On a System of Two Difference Equations of Exponential Form

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Abstract

The goal of this paper is to study the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form:

\[ x_{n+1} = \frac{a + be^{-x_n}}{c + y_n}, \quad y_{n+1} = \frac{a + be^{-y_n}}{c + x_n} \]

where \(a, b, c\) are positive constants and the initial values \(x_0, y_0\) are positive real values. Also, we determine the rate of convergence of a solution that converges to the equilibrium \(E = (\bar{x}, \bar{y})\) of this system.

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1 Introduction

In [10], the authors studied the boundedness, the asymptotic behavior, the periodicity and the stability of the positive solutions of the difference equation:

\[ y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}} \]

where \(\alpha, \beta, \gamma\) are positive constants and the initial values \(y_{-1}, y_0\) are positive numbers.
Motivated by the above paper, we will investigate the boundedness, the persistence and the asymptotic behavior of the positive solutions of the following system of exponential form:

\[
\begin{align*}
x_{n+1} &= \frac{a + be^{-x_n}}{c + y_n}, \\
y_{n+1} &= \frac{a + be^{-y_n}}{c + x_n}
\end{align*}
\]  

(1.1)

where \(a, b, c\) are positive constants and the initial values \(x_0, y_0\) are positive real values.

Difference equations and systems of difference equations of exponential form can be found in the following papers: [1, 3–5, 7]. Moreover, as difference equations have many applications in applied sciences, there are many papers and books that can be found concerning the theory and applications of difference equations, see [2, 6, 9] and the references cited therein.

2 Global Behavior of Solutions of the System

In the first lemma we study the boundedness and persistence of the positive solutions of (1.1).

**Lemma 2.1.** Every positive solution of (1.1) is bounded and persists.

**Proof.** Let \((x_n, y_n)\) be an arbitrary solution of (1.1). From (1.1), we can see that

\[
x_n \leq \frac{a + b}{c}, \quad y_n \leq \frac{a + b}{c}, \quad n = 1, 2, \ldots
\]

(2.1)

In addition, from (1.1) and (2.1) we get

\[
x_n \geq \frac{a + be^{-\frac{a+b}{c}}}{c + \frac{a+b}{c}}, \quad y_n \geq \frac{a + be^{-\frac{a+b}{c}}}{c + \frac{a+b}{c}}, \quad n = 2, 3, \ldots
\]

(2.2)

Therefore, from (2.1) and (2.2) the proof of lemma is complete.

In order to prove the main result of this section, we recall the next theorem without its proof. See [11] and [12].

**Theorem 2.2.** Let \(\mathcal{R} = [a_1, b_1] \times [c_1, d_1]\) and

\[
f : \mathcal{R} \to [a_1, b_1], \quad g : \mathcal{R} \to [c_1, d_1]
\]

be a continuous functions such that:

(a) \(f(x, y)\) is decreasing in both variables and \(g(x, y)\) is decreasing in both variables for each \((x, y) \in \mathcal{R} \);
(b) If \((m_1, M_1, m_2, M_2) \in \mathbb{R}^2\) is a solution of

\[
\begin{aligned}
M_1 &= f(m_1, m_2), \quad m_1 = f(M_1, M_2), \\
M_2 &= g(m_1, m_2), \quad m_2 = g(M_1, M_2),
\end{aligned}
\]

then \(m_1 = M_1\) and \(m_2 = M_2\).

Then the system of difference equations

\[
x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n)
\]

has a unique equilibrium \((\bar{x}, \bar{y})\) and every solution \((x_n, y_n)\) of the system (2.4) with \((x_0, y_0) \in \mathbb{R}\) converges to the unique equilibrium \((\bar{x}, \bar{y})\). In addition, the equilibrium \((\bar{x}, \bar{y})\) is globally asymptotically stable.

Now we state the main theorem of this section.

**Theorem 2.3.** Consider system (1.1). Suppose that the following relation holds true:

\[
b < c.
\]

Then system (1.1) has a unique positive equilibrium \((\bar{x}, \bar{y})\) and every positive solution of (1.1) tends to the unique positive equilibrium \((\bar{x}, \bar{y})\) as \(n \to \infty\). In addition, the equilibrium \((\bar{x}, \bar{y})\) is globally asymptotically stable.

**Proof.** We consider the functions

\[
f(u, v) = \frac{a + be^{-u}}{c + v}, \quad g(u, v) = \frac{a + be^{-v}}{c + u}
\]

where

\[
u, v \in I = \left[ \frac{a + be^{-\frac{a+b}{c}}}{c + \frac{a+b}{c}}, \frac{a+b}{c} \right].
\]

It is easy to see that \(f(u, v), g(u, v)\) are decreasing in both variables for each \((u, v) \in I \times I\). In addition, from (2.6) and (2.7) we have \(f(u, v) \in I, g(u, v) \in I\) as \((u, v) \in I \times I\) and so \(f : I \times I \to I, g : I \times I \to I\).

Now let \(m_1, M_1, m_2, M_2\) be positive real numbers such that

\[
M_1 = \frac{a + be^{-m_1}}{c + m_2}, \quad M_2 = \frac{a + be^{-m_2}}{c + m_1}, \quad m_1 = \frac{a + be^{-M_1}}{c + M_2}, \quad m_2 = \frac{a + be^{-M_2}}{c + M_1}.
\]

Moreover arguing as in the proof of Theorem 2.2, it suffices to assume that

\[
m_1 \leq M_1, \quad m_2 \leq M_2.
\]
From (2.8), we get
\[
be^{-m_1} = (c + m_2)M_1 - a, \quad be^{-M_1} = (c + M_2)m_1 - a, \\
be^{-m_2} = (c + m_1)M_2 - a, \quad be^{-M_2} = (c + M_1)m_2 - a.
\]
which imply that
\[
c(M_1 - m_1) + M_1m_2 - M_2m_1 = b(e^{-m_1} - e^{-M_1}) = be^{-m_1-M_1}(e^{M_1} - e^{m_1}), \\
c(M_2 - m_2) + M_2m_1 - M_1m_2 = b(e^{-m_2} - e^{-M_2}) = be^{-m_2-M_2}(e^{M_2} - e^{m_2}).
\]
Moreover, we get
\[
e^{M_1} - e^{m_1} = e^{\alpha}(M_1 - m_1), \quad m_1 \leq \alpha \leq M_1, \\
e^{M_2} - e^{m_2} = e^{\beta}(M_2 - m_2), \quad m_2 \leq \beta \leq M_2.
\]
Then by adding the two relations (2.11) we obtain
\[
c(M_1 - m_1) + c(M_2 - m_2) = be^{-m_1-M_1+\alpha}(M_1 - m_1) + be^{-m_2-M_2+\beta}(M_2 - m_2).
\]
Therefore from (2.13) we have
\[
(M_1 - m_1)(c - be^{-m_1-M_1+\alpha}) + (M_2 - m_2)(c - be^{-m_2-M_2+\beta}) = 0.
\]
Then using (2.5), (2.9) and (2.14), gives us \(m_1 = M_1\) and \(m_2 = M_2\). Hence from Theorem 2.2 system (1.1) has a unique positive equilibrium \((\bar{x}, \bar{y})\) and every positive solution of (1.1) tends to the unique positive equilibrium \((\bar{x}, \bar{y})\) as \(n \to \infty\). In addition, the equilibrium \((\bar{x}, \bar{y})\) is globally asymptotically stable. This completes the proof of the theorem.

3 Rate of Convergence

In this section we give the rate of convergence of a solution that converges to the equilibrium \(E = (\bar{x}, \bar{y})\) of the systems (1.1) for all values of parameters. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [13] and [14].

The following results give the rate of convergence of solutions of a system of difference equations
\[
x_{n+1} = [A + B(n)]x_n
\]
where \(x_n\) is a \(k\)-dimensional vector, \(A \in \mathbb{C}^{k \times k}\) is a constant matrix, and \(B : \mathbb{Z}^+ \to \mathbb{C}^{k \times k}\) is a matrix function satisfying
\[
\|B(n)\| \to 0 \text{ when } n \to \infty,
\]
where \(\|\cdot\|\) denotes any matrix norm which is associated with the vector norm; \(\|\cdot\|\) also denotes the Euclidean norm in \(\mathbb{R}^2\) given by
\[
\|x\| = \|(x, y)\| = \sqrt{x^2 + y^2}.
\]
Theorem 3.1 (See [15]). Assume that condition (3.2) holds. If $x_n$ is a solution of system (3.1), then either $x_n = 0$ for all large $n$ or
\[
\rho = \lim_{n \to \infty} \sqrt[n]{\|x_n\|}
\] (3.4)
exists and is equal to the modulus of one of the eigenvalues of matrix $A$.

Theorem 3.2 (See [15]). Assume that condition (3.2) holds. If $x_n$ is a solution of system (3.1), then either $x_n = 0$ for all large $n$ or
\[
\rho = \lim_{n \to \infty} \frac{\|x_{n+1}\|}{\|x_n\|}
\] (3.5)
exists and is equal to the modulus of one of the eigenvalues of matrix $A$.

The equilibrium point of the system (1.1) satisfies the following system of equations
\[
\begin{align*}
\bar{x} &= \frac{a + be^{-\bar{x}}}{c + \bar{y}} \\
\bar{y} &= \frac{a + be^{-\bar{y}}}{c + \bar{x}}
\end{align*}
\] (3.6)

We can easily see that the system (3.6) has an unique equilibrium $E = (\bar{x}, \bar{y})$.

The map $T$ associated to the system (1.1) is
\[
T(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} a + be^{-x} \\ \frac{c + y}{a + be^{-y}} \end{pmatrix}.
\] (3.7)

The Jacobian matrix of $T$ is
\[
J_T = \begin{pmatrix}
-\frac{be^{-x}}{c + y} & -\frac{(a + be^{-x})}{(c + x)^2} \\
\frac{c + y}{(a + be^{-y})} & -\frac{(c + y)^2}{(c + x)}
\end{pmatrix}.
\] (3.8)

By using the system (3.6), value of the Jacobian matrix of $T$ at the equilibrium point $E = (\bar{x}, \bar{y}) = (\bar{x}, \bar{x})$ is
\[
J_T = \begin{pmatrix}
-\frac{be^{-x}}{c + \bar{x}} & -\frac{(a + be^{-x})}{(c + \bar{x})^2} \\
\frac{c + \bar{x}}{(a + be^{-\bar{x}})} & -\frac{(c + \bar{x})^2}{c + \bar{x}}
\end{pmatrix}.
\] (3.9)

Our goal in this section is to determine the rate of convergence of every solution of the system (1.1) in the regions where the parameters $a, b, c \in (0, \infty)$, $(b < c)$ and initial conditions $x_0$ and $y_0$ are arbitrary, nonnegative numbers.
**Theorem 3.3.** The error vector $e_n = \begin{pmatrix} e_1^n \\ e_2^n \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \end{pmatrix}$ of every solution $x_n \neq 0$ of (1.1) satisfies both of the following asymptotic relations:

$$
\lim_{n \to \infty} \sqrt{\|e_n\|} = |\lambda_i(J_T(E))| \text{ for some } i = 1, 2,
$$

(3.10)

and

$$
\lim_{n \to \infty} \|e_{n+1}\| = |\lambda_i(J_T(E))| \text{ for some } i = 1, 2,
$$

(3.11)

where $|\lambda_i(J_T(E))|$ is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium $J_T(E)$.

**Proof.** First, we will find a system satisfied by the error terms. The error terms are given as

$$
x_{n+1} - \bar{x} = \frac{a + be^{-x_n}}{c + y_n} - \frac{a + be^{-\bar{x}}}{c + \bar{y}} = \frac{(a + be^{-x_n})(c + \bar{y}) - (a + be^{-\bar{x}})(c + y_n)}{(c + y_n)(c + \bar{y})}
$$

$$
= \frac{b(c^{-x_n} - e^{-\bar{x}}) + a(\bar{y} - y_n) + b(e^{-x_n}\bar{y} - e^{-\bar{x}}y_n)}{(c + y_n)(c + \bar{y})}\left(e^{-x_n}\bar{y} - e^{-\bar{x}}y_n + e^{-x_n}y_n - e^{-\bar{x}}y_n\right)
$$

$$
= \frac{a}{c + y_n}(y_n - \bar{y})
$$

$$
= \frac{a + be^{-x_n}}{c + y_n}(y_n - \bar{y})
$$

$$
= \frac{e^{-x_n}(c + \bar{y})(e^{x_n} - e^{\bar{x}}) - \frac{a + be^{-x_n}}{c + y_n}(y_n - \bar{y})}{c + y_n}\left(\frac{a + be^{-x_n}}{c + y_n}(y_n - \bar{y})\right)
$$

$$
= \frac{e^{-x_n}(c + \bar{y})(e^{x_n} - e^{\bar{x}}) - \frac{a + be^{-x_n}}{c + y_n}(y_n - \bar{y})}{c + y_n}\left(\frac{a + be^{-x_n}}{c + y_n}(y_n - \bar{y})\right)
$$

$$
= \frac{e^{-x_n}(c + \bar{y})(x_n - \bar{x}) - \frac{a + be^{-x_n}}{c + y_n}(y_n - \bar{y})}{c + y_n}\left(\frac{a + be^{-x_n}}{c + y_n}(y_n - \bar{y})\right)
$$

(3.12)

By calculating similarly, we get

$$
y_{n+1} - \bar{y} = \frac{-b}{e^{y_n}(c + x})(y_n - \bar{y}) - \frac{a + be^{-y_n}}{(e + x_n)(c + x)}(x_n - \bar{x}) + O_2 \left((y_n - \bar{y})^2\right).
$$

(3.13)
From (3.12) and (3.13) we have

\[ x_{n+1} - \bar{x} \approx -\frac{b}{e^{x_n(c + y)}} (x_n - \bar{x}) - \frac{a + be^{-x_n}}{(c + y_n)(c + \bar{y})} (y_n - \bar{y}) \]

\[ y_{n+1} - \bar{y} \approx -\frac{b}{e^{y_n(c + \bar{x})}} (y_n - \bar{y}) - \frac{a + be^{-y_n}}{(c + x_n)(c + \bar{x})} (x_n - \bar{x}). \]  

(3.14)

Set

\[ e_1^n = x_n - \bar{x} \text{ and } e_2^n = y_n - \bar{y}. \]

Then system (3.14) can be represented as

\[ e_{1,n+1} \approx a_ne_1^n + b_ne_2^n \]

\[ e_{2,n+1} \approx c_ne_1^n + d_ne_2^n \]

where

\[ a_n = -\frac{b}{e^{x_n(c + y)}}, \quad b_n = -\frac{a + be^{-x_n}}{(c + y_n)(c + \bar{y})}, \]

\[ c_n = -\frac{a + be^{-y_n}}{(c + x_n)(c + \bar{x})}, \quad d_n = -\frac{b}{e^{y_n(c + \bar{x})}}. \]

Taking the limits of \( a_n, b_n, c_n \) and \( d_n \) as \( n \to \infty \), we obtain

\[ \lim_{n \to \infty} a_n = -\frac{b}{e^{x(c + y)}}, \quad \lim_{n \to \infty} b_n = -\frac{a + be^{-x}}{(c + x)^2}, \]

\[ \lim_{n \to \infty} c_n = -\frac{a + be^{-y}}{(c + x)^2}, \quad \lim_{n \to \infty} d_n = -\frac{b}{e^{y(c + \bar{x})}}, \]

that is

\[ a_n = -\frac{b}{e^{x(c + y)}} + \alpha_n, \quad b_n = -\frac{a + be^{-x}}{(c + x)^2} + \beta_n, \]

\[ c_n = -\frac{a + be^{-y}}{(c + x)^2} + \gamma_n, \quad d_n = -\frac{b}{e^{y(c + \bar{x})}} + \delta_n, \]

where \( \alpha_n \to 0, \beta_n \to 0, \gamma_n \to 0 \) and \( \delta_n \to 0 \) as \( n \to \infty \).

Now, we have system of the form (3.1):

\[ e_{n+1} = (A + B(n))e_n, \]

where

\[ A = \begin{pmatrix} -\frac{b e^{-x}}{c + x} & -\frac{(a + be^{-x})}{(c + x)^2} \\ -\frac{(a + be^{-x})}{(c + x)^2} & -\frac{b e^{-x}}{c + x} \end{pmatrix}, \quad B(n) = \begin{pmatrix} \alpha_n & \beta_n \\ \delta_n & \gamma_n \end{pmatrix} \]

and

\[ \|B(n)\| \to 0 \text{ as } n \to \infty. \]
Thus, the limiting system of error terms can be written as:

\[
\begin{pmatrix}
  e_{1n+1}^1 \\
  e_{2n+1}^1
\end{pmatrix}
= A \begin{pmatrix}
  e_{1n}^1 \\
  e_{2n}^1
\end{pmatrix}.
\]

The system is exactly linearized system of (1.1) evaluated at the equilibrium \( E = (\bar{x}, \bar{y}) = (\bar{x}, \bar{x}). \) Then Theorem 3.1 and Theorem 3.2 imply the result.

\[\square\]

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