbb{N}_0} \cup \{0\}$ for $q = \dfrac{1}{2}$. Then consider

$$-u^{\Delta\Delta} = f(t, u^\sigma) \qquad 0 \leq t \leq \frac{1}{4}$$

with boundary conditions

$$u(0) = 0 = u^\Delta\left(\frac{1}{2}\right),$$

where

$$f(t,u) = \begin{cases} \sqrt{u} & 0 \leq u \leq \dfrac{1}{36} \\ 216u^2 & \dfrac{1}{36} \leq u \leq 1 \\ 216\sqrt{u} & 1 \leq u \leq 1296 \\ \dfrac{1}{216}u^2 & u \geq 1296. \end{cases}$$

Then, $f \geq 0$ is continuous for $u \geq 0$, and $m(0) = \infty = m(\infty)$. The Green's function for this BVP is

$$G(t,s) = \begin{cases} t & t \leq s \\ \sigma(s) & \sigma(s) \leq t. \end{cases}$$

Note that $G(t,s)$ is defined on $[0,1] \times [0,1/4]$. We have $\xi = \dfrac{1}{4}$, thus

$$\frac{G(t,s)}{G(\sigma(s),s)} \geq \frac{t}{1/2},$$

therefore, $\dfrac{G(t,s)}{G(\sigma(s),s)} \geq \xi$ when $t \geq \dfrac{1}{8}$. Then the interval I is $I = \left[\dfrac{1}{8}, 1\right]$. Also,

$$\begin{aligned}
\eta^{-1} &= \int_0^{1/2} \sigma(s)\Delta s \\
&= \sum_{t\in[0,1/2)} \mu(s)\sigma(s) \\
&= \sum_{k=2}^{\infty}((1/2)^{k-1} - (1/2)^k)(1/2)^{k-1} \\
&= \sum_{k=2}^{\infty}(1/2)^{2k-1} \\
&= 2\sum_{k=2}^{\infty}(1/4)^k \\
&= \frac{1}{6}.
\end{aligned}$$

Thus, $\eta = 6$. Now let $\tau_0 = \dfrac{1}{4}$. Then

$$\begin{aligned}
\theta^{-1} &= \int_{1/8}^{1} G(1/4, s)\Delta s \\
&= \int_{1/8}^{1/4} \sigma(s)\Delta s + \int_{1/4}^{1} \frac{1}{4}\Delta s \\
&= \mu\left(\frac{1}{8}\right)\sigma\left(\frac{1}{8}\right) + \frac{3}{4}\cdot\frac{1}{4} \\
&= \frac{7}{32}.
\end{aligned}$$

Hence, $\theta = \dfrac{32}{7}$. Thus, we have that $f(t,u)$ satisfies $f(t,u) \leq 6p$ for $u \leq p$ if $p = \dfrac{1}{36}$ and $p = 1296$. Also, $f(t,u) \geq \dfrac{32}{7}q$ for $\dfrac{q}{4} \leq u \leq q$ when $q = \dfrac{64}{189}$. Thus, by Theorem 2.1, the dynamic equation has at least 4 positive solutions satisfying

$$0 < \|u_1\| < \frac{1}{36} < \|u_2\| < \frac{64}{189} < \|u_3\| < 1296 < \|u_4\|.$$

Example 3.2. We give an example of a dynamic equation that has at least 1 positive solution using Theorem 2.11. Let $\mathbb{T}$ be any time scale, and consider

$$-u^{\Delta\Delta} = f(t, u^{\sigma}) \qquad a \leq t \leq b$$

with boundary conditions

$$\begin{cases} \alpha u(a) - \beta u^{\Delta}(a) & = 0 \\ \gamma u(\sigma^2(b)) + \delta u^{\Delta}(\sigma(b)) & = 0. \end{cases}$$

where

$$f(t,u) = \begin{cases} \sqrt{u} & 0 \le u \le 1 \\ \sqrt{u} + 1 & u \ge 1. \end{cases}$$

The f is increasing in u, and $m(0) = \infty$ and $M(\infty) = 0$. Note, for any $A \ge 1$, we have $f(t,u) \le A + \dfrac{1}{A}u$. Then

$$(Fu)(t) \le \int_a^{\sigma(b)} G(t,s)\left(A + \frac{1}{A}u\right)\Delta s = \left(A + \frac{1}{A}u\right)\int_a^{\sigma(b)} G(t,s)\Delta s.$$

Pick $A > \displaystyle\int_a^{\sigma(b)} G(t,s)\Delta s$. Then for sufficiently large u, we get $Fu \le u$. Now, let $N > 0$ and let $u = \dfrac{1}{N^2}$, we have $f(t,u) \ge Nu$. Following the notation of the proof of Theorem 2.11, we have

$$(F(u))(t) \ge N\,\delta_0\,\delta_1\,u.$$

Pick N large enough so that $N\,\delta_0\,\delta_1 \ge 1$. Then $Fu \ge u$. By Lemma 2.10, we get the desired result.

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