International Journal of Difference Equations
ISSN 0973-6069, Volume 8, Number 2, pp. 215–223 (2013)
http://campus.mst.edu/ijde

# On a System of Two Difference Equations of Exponential Form

**Vu Van Khuong** and **Mai Nam Phong**
University of Transport and Communications
Department of Mathematical Analysis
Hanoi City, Vietnam
vuvankhuong@gmail.com and mnphong@gmail.com

**Abstract**

The goal of this paper is to study the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form:

$$x_{n+1} = \frac{a + be^{-x_n}}{c + y_n},\ y_{n+1} = \frac{a + be^{-y_n}}{c + x_n}$$

where $a, b, c$ are positive constants and the initial values $x_0, y_0$ are positive real values. Also, we determine the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of this system.

**AMS Subject Classifications:** 39A10.
**Keywords:** Difference equations, boundedness, persistence, asymptotic behavior, rate of convergence.

# 1 Introduction

In [10], the authors studied the boundedness, the asymptotic behavior, the periodicity and the stability of the positive solutions of the difference equation:

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}$$

where $\alpha, \beta, \gamma$ are positive constants and the initial values $y_{-1}, y_0$ are positive numbers.

Received November 14, 2012; Accepted May 9, 2013
Communicated by Martin Bohner

Motivated by the above paper, we will investigate the boundedness, the persistence and the asymptotic behavior of the positive solutions of the following system of exponential form:

$$x_{n+1} = \frac{a + be^{-x_n}}{c + y_n},\ y_{n+1} = \frac{a + be^{-y_n}}{c + x_n} \tag{1.1}$$

where $a, b, c$ are positive constants and the initial values $x_0, y_0$ are positive real values.

Difference equations and systems of difference equations of exponential form can be found in the following papers: [1, 3–5, 7]. Moreover, as difference equations have many applications in applied sciences, there are many papers and books that can be found concerning the theory and applications of difference equations, see [2, 6, 9] and the references cited therein.

# 2 Global Behavior of Solutions of the System

In the first lemma we study the boundedness and persistence of the positive solutions of (1.1).

**Lemma 2.1.** *Every positive solution of* (1.1) *is bounded and persists.*

*Proof.* Let $(x_n, y_n)$ be an arbitrary solution of (1.1). From (1.1), we can see that

$$x_n \leq \frac{a+b}{c},\ y_n \leq \frac{a+b}{c},\ n = 1, 2, \dots \tag{2.1}$$

In addition, from (1.1) and (2.1) we get

$$x_n \geq \frac{a + be^{-\frac{a+b}{c}}}{c + \frac{a+b}{c}},\ y_n \geq \frac{a + be^{-\frac{a+b}{c}}}{c + \frac{a+b}{c}},\ n = 2, 3, \dots \tag{2.2}$$

Therefore, from (2.1) and (2.2) the proof of lemma is complete. □

In order to prove the main result of this section, we recall the next theorem without its proof. See [11] and [12].

**Theorem 2.2.** *Let* $\mathcal{R} = [a_1, b_1] \times [c_1, d_1]$ *and*

$$f : \mathcal{R} \longrightarrow [a_1, b_1],\ g : \mathcal{R} \longrightarrow [c_1, d_1]$$

*be a continuous functions such that:*

*(a)* $f(x, y)$ *is decreasing in both variables and* $g(x, y)$ *is decreasing in both variables for each* $(x, y) \in \mathcal{R}$*;*

*(b) If* $(m_1, M_1, m_2, M_2) \in \mathcal{R}^2$ *is a solution of*

$$\begin{cases} M_1 & = f(m_1, m_2),\ m_1 = f(M_1, M_2), \\ M_2 & = g(m_1, m_2),\ m_2 = g(M_1, M_2), \end{cases} \tag{2.3}$$

*then* $m_1 = M_1$ *and* $m_2 = M_2$*.*

*Then the system of difference equations*

$$x_{n+1} = f(x_n, y_n),\ y_{n+1} = g(x_n, y_n) \tag{2.4}$$

*has a unique equilibrium* $(\bar{x}, \bar{y})$ *and every solution* $(x_n, y_n)$ *of the system* (2.4) *with* $(x_0, y_0) \in \mathcal{R}$ *converges to the unique equilibrium* $(\bar{x}, \bar{y})$*. In addition, the equilibrium* $(\bar{x}, \bar{y})$ *is globally asymptotically stable.*

Now we state the main theorem of this section.

**Theorem 2.3.** *Consider system* (1.1)*. Suppose that the following relation holds true:*

$$b < c. \tag{2.5}$$

*Then system* (1.1) *has a unique positive equilibrium* $(\bar{x}, \bar{y})$ *and every positive solution of* (1.1) *tends to the unique positive equilibrium* $(\bar{x}, \bar{y})$ *as* $n \to \infty$*. In addition, the equilibrium* $(\bar{x}, \bar{y})$ *is globally asymptotically stable.*

*Proof.* We consider the functions

$$f(u, v) = \frac{a + be^{-u}}{c + v},\ \ g(u, v) = \frac{a + be^{-v}}{c + u} \tag{2.6}$$

where

$$u, v \in I = \left[ \frac{a + be^{-\frac{a+b}{c}}}{c + \frac{a+b}{c}},\ \frac{a+b}{c} \right]. \tag{2.7}$$

It is easy to see that $f(u, v), g(u, v)$ are decreasing in both variables for each $(u, v) \in I \times I$. In addition, from (2.6) and (2.7) we have $f(u, v) \in I$, $g(u, v) \in I$ as $(u, v) \in I \times I$ and so $f : I \times I \longrightarrow I$, $g : I \times I \longrightarrow I$.

Now let $m_1, M_1, m_2, M_2$ be positive real numbers such that

$$M_1 = \frac{a + be^{-m_1}}{c + m_2},\ M_2 = \frac{a + be^{-m_2}}{c + m_1},\ m_1 = \frac{a + be^{-M_1}}{c + M_2},\ m_2 = \frac{a + be^{-M_2}}{c + M_1}. \tag{2.8}$$

Moreover arguing as in the proof of Theorem.2.2, it suffices to assume that

$$m_1 \leq M_1,\ m_2 \leq M_2. \tag{2.9}$$

From (2.8), we get

$$\begin{aligned} be^{-m_1} &= (c+m_2)M_1 - a,\ be^{-M_1} = (c+M_2)m_1 - a, \\ be^{-m_2} &= (c+m_1)M_2 - a,\ be^{-M_2} = (c+M_1)m_2 - a. \end{aligned} \tag{2.10}$$

which imply that

$$\begin{aligned} c(M_1-m_1) + M_1m_2 - M_2m_1 &= b(e^{-m_1} - e^{-M_1}) = be^{-m_1-M_1}(e^{M_1} - e^{m_1}), \\ c(M_2-m_2) + M_2m_1 - M_1m_2 &= b(e^{-m_2} - e^{-M_2}) = be^{-m_2-M_2}(e^{M_2} - e^{m_2}). \end{aligned} \tag{2.11}$$

Moreover, we get

$$\begin{aligned} e^{M_1} - e^{m_1} &= e^{\alpha}(M_1 - m_1),\ m_1 \le\ \alpha \le M_1, \\ e^{M_2} - e^{m_2} &= e^{\beta}(M_2 - m_2),\ m_2 \le\ \beta \le M_2. \end{aligned} \tag{2.12}$$

Then by adding the two relations (2.11) we obtain

$$c(M_1-m_1)+c(M_2-m_2) = be^{-m_1-M_1+\alpha}(M_1-m_1)+be^{-m_2-M_2+\beta}(M_2-m_2). \tag{2.13}$$

Therefore from (2.13) we have

$$(M_1 - m_1)(c - be^{-m_1-M_1+\alpha}) + (M_2 - m_2)(c - be^{-m_2-M_2+\beta}) = 0. \tag{2.14}$$

Then using (2.5), (2.9) and (2.14), gives us $m_1 = M_1$ and $m_2 = M_2$. Hence from Theorem.2.2 system (1.1) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of (1.1) tends to the unique positive equilibrium $(\bar{x}, \bar{y})$ as $n \to \infty$. In addition, the equilibrium $(\bar{x}, \bar{y})$ is globally asymptotically stable. This completes the proof of the theorem. □

# 3 Rate of Convergence

In this section we give the rate of convergence of a solution that converges to the equilibrium $E = (\bar{x}, \bar{y})$ of the systems (1.1) for all values of parameters. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [13] and [14].

The following results give the rate of convergence of solutions of a system of difference equations

$$\mathbf{x}_{n+1} = [A + B(n)]\mathbf{x}_n \tag{3.1}$$

where $\mathbf{x}_n$ is a $k$-dimensional vector, $A \in \mathbf{C}^{k\times k}$ is a constant matrix, and $B:\ \mathbb{Z}^+ \longrightarrow \mathbf{C}^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to\ \infty, \tag{3.2}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm; $\|.\|$ also denotes the Euclidean norm in $\mathbb{R}^2$ given by

$$\|\mathbf{x}\| = \|(x, y)\| = \sqrt{x^2 + y^2}. \tag{3.3}$$

**Theorem 3.1** (See [15]). *Assume that condition* (3.2) *holds. If* $\mathbf{x}_n$ *is a solution of system* (3.1)*, then either* $\mathbf{x}_n = 0$ *for all large* $n$ *or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|} \tag{3.4}$$

*exists and is equal to the modulus of one of the eigenvalues of matrix* $A$.

**Theorem 3.2** (See [15]). *Assume that condition* (3.2) *holds. If* $\mathbf{x}_n$ *is a solution of system* (3.1)*, then either* $\mathbf{x}_n = 0$ *for all large* $n$ *or*

$$\rho = \lim_{n\to\infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|} \tag{3.5}$$

*exists and is equal to the modulus of one of the eigenvalues of matrix* $A$.

The equilibrium point of the system (1.1) satisfies the following system of equations

$$\begin{cases} \bar{x} &= \dfrac{a+be^{-\bar{x}}}{c+\bar{y}} \\ \bar{y} &= \dfrac{a+be^{-\bar{y}}}{c+\bar{x}} \end{cases}. \tag{3.6}$$

We can easily see that the system (3.6) has an unique equilibrium $E = (\bar{x}, \bar{x})$.

The map $T$ associated to the system (1.1) is

$$T(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \dfrac{a+be^{-x}}{c+y} \\ \dfrac{a+be^{-y}}{c+x} \end{pmatrix}. \tag{3.7}$$

The Jacobian matrix of $T$ is

$$J_T = \begin{pmatrix} \dfrac{-be^{-x}}{c+y} & \dfrac{-(a+be^{-x})}{(c+y)^2} \\ \dfrac{-(a+be^{-y})}{(c+x)^2} & \dfrac{-be^{-y}}{c+x} \end{pmatrix}. \tag{3.8}$$

By using the system (3.6), value of the Jacobian matrix of $T$ at the equilibrium point $E = (\bar{x}, \bar{y}) = (\bar{x}, \bar{x})$ is

$$J_T = \begin{pmatrix} \dfrac{-be^{-\bar{x}}}{c+\bar{x}} & \dfrac{-(a+be^{-\bar{x}})}{(c+\bar{x})^2} \\ \dfrac{-(a+be^{-\bar{x}})}{(c+\bar{x})^2} & \dfrac{-be^{-\bar{x}}}{c+\bar{x}} \end{pmatrix}. \tag{3.9}$$

Our goal in this section is to determine the rate of convergence of every solution of the system (1.1) in the regions where the parameters $a, b, c \in (0, \infty)$, $(b < c)$ and initial conditions $x_0$ and $y_0$ are arbitrary, nonnegative numbers.

**Theorem 3.3.** *The error vector* $\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \end{pmatrix}$ *of every solution* $\mathbf{x}_n \neq \mathbf{0}$ *of* (1.1) *satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \text{ for some } i = 1, 2, \tag{3.10}$$

*and*

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = |\lambda_i(J_T(E))| \text{ for some } i = 1, 2, \tag{3.11}$$

*where* $|\lambda_i(J_T(E))|$ *is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium* $J_T(E)$*.*

*Proof.* First, we will find a system satisfied by the error terms. The error terms are given as

$$\begin{aligned}
x_{n+1} - \bar{x} &= \frac{a + be^{-x_n}}{c + y_n} - \frac{a + be^{-\bar{x}}}{c + \bar{y}} = \frac{(a + be^{-x_n})(c + \bar{y}) - (a + be^{-\bar{x}})(c + y_n)}{(c + y_n)(c + \bar{y})} \\
&= \frac{bc(e^{-x_n} - e^{-\bar{x}}) + a(\bar{y} - y_n) + b(e^{-x_n}\bar{y} - e^{-\bar{x}}y_n)}{(c + y_n)(c + \bar{y})} \\
&= \frac{-bc(e^{x_n} - e^{\bar{x}})}{e^{x_n+\bar{x}}(c + y_n)(c + \bar{y})} + \frac{b}{(c + y_n)(c + \bar{y})}(e^{-x_n}\bar{y} - e^{-x_n}y_n + e^{-x_n}y_n - e^{-\bar{x}}y_n) \\
&\quad - \frac{a}{(c + y_n)(c + \bar{y})}(y_n - \bar{y}) \\
&= \frac{-bc}{e^{x_n+\bar{x}}(c + y_n)(c + \bar{y})}(e^{x_n} - e^{\bar{x}}) + \frac{by_n}{e^{x_n+\bar{x}}(c + y_n)(c + \bar{y})}(e^{x_n} - e^{\bar{x}}) \\
&\quad - \frac{a + be^{-x_n}}{(c + y_n)(c + \bar{y})}(y_n - \bar{y}) \\
&= \frac{-b}{e^{x_n+\bar{x}}(c + \bar{y})}(e^{x_n} - e^{\bar{x}}) - \frac{a + be^{-x_n}}{(c + y_n)(c + \bar{y})}(y_n - \bar{y}) \\
&= \frac{-be^{\bar{x}}}{e^{x_n+\bar{x}}(c + \bar{y})}(e^{x_n-\bar{x}} - 1) - \frac{a + be^{-x_n}}{(c + y_n)(c + \bar{y})}(y_n - \bar{y}) \\
&= \frac{-b}{e^{x_n}(c + \bar{y})}\left[(x_n - \bar{x}) + \mathcal{O}_1\left((x_n - \bar{x})^2\right)\right] - \frac{a + be^{-x_n}}{(c + y_n)(c + \bar{y})}(y_n - \bar{y}) \\
&= \frac{-b}{e^{x_n}(c + \bar{y})}(x_n - \bar{x}) - \frac{a + be^{-x_n}}{(c + y_n)(c + \bar{y})}(y_n - \bar{y}) + \mathcal{O}_1\left((x_n - \bar{x})^2\right).
\end{aligned} \tag{3.12}$$

By calculating similarly, we get

$$y_{n+1} - \bar{y} = \frac{-b}{e^{y_n}(c + \bar{x})}(y_n - \bar{y}) - \frac{a + be^{-y_n}}{(c + x_n)(c + \bar{x})}(x_n - \bar{x}) + \mathcal{O}_2\left((y_n - \bar{y})^2\right). \tag{3.13}$$

From (3.12) and (3.13) we have

$$\begin{aligned} x_{n+1} - \bar{x} &\approx \frac{-b}{e^{x_n}(c+\bar{y})}(x_n - \bar{x}) - \frac{a + be^{-x_n}}{(c+y_n)(c+\bar{y})}(y_n - \bar{y}) \\ y_{n+1} - \bar{y} &\approx \frac{-b}{e^{y_n}(c+\bar{x})}(y_n - \bar{y}) - \frac{a + be^{-y_n}}{(c+x_n)(c+\bar{x})}(x_n - \bar{x}). \end{aligned} \tag{3.14}$$

Set

$$e_n^1 = x_n - \bar{x} \text{ and } e_n^2 = y_n - \bar{y}.$$

Then system (3.14) can be represented as

$$\begin{aligned} e_{n+1}^1 &\approx a_n e_n^1 + b_n e_n^2 \\ e_{n+1}^2 &\approx c_n e_n^1 + d_n e_n^2 \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{-b}{e^{x_n}(c+\bar{y})}, \quad b_n = -\frac{a + be^{-x_n}}{(c+y_n)(c+\bar{y})}, \\ c_n &= -\frac{a + be^{-y_n}}{(c+x_n)(c+\bar{x})}, \quad d_n = \frac{-b}{e^{y_n}(c+\bar{x})}. \end{aligned}$$

Taking the limits of $a_n, b_n, c_n$ and $d_n$ as $n \to \infty$, we obtain

$$\begin{aligned} \lim_{n\to\infty} a_n &= \frac{-b}{e^{\bar{x}}(c+\bar{y})}, \ \lim_{n\to\infty} b_n = -\frac{a + be^{-\bar{x}}}{(c+\bar{x})^2}, \\ \lim_{n\to\infty} c_n &= -\frac{a + be^{-\bar{x}}}{(c+\bar{x})^2}, \ \lim_{n\to\infty} d_n = \frac{-b}{e^{\bar{x}}(c+\bar{x})}, \end{aligned}$$

that is

$$\begin{aligned} a_n &= \frac{-b}{e^{\bar{x}}(c+\bar{y})} + \alpha_n, \ b_n = -\frac{a + be^{-\bar{x}}}{(c+\bar{x})^2} + \beta_n, \\ c_n &= -\frac{a + be^{-\bar{x}}}{(c+\bar{x})^2} + \gamma_n, \ d_n = \frac{-b}{e^{\bar{x}}(c+\bar{x})} + \delta_n, \end{aligned}$$

where $\alpha_n \to 0$, $\beta_n \to 0$, $\gamma_n \to 0$ and $\delta_n \to 0$ as $n \to \infty$.

Now, we have system of the form (3.1):

$$\mathbf{e}_{n+1} = (A + B(n))\mathbf{e}_n,$$

where $A = \begin{pmatrix} \dfrac{-be^{-\bar{x}}}{c+\bar{x}} & \dfrac{-(a+be^{-\bar{x}})}{(c+\bar{x})^2} \\ \dfrac{-(a+be^{-\bar{x}})}{(c+\bar{x})^2} & \dfrac{-be^{-\bar{x}}}{c+\bar{x}} \end{pmatrix}$, $B(n) = \begin{pmatrix} \alpha_n & \beta_n \\ \delta_n & \gamma_n \end{pmatrix}$ and

$$\|B(n)\| \to 0 \text{ as } n \to \infty.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \end{pmatrix} = A \begin{pmatrix} e^1_n \\ e^2_n \end{pmatrix}.$$

The system is exactly linearized system of (1.1) evaluated at the equilibrium $E = (\bar{x}, \bar{y}) = (\bar{x}, \bar{x})$. Then Theorem 3.1 and Theorem 3.2 imply the result. $\square$

# References


[1] D.C. Zhang, B. Shi, Oscillation and global asymptotic stability in a discrete epidemic model, J. Math. Anal. Appl. 278 (2003), 194–202.

[2] E.A. Grove, G.Ladas, Periodicities in Nonlinear Difference Equations, Chapman and Hall, CRC, 2005.

[3] E. El-Metwally, E.A. Grove, G.Ladas, R. Levins, M. Radin, On the difference equation, $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$, Nonlinear Anal. 47 (2001) 4623–4634.

[4] G. Papaschinopoluos, M.A. Radin, C.J. Schinas, On a system of two difference equations of exponential form: $x_{n+1} = a + b x_{n-1} e^{-y_n},\ y_{n+1} = c + d y_{n-1} e^{-x_n}$, Math. Comput. Model. 54 (2011), 2969–12977.

[5] G. Stefanidou, G. Papaschinopoluos, C.J. Schinas, On a system of two exponential type difference equations, Com. Appl. Nonlinear Anal. 17(2) (2010), 1–13.

[6] R. P. Agarwal, Difference Equations and Inequalities, Second Ed. Dekker, New York, 1992, 2000.

[7] S. Stević, On a discrete epidemic model, Discrete Dyn. Nat. Soc. 2007 (2007) Article ID 87519, 10 pages.

[8] S. Stević, On the recursive sequence $x_{n+1} = \dfrac{\alpha + \beta x_{n-1}}{1 + g(x_n)}$, Indian J. Pure Appl. Math. 33(12)(2002), 1767–1774.

[9] V. L. Kocic, G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic, Dordrecht, 1993.

[10] I. Ozturk, F. Bozkurt, S. Ozen, On the difference equation $y_{n+1} = (\alpha + \beta e^{-y_n})/(\gamma + y_{n-1})$, *Appl. Math. Comput.*, **181** (2006), 1387–1393.

[11] M. R. S. Kulenović, G.Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/ CRC, Boca Raton, London, 2001.

[12] DŽ. Burgić and Z. Nurkanović, An example of globally asymptotically stable anti-monotonic system of rational difference equations in the plane, Sarajevo Journal of Mathematics, 5 (18) (2009), 235–245.

[13] M. R. S. Kulenović and M. Nurkanović, Asymptotic behavior of a competitive system of linear fractional difference equations, Adv. Difference Equ. , (2006), Art. ID 19756, 13pp.

[14] M. R. S. Kulenović and Z. Nurkanović, The rate of convergence of solution of a three dimensional linear fractional systems of difference equations, Zbornik radova PMF Tuzla - Svezak Matematika, 2 (2005), 1–6.

[15] M. Pituk, More on Poincare's and Peron's theorems for difference equations, J. Difference Equ. Appl., 8 (2002), 201–216.