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Boundedness of Solutions of a Rational Equation with a Positive Real Power

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Abstract

In this paper, we investigate the boundedness character of solutions of a second-order rational difference equation, in which the two arguments of the rational expression of the equation, x_n and x_{n-1}, are raised to a positive real power k. This equation contains 49 special cases each with positive parameters. We establish the boundedness characterizations of 44 out of the 49 special cases of the equation. More precisely, we establish that in 24 special cases of the equation every solution is bounded and in 20 special cases there exist unbounded solutions in some range of the parameters and for some initial conditions. For the remaining five special cases we establish that every solution is bounded when $k \in (0, 4)$ and we conjecture that there exist unbounded solutions for each $k \in [4, \infty)$. Furthermore, when $k \in (0, 4)$, we present the boundedness characterizations of each one of the 49 special cases of the equation. Actually, we establish that in 30 special cases of the equation every solution is bounded and in the remaining 19 special cases there exist unbounded solutions in some range of the parameters and for some initial conditions. Finally, we conjecture that, when $k \in \left(0, \frac{1}{2}\right]$, every solution of the equation is bounded. In fact, we establish this conjecture in 48 out of the 49 special cases of the equation.

AMS Subject Classifications: 39A10.
Keywords: Boundedness, boundedness characterization, rational difference equation.

1 Introduction

In this paper, we investigate the boundedness character of solutions of the second-order rational difference equation,

$$x_{n+1} = \frac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{A + B x_n^k + C x_{n-1}^k}, \quad n = 0, 1, \ldots, \tag{1.1}$$

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with nonnegative parameters, arbitrary nonnegative initial conditions such that the denominator is always positive, and $k \in (0, \infty)$. The boundedness character of solutions of a difference equation is one of the main ingredients in understanding the global behavior of the equation including its global stability.

The boundedness character of solutions of (1.1) when $k = 1$, that is the rational equation,

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.2}$$

has been extensively studied in [14] and in the references cited therein. See also [5] and the references cited therein.

If we allow one or more of the parameters of (1.1) to be zero, it contains 49 **special cases** of equations each with positive parameters. For the complete list of each one of these special cases, see the Appendix at the end of this paper. Each one of these 49 special cases may have exactly one of the following four boundedness characterizations:

$$\mathbf{B}, \ \mathbf{B}^*, \ \mathbf{U}, \ \text{or} \ \mathbf{U}^*.$$

A special case of (1.1) has the boundedness characterization **(B)** means: For all positive values of the parameters and for each pair of initial conditions of this special case the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ is bounded.

A special case of (1.1) has the boundedness characterization **(U)** means: There exists at least one set of positive values of the parameters and at least one pair of initial conditions of this special case, for which the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

A special case of (1.1) has the symbol **(B*)** next to it means: We conjecture that the boundedness characterization of this special case is **B**, but for the time being, we are not able to establish it.

A special case of (1.1) has the symbol **(U*)** next to it means: We conjecture that the boundedness characterization of this special case is **U**, but for the time being, we are not able to establish it.

Remark 1.1. It is important to mention that it takes only one positive value of k for a special case of (1.1) to be characterized as **U**. On the other hand, a special case of (1.1) is characterized as **B** if and only if it is characterized as **B** for all $k \in (0, \infty)$.

Remark 1.2. We should point out to the reader that when we investigate a special case, for simplicity and convenience we may write the equation in normalized form, by using an appropriate change of variables of the form:

$$x_n = \lambda y_n.$$

By doing so, some of the parameters of the equation may be chosen equal to one. For example, a normalized form of the special case 14 of (1.1),

$$14: \quad x_{n+1} = \frac{\beta x_n^k}{A + Cx_{n-1}^k}, \quad n = 0, 1, \ldots,$$

may be obtained by choosing $A = C = 1$. In fact, one may easily verify this, by using the change of variables, $x_n = \left(\frac{A}{C}\right)^{\frac{1}{k}} y_n$.

In this paper, we establish that the boundedness characterization of each one of the following 24 special cases of (1.1):

$$\begin{array}{cccccccc} 1, & 5, & 9, & 10, & 11, & 13, & 15, & 17, \\ 18, & 22, & 26, & 28, & 30, & 32, & 33, & 36, \\ 37, & 38, & 39, & 43, & 44, & 45, & 48, & 49, \end{array} \tag{1.3}$$

is **B** and that the boundedness characterization of each one of the following 20 special cases of (1.1):

$$\begin{array}{cccccccccc} 2, & 3, & 4, & 6, & 7, & 8, & 12, & 14, & 16, & 19, \\ 20, & 21, & 23, & 24, & 25, & 31, & 34, & 40, & 41, & 46, \end{array} \tag{1.4}$$

is **U**.

For each one of the remaining five special cases of (1.1):

$$27, \quad 29, \quad 35, \quad 42, \quad 47, \tag{1.5}$$

we conjecture:

Conjecture 1.3. Show that the boundedness characterization of each one of the five special cases of (1.1), which are listed in (1.5), is **U**. More precisely, prove that when

$$k \in [4, \infty),$$

for each one of these five special cases there exist unbounded solutions in a certain parameter range and for some initial conditions.

The results of this paper are presented as follows:

In Section 2, we establish that the boundedness characterization of each one of the 24 special cases of (1.1) listed in (1.3) is **B**.

In Section 3, we establish that every solution of each one of the following six special cases of (1.1):

$$14, \quad 27, \quad 29, \quad 35, \quad 42, \quad 47,$$

is bounded when

$$k \in (0, 4).$$

The special case 14, which in normalized form can be written as:

$$x_{n+1} = \frac{\beta x_n^k}{1 + x_{n-1}^k}, \quad n = 0, 1, \dots, \tag{1.6}$$

has been investigated in [4]. When $k = 2$, see also [8]. The next theorem has been established in [4].

Theorem 1.4 (See [4])**.** *Every positive solution of* (1.6) *is bounded when*

$$0 < k < 4.$$

When,

$$k \geq 4$$

then (1.6) *has an increasing unbounded solution. Furthermore, every solution of* (1.6) *is bounded from above and from below by positive constants when*

$$k = 1 \text{ and } \beta > 1$$

or when

$$0 < k < 1 \text{ and } \beta > 0.$$

In Section 4, we establish that the boundedness characterization of each one of the following 14 special cases of (1.1):

$$\begin{array}{ccccccc} 4, & 7, & 8, & 16, & 19, & 20, & 21, \\ 23, & 24, & 31, & 34, & 40, & 41, & 46, \end{array}$$

is **U**. For each $k \in [1, \infty)$, we present a range of parameters and a set of initial conditions for which there exist unbounded solutions. In addition, when $k \in (0, 1)$, we prove that every solution is bounded, except for the two special cases:

$$8 \text{ and } 23.$$

The boundedness character of solutions of these two special cases is examined separately when $k \in (0, 1)$.

In Sections 5.1, 5.2, and 5.3, we establish that the boundedness characterization, of each one of the following four special cases of (1.1):

$$2, \ 3, \ 6, \ 12,$$

is **U**. For each $k \in (1, \infty)$ we prove that, each one of these four special cases possesses unbounded solutions, for some range of the parameters and for some initial conditions. In addition, when $k \in (0, 1]$, we prove that every solution of each one of these four special cases is bounded.

In Section 5.4, we establish the **U** boundedness characterization of the special case 25. Actually, we present the existence of unbounded solutions of the special case 25, only when $k = 2$.

In Section 6, we pose some interesting open problems on the global character of solutions of (1.1).

Finally, at the end of this paper we present an Appendix that summarizes the results of this paper.

By combining the results of this paper the following theorem is also established.

Theorem 1.5. *Assume that $k \in (0, 4)$. Then the following statements are true:*

1. *When*

$$\gamma + \beta B + A > 0 \ \ \textit{and} \ \ C > 0, \tag{1.7}$$

or

$$\gamma = C = 0, \ \ \beta + A > 0, \ \ \textit{and} \ \ (\beta = 0 \ \ \textit{or} \ \ B > 0), \tag{1.8}$$

(1.1) *contains only the following 30 special cases:*

$$\begin{array}{cccccccccc} \textit{1,} & \textit{5,} & \textit{9,} & \textit{10,} & \textit{11,} & \textit{13,} & \textit{14,} & \textit{15,} & \textit{17,} & \textit{18,} \\ \textit{22,} & \textit{26,} & \textit{27,} & \textit{28,} & \textit{29,} & \textit{30,} & \textit{32,} & \textit{33,} & \textit{35,} & \textit{36,} \\ \textit{37,} & \textit{38,} & \textit{39,} & \textit{42,} & \textit{43,} & \textit{44,} & \textit{45,} & \textit{47,} & \textit{48,} & \textit{49,} \end{array} \tag{1.9}$$

*Each one of them has the boundedness characterization **B**.*

2. *When*

$$C = 0 \ \ \textit{and} \ \ \gamma > 0, \tag{1.10}$$

or

$$\gamma = \beta B = A = 0, \tag{1.11}$$

or

$$\gamma = B = C = 0, \ \ \textit{and} \ \ \beta > 0, \tag{1.12}$$

(1.1) *contains only the following 19 special cases:*

$$\begin{array}{cccccccccc} \textit{2,} & \textit{3,} & \textit{4,} & \textit{6,} & \textit{7,} & \textit{8,} & \textit{12,} & \textit{16,} & \textit{19,} & \textit{20,} \\ \textit{21,} & \textit{23,} & \textit{24,} & \textit{25,} & \textit{31,} & \textit{34,} & \textit{40,} & \textit{41,} & \textit{46.} & \end{array} \tag{1.13}$$

*Each one of them has the boundedness characterization **U**.*

Question: Is there a subset of values of k, for which every solution of (1.1) is bounded?.

The answer to this question will be positive if the following conjecture is established.

Conjecture 1.6. Assume that $k \in \left(0, \frac{1}{2}\right]$. Show that every solution of (1.1) is bounded.

We should point out to the reader that in this paper we establish Conjecture 1.6 in all special cases of (1.1), except for the special case 25. Thus, Conjecture 1.6 will be established if the following conjecture for the special case 25 is established.

Conjecture 1.7. Assume that $k \in (0, 1]$. Prove that every solution of the special case:

$$25: \quad x_{n+1} = \frac{\alpha + x_n^k}{x_{n-1}^k}, \ n = 0, 1, \ldots,$$

is bounded.

When $k = 1$, the special case 25 becomes,

$$x_{n+1} = \frac{\alpha + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.14}$$

which is the so called **Lyness** equation and has been investigated by many authors. See [1–3, 10, 11, 13, 14, 17, 19]. The special case of (1.14), when $\alpha = 1$, was noted by Lyness. See [15, 16].

Finally, at the end of this introduction we present three theorems, which will be useful in the sequel.

Theorem 1.8 (See [5] and [6]). *Let I be a set of real numbers and let*

$$F : \; I \times I \to I$$

be a function $F(u, v)$, which decreases in u and increases in v. Then for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}), \quad n = 0, 1, \ldots,$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:

(i) They are both monotonically increasing.

(ii) They are both monotonically decreasing.

(iii) Eventually, one of them is monotonically increasing and the other is monotonically decreasing.

Theorem 1.9 (See [12] and [14]). *Let I be an open interval of real numbers, let $F \in C(I^{k+1}, I)$, and let $\bar{x} \in I$ be an equilibrium point of the equation*

$$x_{n+1} = F(x_n, x_{n-1}), \quad n = 0, 1, \ldots \,. \tag{1.15}$$

Assume that F satisfies the following two conditions:

1. *F is increasing in each of its arguments.*
2. *F satisfies the **negative feedback** property:*

$$(u - \bar{x})[F(u, u, \ldots, u) - u] < 0, \qquad \textit{for all} \qquad u \in I - \{\bar{x}\}.$$

Then the equilibrium point $\bar{x}$ is a global attractor of all solutions of (1.15).

Theorem 1.10 (See [9]). *Assume that $B > 0$ and that $k \in (0, 1)$. Then every positive solution $\{y_n\}_{n=-1}^{\infty}$ of the equation*

$$y_{n+1} = \frac{B}{y_n^k} + \frac{1}{y_{n-1}^k}, \quad n = 0, 1, \ldots, \tag{1.16}$$

is bounded.

2 The Special Cases of (1.1) with Bounded Solutions for all $k \in (0, \infty)$ – The B Cases

In this section, we establish that each one of the following 24 special cases of (1.1):

$$\begin{array}{cccccccccccc} 1, & 5, & 9, & 10, & 11, & 13, & 15, & 17, & 18, & 22, & 26, & 28, \\ 30, & 32, & 33, & 36, & 37, & 38, & 39, & 43, & 44, & 45, & 48, & 49, \end{array}$$

has the boundedness characterization **B**. When $k = 1$, the global character of solutions of each one of these 24 special cases has been extensively studied in [14] and in the references cited therein. See also [5] and the references cited therein.

The next theorem establishes that each one of the following 19 special cases of (1.1):

$$\begin{array}{cccccccccc} 1, & 5, & 9, & 10, & 11, & 13, & 15, & 17, & 18, & \\ 28, & 32, & 36, & 37, & 38, & 39, & 43, & 44, & 45, & 49, \end{array} \tag{2.1}$$

has the boundedness characterization **B**.

Theorem 2.1. *Assume that $k \in (0, \infty)$ and that*

$$(\alpha = 0 \ \textit{ or } \ A > 0), \ \ (\beta = 0 \ \textit{ or } \ B > 0), \ \ \textit{and} \ \ (\gamma = 0 \ \textit{ or } \ C > 0). \tag{2.2}$$

Then every solution of (1.1) *is bounded. Furthermore,* (1.1) *contains only the 19 special cases listed in* (2.1) *and each one of them has the boundedness characterization* ***B***.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary solution of (1.1). In view of (2.2), there exists a positive real number M such that:

$$\alpha \leq AM, \ \ \beta \leq BM, \ \text{ and } \ \gamma \leq CM.$$

Then, for each $n \geq 0$,

$$x_{n+1} = \frac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{A + Bx_n^k + Cx_{n-1}^k} \leq \frac{AM + BMx_n^k + CMx_{n-1}^k}{A + Bx_n^k + Cx_{n-1}^k} = M.$$

The proof is complete. □

Remark 2.2. One may easily verify that, every solution of each one of the following 12 special cases of (1.1):

$$\begin{array}{cccccc} 1, & 5, & 9, & 10, & 11, & 28, \\ 32, & 36, & 37, & 43, & 44, & 49, \end{array} \tag{2.3}$$

is bounded from above and from below by positive constants.

The next theorem establishes that, each one of the following three special cases of (1.1):

$$22, \ \ 26, \ \ 48, \tag{2.4}$$

has the boundedness characterization **B**.

Theorem 2.3. *Assume that* $k \in (0, \infty)$ *and that*

$$A = 0,\ \alpha, \beta + \gamma > 0,\ \ (\beta = B = 0\ \ or\ \ \beta B > 0),\ \ and\ \ (\gamma = C = 0\ \ or\ \ \gamma C > 0). \tag{2.5}$$

Then every solution of (1.1) *is bounded from above and from below by positive constants. Furthermore,* (1.1) *contains only the three special cases listed in* (2.4) *and each one of them has the boundedness characterization* ***B***.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary solution of (1.1). In view of (2.5), there exist two positive real numbers m and M such that:

$$Bm \leq \beta \leq BM \ \text{ and } \ Cm \leq \gamma \leq CM.$$

Then, for each $n \geq 0$,

$$x_{n+1} = \frac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{Bx_n^k + Cx_{n-1}^k} > \frac{\beta x_n^k + \gamma x_{n-1}^k}{Bx_n^k + Cx_{n-1}^k} \geq \frac{Bmx_n^k + Cmx_{n-1}^k}{Bx_n^k + Cx_{n-1}^k} = m.$$

Also, for each $n \geq 2$,

$$x_{n+1} = \frac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{Bx_n^k + Cx_{n-1}^k} = \frac{\alpha}{Bx_n^k + Cx_{n-1}^k} + \frac{\beta x_n^k + \gamma x_{n-1}^k}{Bx_n^k + Cx_{n-1}^k}$$

$$< \frac{\alpha}{m^k(B+C)} + \frac{BMx_n^k + CMx_{n-1}^k}{Bx_n^k + Cx_{n-1}^k} = \frac{\alpha}{m^k(B+C)} + M.$$

The proof is complete. □

Next, we investigate the boundedness character of solutions of the special case 30 of (1.1), which in normalized form can be written as:

$$x_{n+1} = \frac{\alpha + x_n^k}{x_n^k + Cx_{n-1}^k}, \quad , n = 0, 1, \ldots \tag{2.6}$$

with positive parameters α, C, and nonnegative initial conditions x_{-1} and x_0 such that the denominator is positive.

In the theorem below, we prove that every solution of the special case 30 of (1.1) is bounded, for all $k \in (0, \infty)$.

Theorem 2.4. *Assume that* $k \in (0, \infty)$. *Every solution of* (2.6) *is bounded from above and from below by positive constants.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary solution of (2.6). Assume for the sake of contradiction that there exists an infinite sequence of indices $\{n_i\}_{i=1}^{\infty}$ such that:

$$\lim_{i \to \infty} x_{n_i+1} = \infty.$$

We claim that

$$x_{n_i+1} \to \infty \Rightarrow x_{n_i}, x_{n_i-1} \to 0. \tag{2.7}$$

Indeed,

$$x_{n_i+1} = \frac{\alpha + x_{n_i}^k}{x_{n_i}^k + Cx_{n_i-1}^k} = \frac{\alpha}{x_{n_i}^k + Cx_{n_i-1}^k} + \frac{x_{n_i}^k}{x_{n_i}^k + Cx_{n_i-1}^k} \to \infty$$

implies

$$x_{n_i}^k + Cx_{n_i-1}^k \to 0$$

and so,

$$\lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} x_{n_i-1} = 0.$$

In addition,

$$x_{n_i} = \frac{\alpha + x_{n_i-1}^k}{x_{n_i-1}^k + Cx_{n_i-2}^k} = \frac{\alpha}{x_{n_i-1}^k + Cx_{n_i-2}^k} + \frac{x_{n_i-1}^k}{x_{n_i-1}^k + Cx_{n_i-2}^k} \to 0$$

implies

$$x_{n_i-1}^k + Cx_{n_i-2}^k \to \infty,$$

and so, in view of the fact that $x_{n_i-1} \to 0$, we get

$$\lim_{i\to\infty} x_{n_i-2} = \infty.$$

Similarly, as in (2.7),

$$x_{n_i-2} \to \infty \Rightarrow x_{n_i-3}, x_{n_i-4} \to 0.$$

However,

$$x_{n_i-1} = \frac{\alpha + x_{n_i-2}^k}{x_{n_i-2}^k + Cx_{n_i-3}^k} = \frac{\alpha}{x_{n_i-2}^k + Cx_{n_i-3}^k} + \frac{\frac{x_{n_i-2}^k}{x_{n_i-3}^k}}{\frac{x_{n_i-2}^k}{x_{n_i-3}^k} + C} \to 0$$

implies

$$\frac{x_{n_i-2}}{x_{n_i-3}} \to 0.$$

By combining this with the fact that $x_{n_i-2} \to \infty$, it follows that

$$x_{n_i-3} \to \infty.$$

This is a contradiction. Therefore, the solution $\{x_n\}_{n=-1}^{\infty}$ of (2.6) is bounded. In addition, it is easy to see that

$$\limsup_{n\to\infty} x_n = S \in (0,\infty) \text{ and } I = \liminf_{n\to\infty} x_n \geq \frac{\alpha}{(1+C)S} > 0.$$

The proof is complete.

□

Next, we investigate the boundedness character of solutions of the special case 33 of (1.1), which in normalized form can be written as:

$$x_{n+1} = \frac{\alpha + x_{n-1}^k}{Bx_n^k + x_{n-1}^k}, \quad n = 0, 1, \dots, \tag{2.8}$$

with positive parameters α, B, and nonnegative initial conditions x_{-1} and x_0 such that the denominator is positive.

In the theorem below, we prove that every solution of the special case 30 of (1.1) is bounded, for all $k \in (0, \infty)$.

Theorem 2.5. *Assume that* $k \in (0, \infty)$. *Every solution of* (2.8) *is bounded from above and from below by positive constants.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary solution of (2.8). Assume for the sake of contradiction that there exists an infinite sequence of indices $\{n_i\}_{i=1}^{\infty}$ such that:

$$\lim_{i\to\infty} x_{n_i+1} = \infty.$$

We claim that

$$x_{n_i+1} \to \infty \Rightarrow x_{n_i}, x_{n_i-1} \to 0.$$

Indeed,

$$x_{n_i+1} = \frac{\alpha + x_{n_i-1}^k}{Bx_{n_i}^k + x_{n_i-1}^k} = \frac{\alpha}{Bx_{n_i}^k + x_{n_i-1}^k} + \frac{x_{n_i-1}^k}{Bx_{n_i}^k + x_{n_i-1}^k} \to \infty$$

implies

$$Bx_{n_i}^k + x_{n_i-1}^k \to 0$$

and so,

$$\lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} x_{n_i-1} = 0.$$

In addition, from

$$x_{n_i} = \frac{\alpha + x_{n_i-2}^k}{Bx_{n_i-1}^k + x_{n_i-2}^k} = \frac{\alpha}{Bx_{n_i-1}^k + x_{n_i-2}^k} + \frac{\frac{x_{n_i-2}^k}{x_{n_i-1}^k}}{B + \frac{x_{n_i-2}^k}{x_{n_i-1}^k}} \to 0$$

it follows that

$$x_{n_i-1}^k + Cx_{n_i-2}^k \to \infty \text{ and } \frac{x_{n_i-2}}{x_{n_i-1}} \to 0,$$

which is a contradiction and establishes that the solution $\{x_n\}$ of (2.8) is bounded. In addition, it is easy to see that

$$\limsup_{n\to\infty} x_n = S \in (0, \infty) \text{ and } I = \liminf_{n\to\infty} x_n \geq \frac{\alpha}{(B+1)S} > 0.$$

The proof is complete. □

3 The Six Special Cases where Boundedness is Preserved when $k \in (0,4)$ – The U* Cases

In this section we establish that each one of the following six special cases of (1.1):

$$14, \ \ 27, \ \ 29, \ \ 35, \ \ 42, \ \ 47, \tag{3.1}$$

has the boundedness characterization **B**, when $k \in (0,4)$. We should point out to the reader that we conjecture that each one of the six special cases of (1.1), which are listed in (3.1), has the boundedness characterization **U**. See Conjecture 1.3. This conjecture has been established only for the special case 14. See [4]. The global character of solutions of the special case 14, when $k \in (0,4)$, has been investigated in [4] and [8]. When $k = 1$, each one of the six special cases listed in (3.1) has been extensively studied in [14] and in the references cited therein. See also [5] and the references cited therein.

The proof that every solution of each of the six special cases is bounded, when $k \in (0,4)$, will be a direct application of the next theorem.

Theorem 3.1. *Assume that* $a \in [0,\infty)$. *Let* $f \in C((a,\infty)\times(a,\infty),(a,\infty))$. *Assume that there exist nonnegative real numbers* A, B_1, B_2, *such that*

$$A + B_1B_2 > 0,$$

and a positive real power $k \in (0,4)$ *such that:*

$$f(x,y) \leq \min\{B_1x^k, \frac{B_2x^k}{y^k}\} + A, \ \ \textit{for all} \ \ x,y \in (a,\infty). \tag{3.2}$$

Then every solution $\{x_n\}_{n=-1}^{\infty}$ *of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \ \ n = 0,1,\ldots, \tag{3.3}$$

is bounded.

Proof. Assume without loss of generality that $B_1, B_2 > 0$. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary solution of (3.3). The proof will be by contradiction. Assume that there exists an infinite sequence of indices $\{n_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} x_{n_i+1} = \infty.$$

We claim that

$$x_{n_i+1} \to \infty \Rightarrow x_{n_i-s}, \frac{x_{n_i-s}}{x_{n_i-s-1}} \to \infty, \ \ \text{for} \ \ s = 0,1,\ldots \ .$$

Indeed, (3.2) implies

$$x_{n_i+1} = f(x_{n_i}, x_{n_i-1}) \leq \min\{B_1x_{n_i}^k, \frac{B_2x_{n_i}^k}{x_{n_i-1}^k}\} + A,$$

from which it follows

$$x_{n_i}, \frac{x_{n_i}}{x_{n_i-1}} \to \infty,$$

and so, our claim follows by induction. Now we divide the proof into the following two cases:

Case 1:

$$0 < k \leq 1.$$

Then

$$\frac{x_{n_i}}{x_{n_i-1}} = \frac{f(x_{n_i}, x_{n_i-1})}{x_{n_i-1}} \leq \frac{B_2}{x_{n_i-1}^{1-k} x_{n_i-2}^{k}} + \frac{1}{x_{n_i-1}} \cdot A \to 0,$$

yields a contradiction and completes the proof in this case.

Case 2:

$$1 < k < 4.$$

Set

$$a_1 = 1$$

and

$$a_{s+1}(k - a_s) = k, \quad s = 1, 2, \ldots .$$

We claim that there exists a positive integer s_0, such that

$$a_{s_0+2} < 0 < a_1 \leq \ldots \leq a_s < k < a_{s+1}, \quad \text{when} \quad s \in \{1, \ldots, s_0\} \tag{3.4}$$

or

$$0 < a_1 \leq \ldots \leq a_s < k = a_{s_0+1}, \quad \text{when} \quad s \in \{1, \ldots, s_0\}. \tag{3.5}$$

Otherwise, for all $s \geq 1$,

$$0 < a_s < k$$

and in view of our assumption, that $k \in (0, 4)$,

$$0 < a_s < \frac{k}{k - a_s} = a_{s+1} < k,$$

from which it follows that the sequence $\{a_s\}_{s=1}^{\infty}$ converges to a positive number L. Thus, the equation

$$x(k - x) = k,$$

has a positive solution L, which under our assumption, that $k \in (0, 4)$, is a contradiction and proves our claim. Next, for all $s \in \{0, \ldots, s_0\}$, it holds

$$\frac{x_{n_i-s}}{x_{n_i-s-1}^{a_{s+1}}} = \frac{f(x_{n_i}, x_{n_i-1})}{x_{n_i-s-1}^{a_{s+1}}}, \quad \text{for all} \quad i \geq 1.$$

When (3.4) holds, in view of (3.2) we have that, for all $i \geq 1$,

$$\frac{x_{n_i-s}}{x_{n_i-s-1}^{a_{s+1}}} \leq B_2 \cdot \left(\frac{x_{n_i-s-1}}{x_{n_i-s-2}^{a_{s+2}}}\right)^{k-a_{s+1}} + \frac{1}{x_{n_i-s-1}^{a_{s+1}}} \cdot A, \text{ for all } s \in \{0, \ldots, s_0\}. \tag{3.6}$$

When (3.5) holds, in view of (3.2) we have that, for all $i \geq 1$,

$$\frac{x_{n_i-s}}{x_{n_i-s-1}^{a_{s+1}}} \leq B_2 \cdot \left(\frac{x_{n_i-s-1}}{x_{n_i-s-2}^{a_{s+2}}}\right)^{k-a_{s+1}} + \frac{1}{x_{n_i-s-1}^{a_{s+1}}} \cdot A, \text{ for all } s \in \{0, \ldots, s_0 - 1\} \tag{3.7}$$

and

$$\frac{x_{n_i-s_0}}{x_{n_i-s_0-1}^{a_{s_0+1}}} = \frac{x_{n_i-s_0}}{x_{n_i-s_0-1}^{k}} \leq B_2 \cdot \frac{1}{x_{n_i-s_0-2}^{k}} + \frac{1}{x_{n_i-s_0-1}^{k}} \cdot A. \tag{3.8}$$

From (3.4), (3.5), (3.6), (3.7), and the fact that,

$$\frac{x_{n_i}}{x_{n-i-1}^{a_1}} = \frac{x_{n_i}}{x_{n-i-1}} \to \infty \text{ and } x_{n_i-s} \to \infty, \quad \text{for all } \ s \in \{0, \ldots, s_0\},$$

we find

$$\frac{x_{n_i-s}}{x_{n_i-s-1}^{a_{s+1}}} \to \infty, \quad \text{for all } \ s \in \{0, \ldots, s_0\}.$$

When $s = s_0$, (3.6) yields,

$$\frac{x_{n_i-s_0}}{x_{n_i-s_0-1}^{a_{s_0+1}}} \leq B_2 \cdot \left(\frac{x_{n_i-s_0-1}}{x_{n_i-s_0-2}^{a_{s_0+2}}}\right)^{k-a_{s_0+1}} + \frac{1}{x_{n_i-s_0-1}^{a_{s_0+1}}} \cdot A, \quad \text{for all } \ i \geq 1,$$

from which it follows

$$\left(\frac{x_{n_i-s_0-1}}{x_{n_i-s_0-2}^{a_{s_0+2}}}\right)^{k-a_{s_0+1}} \to \infty.$$

This together, with

$$x_{n_i-s_0-1}, x_{n_i-s_0-2} \to \infty$$

and

$$a_{s_0+2}, k - a_{s_0+1} < 0$$

yields a contradiction. When $s = s_0 - 1$, (3.7) becomes,

$$\frac{x_{n_i-s_0+1}}{x_{n_i-s_0}^{a_{s_0}}} \leq B_2 \cdot \left(\frac{x_{n_i-s_0}}{x_{n_i-s_0-1}^{a_{s_0+1}}}\right)^{k-a_{s_0}} + \frac{1}{x_{n_i-s_0}^{a_{s_0}}} \cdot A, \text{ for all } i \geq 1,$$

from which it follows

$$\frac{x_{n_i-s_0}}{x_{n_i-s_0-1}^{a_{s_0+1}}} = \frac{x_{n_i-s_0}}{x_{n_i-s_0-1}^{k}} \to \infty.$$

This, in view of (3.8), yields a contradiction. The proof is complete.

□

Theorem 3.1 may be extended to the following theorem. The proof is along the same lines and is omitted.

Theorem 3.2. *Assume that* $a \in [0, \infty)$. *Let* $f \in C((a, \infty) \times (a, \infty), (a, \infty))$. *Assume that there exist nonnegative real numbers* A, B_1, B_2, *such that*

$$A + B_1 B_2 > 0,$$

and two positive real powers k_1 *and* k_2, *with*

$$0 < k_1^2 < 4k_2,$$

$$f(x, y) \leq \min\{B_1 x^{k_1}, \frac{B_2 x^{k_1}}{y^{k_2}}\} + A, \quad \textit{for all} \quad x, y \in (a, \infty).$$

Then every solution $\{x_n\}_{n=-1}^{\infty}$ *of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots$$

is bounded.

The proof of the next theorem is a direct application of Theorem 3.1.

Theorem 3.3. *Assume that* $k \in (0, 4)$. *Then every solution of each one of the six special cases:*

$$14, \quad 27, \quad 29, \quad 35, \quad 42, \quad 47, \tag{3.9}$$

is bounded from above by a positive constant. In addition, for each one of the following four special cases:

$$27, \quad 29, \quad 42, \quad 47,$$

every solution is bounded from below by a positive constant.

Proof. Consider a function f of two positive real variables x and y, with positive real values,

$$f(x, y) = \frac{\alpha + \beta x^k + \gamma y^k}{A + y^k},$$

with

$$\beta > 0 \text{ and } [A > 0 \text{ or } (A = 0 \text{ and } \gamma > 0)].$$

Clearly, when $\{x_n\}_{n=-1}^{\infty}$ is an arbitrary solution of one of the six special cases listed in (3.9) which is chosen arbitrarily,

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \, . \tag{3.10}$$

We establish that every solution $\{x_n\}_{n=-1}^{\infty}$ of (3.10) is bounded. Actually, we do this by proving that the function f satisfies the conditions of Theorem 3.1. We divide the proof into the following two cases:

Case 1: When

$$\beta A > 0,$$

that is for each one of the four special cases:

$$14, \quad 29, \quad 35, \quad 47,$$

$f \in C((0,\infty) \times (0,\infty), (0,\infty))$ and also

$$f(x,y) < \min\{\beta \frac{x^k}{y^k}, \frac{\beta}{A} x^k\} + \frac{\alpha}{A} + \gamma, \quad \text{for all} \quad x, y \in (0,\infty).$$

Case 2: When

$$\beta\gamma > 0 \quad \text{and} \quad A = 0,$$

that is for each one of the two special cases:

$$27, \quad \text{and} \quad 42,$$

$f \in C((\gamma,\infty) \times (\gamma,\infty), (\gamma,\infty))$ and

$$f(x,y) \leq \min\{\beta \frac{x^k}{y^k}, \max\{\beta, \frac{\beta}{\gamma^4}\} x^k\} + \max\{\alpha, \frac{\alpha}{\gamma^4}\} + \gamma, \quad \text{for all,} \quad x, y \in (\gamma,\infty).$$

This completes the proof that every solution of each one of the six special cases listed in (3.9) is bounded from above by a positive constant.

In addition, for the two special cases:

$$27 \quad \text{and} \quad 42,$$

clearly,

$$x_{n+1} > \gamma, \quad \text{for} \quad n \geq 0.$$

Also, for the two special cases:

$$29 \quad \text{and} \quad 47,$$

it is easy to see that

$$S = \limsup_{n\to\infty} x_n \in (0,\infty) \quad \text{and} \quad I = \liminf_{n\to\infty} x_n \geq \frac{\alpha}{A + S^k} > 0.$$

The proof is complete.

□

4 Existence of Unbounded Solutions when $k \in [1, \infty)$ – 14 U Cases

In this section we establish the **U** characterization of each one of the following 14 special cases of (1.1):

$$\begin{array}{ccccccc} 4, & 7, & 8, & 16, & 19, & 20, & 21, \\ 23, & 24, & 31, & 34, & 40, & 41, & 46. \end{array}$$

In particular, when $k \in [1, \infty)$, we establish that each one of these 14 special cases of (1.1) possesses unbounded solutions in a certain range of parameters and for some initial conditions. We also present a detailed examination of the boundedness character of solutions when $k \in (0, 1)$. When $k = 1$, the global character of solutions of each one of these 14 special cases has been extensively studied in [14] and in the references cited therein. See also [5] and the references cited therein.

4.1 Six Straightforward Special Cases

In this section, we establish the existence of unbounded solutions in each one of the following six special cases of (1.1):

$$4, \quad 7, \quad 19, \quad 20, \quad 21, \quad 40, \tag{4.1}$$

in a certain range of the parameters and for some initial conditions when $k \in [1, \infty)$. In addition, when $k \in (0, 1)$, we prove that the boundedness characterization of each one of these six special cases is **B**.

These six special cases comprise the equation

$$x_{n+1} = \alpha + \beta x_n^k + \gamma x_{n-1}^k, \quad n = 0, 1, \ldots \tag{4.2}$$

with

$$\alpha, \beta, \gamma \geq 0, \quad \beta + \gamma > 0,$$

and nonnegative initial conditions x_{-1} and x_0.

The next theorem establishes the **U** characterization of each one of the six special cases listed in (4.1). The proof is straightforward and is omitted.

Theorem 4.1. *Assume that $k \in [1, \infty)$ and that*

$$\alpha \geq 0 \ \textit{ and } \ (\beta > 1 \ \textit{ or } \ \gamma > 1).$$

Then every solution $\{x_n\}_{n=-1}^{\infty}$ of (4.2) with

$$x_{-1} > 1 \quad \textit{and} \ \ x_0 > 1$$

is unbounded.

The theorem below, establishes that the boundedness characterization of each one of the six special cases listed in (4.1) is **B**, when $k \in (0, 1)$.

Theorem 4.2. *Assume that* $k \in (0, 1)$ *and that*

$$\alpha, \beta, \gamma \geq 0 \;\; \textit{and} \;\; \beta + \gamma > 0.$$

Then every solution $\{x_n\}_{n=-1}^{\infty}$ *of* (4.2) *converges to a finite limit.*

Proof. The proof is a direct application of Theorem 1.9. □

4.2 The Eight Special Cases 8, 16, 23, 24, 31, 34, 41, 46

In this section, we establish the existence of unbounded solutions in each one of the eight special cases in the title, when $k \in [1, \infty)$. Also, when $k \in (0, 1)$, we prove that the boundedness characterization of each one of the following six special cases of (1.1):

$$16, \;\; 24, \;\; 31, \;\; 34, \;\; 41, \;\; 46, \tag{4.3}$$

is **B**. For the two remaining special cases:

$$8 \;\; \text{and} \;\; 23,$$

we present a detailed analysis of the boundedness character of their solutions when $k \in (0, 1)$.

The eight special cases in the title comprise an equation, which in normalized form can be written as:

$$x_{n+1} = \frac{\alpha + \beta x_n^k + x_{n-1}^k}{A + x_n^k}, \;\; n = 0, 1, \ldots, \tag{4.4}$$

with nonnegative parameters α, β, A, and nonnegative initial conditions x_{-1} and x_0 such that the denominator is positive.

The next theorem establishes the **U** characterization of each one of the eight special cases in the title.

Theorem 4.3. *Assume that*

$$k \in [1, \infty), \;\; 0 \leq A + \beta^k < 1, \;\; \textit{and} \;\; \alpha + \beta^k - \beta A \geq 0. \tag{4.5}$$

Let ϵ *be an arbitrarily small positive real number but fixed such that*

$$0 < \epsilon < (1 - A)^{\frac{1}{k}} - \beta. \tag{4.6}$$

Assume that $\{x_n\}_{n=-1}^{\infty}$ *is a positive solution of* (4.4)*, with initial conditions* x_{-1} *and* x_0 *such that:*

$$x_{-1}^k > \max\{\frac{\alpha + (\beta + \epsilon)^k - (\beta + \epsilon)A}{\epsilon}, 1\} \tag{4.7}$$

and

$$\beta < x_0 < \beta + \epsilon. \tag{4.8}$$

Then

$$\lim_{n\to\infty} x_{2n} = \beta \;\; \textit{and} \;\; \lim_{n\to\infty} x_{2n+1} = \infty.$$

Proof. In view of (4.5), (4.6), (4.7), and (4.8)

$$x_1 = \frac{\alpha + \beta x_0^k + x_{-1}^k}{A + x_0^k} \geq \frac{x_{-1}^k}{A + x_0^k} > \frac{1}{A + (\beta+\epsilon)^k} \cdot x_{-1} > x_{-1}. \tag{4.9}$$

In view of (4.7), (4.8), and (4.9)

$$x_2 - (\beta + \epsilon) = \frac{\alpha + \beta x_1^k + x_0^k - (\beta+\epsilon)(A + x_1^k)}{A + x_1^k}$$

$$< \frac{\alpha + (\beta+\epsilon)^k - (\beta+\epsilon)A - \epsilon x_1^k}{A + x_1^k} < 0.$$

Finally, in view of (4.6), (4.7), and (4.8)

$$x_2 - \beta = \frac{\alpha + \beta x_1^k + x_0^k - \beta(A + x_1^k)}{A + x_1^k} > \frac{\alpha + \beta^k - \beta A}{A + x_1^k} \geq 0,$$

that is

$$x_1 > \frac{1}{A + (\beta+\epsilon)^k} \cdot x_{-1} > x_{-1}, \;\; \beta + \epsilon > x_2 > \beta,$$

and

$$x_1^k > x_{-1}^k > \max\{\frac{\alpha + (\beta+\epsilon)\left[(\beta+\epsilon)^{k-1} - A\right]}{\epsilon}, 1\}.$$

Inductively, we see that for all $n \geq 0$,

$$\beta < x_{2n} < \beta + \epsilon \;\; \text{and} \;\; x_{2n+1} > \left(\frac{1}{A + (\beta+\epsilon)^k}\right)^{n+1} \cdot x_{-1}.$$

Clearly,

$$\lim_{n\to\infty} x_{2n+1} = \infty.$$

By taking limits on both sides of

$$x_{2n+2} = \frac{\alpha + \beta x_{2n+1}^k + x_{2n}^k}{A + x_{2n+1}^k}, \;\; n = 0, 1, \ldots$$

as $n \to \infty$, we find that

$$\lim_{n\to\infty} x_{2n} = \beta.$$

The proof is complete.

□

The next theorem establishes that the boundedness characterization of each one of the six special cases of (1.1):

$$16, \quad 24, \quad 31, \quad 34, \quad 41, \quad 46,$$

is **B** when $k \in (0,1)$.

Theorem 4.4. *Assume that*

$$k \in (0,1) \text{ and } \beta + A > 0. \tag{4.10}$$

Then every positive $\{x_n\}_{n=-1}^{\infty}$ *solution of* (4.4) *is bounded.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary positive solution of (4.4). The proof will be by contradiction. Assume that there exists an infinite sequence of indices $\{n_i\}_{i=1}^{\infty}$ such that

$$x_{n_i+1} \to \infty \text{ and } x_{n_i+1} > x_j, \text{ for all } j < n_i + 1, \text{ and for all } i \geq 1. \tag{4.11}$$

In view of (4.10), one may easily verify that there exist positive constants m and M such that

$$A + x_n^k > m \text{ and } \frac{\alpha + \beta x_n^k}{A + x_n^k} < M, \text{ for all } n \geq 0.$$

Therefore,

$$x_{n_i+1} = \frac{\alpha + \beta x_{n_i}^k + x_{n_i-1}^k}{A + x_{n_i}^k} < M + \frac{x_{n_i-1}^k}{m}, \text{ for all } i \geq 1,$$

and so

$$x_{n_i-1} \to \infty.$$

We claim that the sequence $\{x_{n_i}\}_{i=1}^{\infty}$ is bounded, because otherwise, eventually

$$x_{n_i+1} = \frac{\alpha + \beta x_{n_i}^k}{A + x_{n_i}^k} + \frac{1}{A + x_{n_i}^k} x_{n_i-1}^k < x_{n_i-1}$$

which contradicts (4.11).

Also, (4.11) implies that

$$x_{n_i+1} = \frac{\alpha + \beta x_{n_i}^k + x_{n_i-1}^k}{A + x_{n_i}^k} > x_{n_i-1}, \text{ for all } i \geq 1,$$

from which it follows

$$\alpha + \beta x_{n_i}^k > x_{n_i-1}^k [x_{n_i-1}^{1-k}(A + x_{n_i}^k) - 1], \text{ for all } i \geq 1. \tag{4.12}$$

This is a contradiction. The proof is complete. □

The next theorem establishes that the boundedness characterization of the special case 8 of 1.1, which in normalized form can be written as:

$$8: \quad x_{n+1} = \frac{x_{n-1}^k}{x_n^k}, \quad n = 0, 1, \dots, \tag{4.13}$$

is **B** when

$$k \in (0, \frac{1}{2}].$$

It also proves that this special cases loses its **B** characterization for each $k \in (\frac{1}{2}, 1)$.

Theorem 4.5. *The following statements are true:*

1. *Every positive solution of* (4.13) *converges to 1 when* $k \in \left(0, \frac{1}{2}\right]$.
2. *Every positive solution of* (4.13) *converges to a (not necessarily prime) period-two solution when* $k = \frac{1}{2}$.
3. (4.13) *possesses unbounded solutions when* $k \in \left(\frac{1}{2}, 1\right)$.

Proof. By using the change of variables

$$x_n = e^{y_n}, \quad n = -1, 0, \dots,$$

we see that the sequence $\{y_n\}_{n=-1}^{\infty}$ satisfies the second order linear equation

$$y_{n+1} + ky_n - ky_{n-1} = 0, \quad n = -1, 0, \dots .$$

The corresponding characteristic equation is

$$\lambda^2 + k\lambda - k = 0.$$

Thus, the solutions of the characteristic equation are:

$$\lambda_1 = \frac{-k - \sqrt{k^2 + 4k}}{2} \quad \text{and} \quad \lambda_2 = \frac{-k + \sqrt{k^2 + 4k}}{2}.$$

When $k \in \left(0, \frac{1}{2}\right)$ clearly, $\lambda_1, \lambda_2 \in (-1, 1)$, from which the proof of Statement 1 follows.

When $k \in \left(\frac{1}{2}, \infty\right)$ clearly, $\lambda_1 \in (-\infty, -1)$, from which the proof of Statement 3 follows.

Finally, when $k = \frac{1}{2}$, in order to establish Statement 2, we may consider the following three cases:

Case 1:

$$x_0 x_{-1} > 1 \Rightarrow x_{2n+1} < x_{2n-1} \text{ and } x_{2n+2} < x_{2n}, \text{ for all } n \geq 0.$$

In this case it is easy to see that the two subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$ and $\{x_{2n}\}_{n=0}^{\infty}$ of the solution converge to positive and finite limits.

Case 1:

$$x_0 x_{-1} = 1.$$

In this case the solution is the period-two cycle

$$x_{-1}, x_0, x_{-1}, x_0, \ldots, \ .$$

Case 3:

$$x_0 x_{-1} < 1 \Rightarrow x_{2n+1} > x_{2n-1} \text{ and } x_{2n+2} > x_{2n}, \text{ for all } n \geq 0.$$

In this case it is easy to see that the two subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$ and $\{x_{2n}\}_{n=0}^{\infty}$ of the solution converge to positive and finite limits. The proof is complete.

□

The next theorem establishes that the boundedness characterization of the special case 23 of (1.1), which in normalized form can be written as:

$$23: \quad x_{n+1} = \frac{\alpha + x_{n-1}^k}{x_n^k}, \quad n = 0, 1, \ldots, \tag{4.14}$$

is **B** when

$$k \in \left(0, \frac{\sqrt{5}-1}{2}\right). \tag{4.15}$$

Theorem 4.6. *Assume that* (4.15) *holds. Then every positive solution of* (4.14) *converges to a (not necessarily prime) period-two solution.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary positive solution of (4.14). In view of Theorem 1.8, one may easily conclude that the two subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$ and $\{x_{2n}\}_{n=0}^{\infty}$ are eventually monotonic. Assume for the sake of contradiction and without loss of generality that

$$\lim_{n\to\infty} x_{2n-1} = \infty.$$

Then eventually,

$$x_{2n+1} = \frac{\alpha + x_{2n-1}^k}{x_{2n}^k} > x_{2n-1},$$

from which we see that

$$x_{2n-1}^{k}(x_{2n-1}^{1-k}x_{2n}^{k}-1)<\alpha \tag{4.16}$$

and so necessarily,

$$\lim_{n\to\infty} x_{2n}=0.$$

Also, in view of (4.15) we have $1-k-k^2>0$, and so

$$x_{2n-1}^{1-k}x_{2n}^{k}=\frac{x_{2n}^{k}}{x_{2n-1}^{k}}x_{2n-1}=x_{2n-1}^{1-k-k^2}\cdot(\alpha+x_{2n-2}^{k})^{k}\to\infty,$$

which contradicts (4.16). The proof is complete. □

Remark 4.7. The statement of the previous Theorem that every solution converges to a (not necessarily prime) period-two solution should not be mistaken. The existence of prime period-two solutions in this case remains an open problem. See Open Problem 6.3.

For the remaining of this section assume that

$$\alpha\in(0,1) \text{ and } k\in\left[\frac{\sqrt{5}-1}{2},1\right). \tag{4.17}$$

The following lemma will be useful in the sequel.

Lemma 4.8. *Assume that* (4.17) *holds. Then there exists a positive number* $S_2=S_2(\alpha,k)$*, such that for each positive number* $x_0>S_2$*, we have*

$$x_0>1 \text{ and } (1-\alpha)x_0-x_0^k-\alpha>0. \tag{4.18}$$

Furthermore, there exists a positive real number $M=M(x_0,\alpha,k)$*, such that for each positive real number* x_{-1} *chosen such that*

$$M<x_{-1}<\left(\frac{\alpha+x_0^k}{x_0}\right)^{\frac{1}{k}}, \tag{4.19}$$

we have

$$x_0>\left(\frac{\alpha+x_{-1}^k}{x_{-1}}\right)^{\frac{1}{k}}. \tag{4.20}$$

Proof. First observe that there exists a positive number $S_1=S_1(\alpha,k)>1$ such that

$$F(x)=(1-\alpha)x-x^k-\alpha>0 \text{ and } F'(x)>0, \text{ for all } x\in(S_1,\infty) \tag{4.21}$$

and so (4.18) holds for all positive real numbers x_0, for which $x_0>S_1$.

Next, set

$$f(x)=x_0^k\cdot x-x^k-\alpha.$$

Clearly, for each $x_0 \in (S_1, \infty)$, there exists a unique positive real number M such that

$$f(x) < 0, \text{ for } x \in [0, M), \quad f(M) = 0, \text{ and } f(x) > 0, \text{ for } x \in (M, \infty). \tag{4.22}$$

We now claim that there exists a positive number, $S_2 = S_2(\alpha, k)$, $S_2 \geq S_1$ such that:

$$0 < M < \left(\frac{\alpha + x_0^k}{x_0}\right)^{\frac{1}{k}}, \text{ for all } x_0 > S_2. \tag{4.23}$$

In view of (4.22), (4.23) holds if and only if

$$f\left(\left(\frac{\alpha + x_0^k}{x_0}\right)^{\frac{1}{k}}\right) > 0, \text{ for all } x_0 > S_2, \tag{4.24}$$

if and only if

$$g(x_0) = x_0^{\frac{k^2+k-1}{k}}(\alpha + x_0^k)^{\frac{1}{k}} - \alpha x_0 - x_0^k - \alpha > 0, \text{ for all } x_0 > S_2. \tag{4.25}$$

We divide the search of finding S_2, into the following two cases:
Case 1:

$$k^2 + k - 1 > 0.$$

Set

$$h(x_0) = x_0^{\frac{k^2+2k-1}{k}} - \alpha x_0 - x_0^k - \alpha.$$

Then

$$h(0) = -\alpha \text{ and } h(\infty) = \infty$$

and so there exists a positive number $M_1 = M_1(\alpha, k)$ such that

$$h(x_0) > 0, \text{ for } x_0 \in (M_1, \infty).$$

Also, for all $x_0 \in (0, \infty)$,

$$g(x_0) > h(x_0)$$

and so

$$g(x_0) > 0, \text{ for } x_0 \in (M_1, \infty).$$

Thus, in this case S_2 may be chosen to be any positive real number greater than or equal to $\max\{M_1, S_1\}$.
Case 2:

$$k^2 + k - 1 = 0.$$

In this case

$$g(x_0) = (\alpha + x_0^k)^{\frac{1}{k}} - \alpha x_0 - x_0^k - \alpha.$$

Then

$$g(0) = \alpha^{\frac{1}{k}} - \alpha < 0 \text{ and } g(\infty) = \infty$$

and so there exists a positive number $M_2 = M_2(\alpha, k)$ such that

$$g(x_0) > 0, \quad \text{for all } \ x_0 \in (M_2, \infty).$$

By combining the results of the two cases we choose

$$S_2 = \max\{S_1, M_1, M_2\},$$

and we see that (4.25) holds. Since, (4.23) holds if and only if (4.25) holds, we have that (4.23) holds. Now choose x_{-1} such that:

$$0 < M < x_{-1} < \left(\frac{\alpha + x_0^k}{x_0}\right)^{\frac{1}{k}}, \quad \text{for all } \ x_0 > S_2. \tag{4.26}$$

Since, $x_{-1} > M$, in view of (4.22), we have

$$f(x_{-1}) = x_0^k x_{-1} - x_{-1}^k - \alpha > 0, \quad \text{for all } \ x_0 > S_2,$$

which implies that

$$x_0 > \left(\frac{\alpha + x_{-1}^k}{x_{-1}}\right)^{\frac{1}{k}}, \quad \text{for all } \ M < x_{-1} < \left(\frac{\alpha + x_0^k}{x_0}\right)^{\frac{1}{k}} \quad \text{and } \ x_0 > S_2. \tag{4.27}$$

Thus, (4.20) holds. Finally, in view of (4.21) and the fact that $S_2 \geq S_1 > 1$, we see that (4.18) holds when $x_0 > S_2$. The proof is complete.

□

The next theorem presents unbounded solutions of (4.14).

Theorem 4.9. *Assume that* (4.17) *holds. Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of* (4.14) *with initial conditions* x_{-1} *and* x_0 *chosen such that:*

$$x_0 > 1, \ \ (1-\alpha)x_0 - x_0^k - \alpha > 0, \tag{4.28}$$

$$0 < x_{-1} < \left(\frac{\alpha + x_0^k}{x_0}\right)^{\frac{1}{k}}, \quad \textit{and } \ x_0 > \left(\frac{\alpha + x_{-1}^k}{x_{-1}}\right)^{\frac{1}{k}}. \tag{4.29}$$

Then

$$\lim_{n\to\infty} x_{2n-1} = 0 \ \ \textit{and} \ \ \lim_{n\to\infty} x_{2n} = \infty.$$

Proof. In view of Lemma 4.8, a solution $\{x_n\}_{n=-1}^{\infty}$ of (4.14) with initial conditions x_{-1} and x_0 as prescribed in (4.28) and (4.29) is feasible. We claim that

$$x_{2n+1} < x_{2n-1} \ \text{ and } \ x_{2n+2} > x_{2n}, \quad \text{for all } \ n \geq 0. \tag{4.30}$$

From the second inequality of (4.29), clearly

$$x_1 < x_{-1}.$$

From this and the first inequality of (4.29), we have

$$x_0 < \frac{\alpha + x_0^k}{x_{-1}^k} < \frac{\alpha + x_0^k}{x_1^k} = x_2$$

and so (4.30) holds when $n = 0$. Assume that (4.30) holds for an arbitrary positive integer n. Then

$$x_{2(n+1)+1} = x_{2n+3} = \frac{\alpha + x_{2n+1}^k}{x_{2n+2}^k} < \frac{\alpha + x_{2n-1}^k}{x_{2n}^k} = x_{2n+1} = x_{2(n+1)-1}$$

and

$$x_{2(n+1)+2} = x_{2n+4} = \frac{\alpha + x_{2n+2}^k}{x_{2n+3}^k} > \frac{\alpha + x_{2n}^k}{x_{2n+1}^k} = x_{2n+2} = x_{2(n+1)}$$

establish (4.30).

Next, we claim that

$$\lim_{n\to\infty} x_{2n} = \infty.$$

Assume for the sake of contradiction that

$$\lim_{n\to\infty} x_{2n-1} = L_{-1} \in (0,\infty) \text{ and } \lim_{n\to\infty} x_{2n} = L_0 \in (0,\infty). \tag{4.31}$$

In view of (4.31), clearly

$$L_0^k = \frac{\alpha + L_{-1}^k}{L_{-1}} \text{ and } L_{-1}^k = \frac{\alpha + L_0^k}{L_0}$$

from which it follows that

$$L_{-1} = \frac{\alpha L_0 + \alpha + L_0^k}{L_0^{k+1}} \text{ and } L_{-1}^k = \frac{(\alpha L_0 + \alpha + L_0^k)^k}{L_0^{k^2+k}} = \frac{\alpha + L_0^k}{L_0}.$$

Thus,

$$\left(\alpha + \frac{\alpha}{L_0} + \frac{1}{L_0^{1-k}}\right)^k = L_0^{k^2+k-1} + \frac{\alpha}{L_0^{1-k^2}}. \tag{4.32}$$

Next, consider the function

$$F(x) = (1-\alpha)x - x^k - \alpha, \quad \text{for } x \in (0,\infty).$$

In view of (4.28),

$$F(x_0) > 0$$

and one may easily verify that

$$F(x) > 0, \quad \text{for all } x > x_0.$$

Furthermore,

$$x_{2n} \uparrow L_0 \Rightarrow L_0 > x_0 > 1.$$

Thus,

$$L_0 > 1 \text{ and } (1-\alpha)L_0 - L_0^k - \alpha > 0,$$

from which it follows that

$$\left(\alpha + \frac{\alpha}{L_0} + \frac{1}{L_0^{1-k}}\right)^k < 1 < L_0^{k^2+k-1} + \frac{\alpha}{L_0^{1-k^2}}.$$

This contradicts (4.32) and completes the proof. □

5 The Five Special Cases with Unbounded Solutions when $k \in (1, \infty)$

In this section we establish the **U** characterization of each one of the five special cases of (1.1):

$$2, \; 3, \; 6, \; 12, \; 25.$$

When $k = 1$, the global character of solutions of each one of these 24 special cases has been extensively studied in [14] and in the references cited therein. See also [5] and the references cited therein. We should also mention that when $k = 1$, the special case 2 is called: period-two equation, the special case 3 is called: period-four equation, the special case 6 is called : period-six equation, and the special case 25 is the well known Lyness equation. For the special case 12 it has been established in [9] that every solution converges to a finite limit.

5.1 The Two Special Cases 2 and 3

In this section we establish the existence of unbounded solutions in each of the following two special cases of (1.1), written in normalized form:

$$2: \quad x_{n+1} = \frac{1}{x_n^k}, \quad n = 0, 1, \ldots, \tag{5.1}$$

$$3: \quad x_{n+1} = \frac{1}{x_{n-1}^k}, \quad n = 0, 1, \ldots, \tag{5.2}$$

when $k \in (1, \infty)$. We also establish that every solution of each one of these two special cases is bounded when $k \in (0, 1]$.

The main result in this section is the following theorem. The proof of the theorem is based on straightforward calculations and is omitted.

Theorem 5.1. *Assume that $k \in (0, \infty)$. Then the following statements are true:*

1. *Let* $\{x_n\}_{n=0}^{\infty}$ *be a solution of* (5.1). *Then*

$$x_{2n} = x_0^{k^{2n}} \text{ and } x_{2n+1} = \left(\frac{1}{x_0}\right)^{k^{2n+1}}, \text{ for all } n \geq 0.$$

Furthermore, when

$$0 < k < 1 \Rightarrow \lim_{n\to\infty} x_n = 1$$

and when

$$k = 1,$$

every solution is the period-two cycle

$$\ldots, x_0, \frac{1}{x_0}, \ldots .$$

On the other hand, when

$$k > 1 \text{ and } 0 < x_0 < 1,$$

$$\lim_{n\to\infty} x_{2n} = 0 \text{ and } \lim_{n\to\infty} x_{2n+1} = \infty$$

and when

$$k > 1 \text{ and } x_0 > 1,$$

$$\lim_{n\to\infty} x_{2n} = \infty \text{ and } \lim_{n\to\infty} x_{2n+1} = 0.$$

2. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of* (5.2). *Then*

$$x_{4n-1} = x_{-1}^{k^{2n}}, \quad x_{4n} = x_0^{k^{2n}}, \text{ for all } n \geq 0,$$

$$x_{4n+1} = \left(\frac{1}{x_{-1}}\right)^{k^{2n+1}}, \text{ and } x_{4n+2} = \left(\frac{1}{x_0}\right)^{k^{2n+1}}, \text{ for all } n \geq 0.$$

Furthermore, when

$$0 < k < 1 \Rightarrow \lim_{n\to\infty} x_n = 1$$

and when

$$k = 1,$$

every solution is the period-four cycle

$$\ldots, x_{-1}, x_0, \frac{1}{x_0}, \frac{1}{x_0}, \ldots .$$

On the other hand, when

$$k > 1 \text{ and } (x_{-1} \neq 1 \text{ or } x_0 \neq 1)$$

at least one of the four subsequences of the solution increases to ∞.

5.2 The Special Case 6

In this section we establish the existence of unbounded solutions of the special case 6 of (1.1), written in normalized form:

$$6: \quad x_{n+1} = \frac{x_n^k}{x_{n-1}^k}, \quad n = 1, 2, \ldots, \tag{5.3}$$

when $k \in (1, \infty)$. We also establish that every solution of (5.3) is bounded when $k \in (0, 1)$.

The next theorem is the main result in this section. The proof is straightforward and is omitted.

Theorem 5.2. *Let $\{x_n\}_{n=0}^{\infty}$ be a solution of* (5.3)*. Then the following statements are true:*

1. *When $k > 4$,*

$$x_n = \frac{x_1^{\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}}}{x_0^{\frac{\lambda_2 \lambda_1^n - \lambda_1 \lambda_2^n}{\lambda_1 - \lambda_2}}}, \quad \textit{for all} \quad n \geq 0\ .$$

   with

$$\lambda_1 = \frac{k + \sqrt{k^2 - 4k}}{2} \quad \text{and} \quad \lambda_2 = \frac{k - \sqrt{k^2 - 4k}}{2}.$$

   Furthermore, when

$$x_0 < 1 \quad \textit{and} \quad x_1 > 1 \Rightarrow \lim_{n \to \infty} x_n = \infty.$$

2. *When $k = 4$,*

$$x_n = \frac{x_1^{n 2^{n-1}}}{x_0^{(n-1) 2^n}}, \quad \textit{for all} \quad n \geq 0\ .$$

   Furthermore, when

$$x_0 < 1 \quad \textit{and} \quad x_1 > 1 \Rightarrow \lim_{n \to \infty} x_n = \infty.$$

3. *When $0 < k < 4$,*

$$x_n = x_0^{k^{\frac{n}{2}} \left(\cos(n\phi) - \frac{k \sin(n\phi)}{\sqrt{4k - k^2}} \right)} \cdot x_1^{\frac{2k^{\frac{n}{2}} \sin(n\phi)}{\sqrt{4k - k^2}}}, \quad \textit{for all} \quad n \geq 0,$$

   where

$$\phi \in \left(0, \frac{\pi}{2}\right), \quad \cos\phi = \frac{\sqrt{k}}{2}, \quad \textit{and} \quad \sin\phi = \frac{\sqrt{4k - k^2}}{2\sqrt{k}}.$$

3a. *When* $1 < k < 4$, *there exist initial conditions* x_0 *and* x_1, *for which the solution* $\{x_n\}_{n=0}^{\infty}$ *is unbounded.*

3b. *When* $k = 1$, *the solution* $\{x_n\}_{n=0}^{\infty}$ *is periodic with period six.*

3c. *When* $0 < k < 1$,

$$\lim_{n\to\infty} x_n = 1.$$

5.3 The Special Case 12

In this section we establish the existence of unbounded solutions of the special case 12, which in normalized form can be written as:

$$x_{n+1} = \frac{1}{Bx_n^k + x_{n-1}^k}, \quad n = 0, 1, \ldots, \tag{5.4}$$

with $B > 0$ and nonnegative initial conditions x_{-1} and x_0 such that the denominator is positive. When $k = 1$, every solution of the equation converges to a finite limit. For the proof of this result see [9] and [14] and the references cited therein. See also [5] and the references cited therein.

The next theorem establishes that every solution of (5.4) is bounded from above and from below by positive constants, when $k \in (0, 1)$. The proof of this theorem is based on the results established in [9].

Theorem 5.3. *Assume that* $k \in (0, 1)$. *Then every solution of* (5.4) *is bounded from above and from below by positive constants.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary positive solution of (5.4). Then it is easy to see that the sequence $\{y_n\}_{n=-1}^{\infty}$, with

$$y_n = \frac{1}{x_n}, \quad n = -1, 0, \ldots,$$

satisfies (1.16), of Theorem 1.10, for which it holds that every positive solution is bounded. Thus, there exists a positive constant, namely $\frac{1}{m}$ such that

$$y_n < \frac{1}{m}, \quad \text{for all } n \geq -1.$$

Consequently,

$$x_n > m, \quad \text{for all } n \geq -1$$

and so

$$x_{n+1} = \frac{1}{Bx_n^k + x_{n-1}^k} < \frac{1}{Bm^k}, \quad \text{for all } n \geq 0.$$

The proof is complete. □

The next theorem establishes the **U** characterization of (5.4). The proof of the **U** characterization of (5.4) also follows from the results of [9].

Theorem 5.4. *Set*

$$U = \max\{\frac{(B+1)^k}{B}, (B+1)^k\}.$$

Choose ϵ_0 *arbitrarily small positive real number but fixed, such that*

$$0 < \epsilon_0 < \left(\frac{1}{2U}\right)^{\frac{1}{k^2-1}}. \tag{5.5}$$

Assume that $\{x_n\}_{n=-1}^{\infty}$ *is a positive solution of* (5.4) *with initial conditions*

$$0 < x_{-1}, x_0 < \epsilon_0. \tag{5.6}$$

Then

$$\lim_{n\to\infty} x_{3n} = \lim_{n\to\infty} x_{3n-1} = 0$$

and

$$\lim_{n\to\infty} x_{3n+1} = \infty.$$

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be an arbitrary solution of (5.4) generated by the initial conditions described by (5.6). Set

$$\epsilon_{n+1} = f(\epsilon_n) = U\epsilon_n^{k^2}, \quad n = 0, 1, \dots . \tag{5.7}$$

Clearly,

$$f(x) < \frac{x}{2}, \quad \text{for} \quad 0 < x < \left(\frac{1}{2U}\right)^{\frac{1}{k^2-1}}$$

and so in view of (5.5),

$$\epsilon_{n+1} < \frac{\epsilon_n}{2} < \left(\frac{1}{2U}\right)^{\frac{1}{k^2-1}}, \quad \text{for all} \quad n \geq 0 \tag{5.8}$$

from which it follows that

$$\lim_{n\to\infty} \epsilon_n = 0.$$

Next, we claim that for all $n \geq 0$,

$$x_{3n-1}, x_{3n} < \epsilon_n. \tag{5.9}$$

The proof will be by induction. In view of our hypothesis (5.6), (5.9) holds when $n = 0$. Now assume that (5.9) holds for an arbitrary positive integer n. Then

$$x_{3n+1} = \frac{1}{Bx_{3n}^k + x_{3n-1}^k} > \frac{1}{(B+1)\epsilon_n^k},$$

$$x_{3(n+1)-1} = x_{3n+2} = \frac{1}{Bx_{3n+1}^k + x_{3n}^k} < \frac{1}{Bx_{3n+1}^k} < \frac{(B+1)^k}{B} \cdot \epsilon_n^{k^2} < U\epsilon_n^{k^2} = \epsilon_{n+1},$$

and

$$x_{3(n+1)} = x_{3n+3} = \frac{1}{Bx_{3n+2}^2 + x_{3n+1}^k} < \frac{1}{x_{3n+1}^k} < (B+1)^k \cdot \epsilon_n^{k^2} < U\epsilon_n^{k^2} = \epsilon_{n+1}$$

establish (5.9), and consequently

$$\lim_{n\to\infty} x_{3n} = \lim_{n\to\infty} .x_{3n-1} = 0.$$

By taking limits on both sides of

$$x_{3n+1} = \frac{1}{Bx_{3n}^k + x_{3n-1}^k}, \quad n = 0, 1, \ldots,$$

as $n \to \infty$, we find that

$$\lim_{n\to\infty} x_{3n+1} = \infty.$$

The proof is complete.

□

5.4 The Special Case 25

In this section we establish the **U** characterization of the special case 25, which in normalized form can be written as:

$$x_{n+1} = \frac{\alpha + x_n^k}{x_{n-1}^k}, \quad n = 0, 1, \ldots . \tag{5.10}$$

When $k = 1$, the equation

$$x_{n+1} = \frac{\alpha + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots, \tag{5.11}$$

is the well known Lyness equation and it has been studied by many authors. See [1–3, 10, 11, 13–17, 19]. When $\alpha = 1$, every solution of the equation

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{5.12}$$

is the period-five cycle

$$x_{-1}, x_0, \frac{1 + x_0}{x_{-1}}, \frac{1 + x_{-1} + x_0}{x_0 x_{-1}}, \frac{1 + x_{-1}}{x_0}, x_{-1}, x_0, \ldots .$$

Equation (5.11), with $k = 1$, possesses the invariant

$$I_n = (\alpha + x_{n-1} + x_n)(1 + \frac{1}{x_{n-1}})(1 + \frac{1}{x_n}) = (\alpha + x_{-1} + x_0)(1 + \frac{1}{x_{-1}})(1 + \frac{1}{x_0}).$$

From this it follows that every solution of (5.11), is bounded from above and from below by positive constants. In [10] it was shown that no nontrivial solution of (5.11) has a limit. Furthermore, in [18] it was shown using KAM theory that the positive equilibrium $\bar{x}$ of (5.11) is stable but not asymptotically stable.

In [7] it has been shown that **periodicity may destroy the boundedness of solutions of** (5.11). Actually, it was shown that the solution of the difference equation

$$x_{n+1} = \frac{\alpha_n + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{5.13}$$

with

$$\alpha_n = \begin{cases} 1, & \text{if } n = 5k + i \text{ with } i \in \{0, 1, 2, 3\} \\ \alpha, & \text{if } n = 5k + 4 \end{cases}, k = 0, 1, \ldots$$

and with initial conditions

$$x_{-1} = x_0 = 1$$

is **unbounded**, if and only

$$\alpha \neq 1.$$

Also, as we establish here, **the appearance of a positive power** k **may destroy the boundedness of solutions of** (5.11)**.** When $k \in (0, 1]$, we conjecture that every solution of (5.10) is bounded. See Conjecture 1.7. When $k \in (1, \infty)$ we pose the second conjecture for (5.10).

Conjecture 5.5. Prove that (5.10) possesses unbounded solutions for each

$$k \in (1, \infty).$$

The next theorem establishes the **U** characterization of the special case 25 and verifies Conjecture 5.5 when $k = 2$.

Theorem 5.6. *Assume that*

$$k = 2 \text{ and } 0 < \alpha < 1.$$

Let ϵ_0 *arbitrarily small positive real number but fixed such that*

$$0 < \epsilon_0 < m = \min\{\alpha, \bar{x}_1, \bar{x}_2\}, \tag{5.14}$$

where $\bar{x}_1$ *is the unique positive solution of the equation*

$$(\alpha x^8 + 1)^5 \cdot \frac{x^8}{\alpha^4} = \frac{x}{2}$$

and $\bar{x}_2$ *is the unique positive solution of*

$$\alpha x^8 + \left(\alpha x^8 + 1\right)^{10} \frac{x^{24}}{\alpha^8} = \frac{x^8}{\alpha^4}.$$

Assume that $\{x_n\}_{n=-1}^{\infty}$ *is a positive solution of* (5.10) *with initial conditions*

$$0 < x_{-1}, x_0 < \epsilon_0. \tag{5.15}$$

Then

$$\lim_{n\to\infty} x_{6n-1} = \lim_{n\to\infty} x_{6n} = 0$$

and

$$\lim_{n\to\infty} x_{6n+1} = \lim_{n\to\infty} x_{6n+2} = \lim_{n\to\infty} x_{6n+3} = \lim_{n\to\infty} x_{6n+4} = \infty.$$

Before we present the proof of the theorem we state a lemma which contains several identities that will be very useful in the proof of the theorem.

Lemma 5.7. *Assume that* $k = 2$. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of* (5.10)*), Then the following hold:*

$$x_{n+1} = \frac{\alpha + x_n^2}{x_n^2} \cdot \left(\frac{x_n}{x_{n-1}}\right)^2, \quad n = 0, 1, \dots . \tag{5.16}$$

$$x_{n+1} = \frac{\alpha + x_n^2}{x_n^2} \cdot \left(\frac{\alpha + x_{n-1}^2}{x_{n-1}^2}\right)^2 \cdot \left(\frac{x_{n-1}}{x_{n-2}^2}\right)^2, \quad n = 1, 2, \dots . \tag{5.17}$$

$$x_{n+1} = \frac{\alpha + x_n^2}{x_n^2} \cdot \left(\frac{\alpha + x_{n-1}^2}{x_{n-1}^2}\right)^2 \cdot \left(\frac{\alpha + x_{n-2}^2}{x_{n-2}^2}\right)^2 \cdot \frac{1}{x_{n-3}^4}, \quad n = 2, 3, \dots, \tag{5.18}$$

$$x_{n+1} > \max\{\frac{\alpha}{x_{n-1}^2}, \frac{x_n^2}{x_{n-1}^2}, \frac{x_{n-1}^2}{x_{n-2}^4}, \frac{1}{x_{n-3}^4}\}, \ \textit{for all} \ n \geq 3, \tag{5.19}$$

and

$$x_{n+1} < \left(\frac{\alpha}{\min\{x_n, x_{n-1}, x_{n-2}\}^2} + 1\right)^5 \cdot \frac{1}{x_{n-3}^4}, \ \textit{for all} \ n \geq 3. \tag{5.20}$$

Proof. The proof is based on straightforward calculations and is omitted. □

Now we present the proof of the theorem.

Proof. Set

$$f(x) = (\alpha x^8 + 1)^5 \cdot \frac{x^8}{\alpha^4} \ \text{ and } \ g(x) = \alpha x^8 + \left(\alpha x^8 + 1\right)^{10} \frac{x^{24}}{\alpha^8}.$$

Clearly, for all $x \in (0, m)$, we have

$$f(x) < \frac{x}{2} \ \text{ and } \ g(x) < \frac{x^8}{\alpha^4} < (\alpha x^8 + 1)^5 \cdot \frac{x^8}{\alpha^4}. \tag{5.21}$$

Consider the difference equation

$$\epsilon_{n+1} = f(\epsilon_n) = (\alpha\epsilon_n^8 + 1)^5 \cdot \frac{\epsilon_n^8}{\alpha^4}, \quad n = 0, 1, \ldots, \tag{5.22}$$

where

$$0 < \epsilon_0 < m.$$

In view of (5.21), we have that

$$\epsilon_1 = f(\epsilon_0) < \frac{\epsilon_0}{2} < m$$

and inductively, we see that

$$\epsilon_{n+1} < \frac{\epsilon_n}{2} < \epsilon_n < m, \quad \text{for all} \quad n \geq 0 \tag{5.23}$$

from which it follows that

$$\lim_{n\to\infty} \epsilon_n = 0. \tag{5.24}$$

Next, we claim that

$$\max\{x_{6n-1}, x_{6n}\} < \epsilon_n, \quad \text{for all} \quad n \geq 0. \tag{5.25}$$

The proof will be by induction. In view of (5.15), which is an assumption, (5.25) holds when $n = 0$. Assume that for an arbitrary positive integer n,

$$x_{6n-1}, x_{6n} < \epsilon_n.$$

From this and (5.19), we have

$$x_{6n+1} > \frac{\alpha}{x_{6n-1}^2} > \frac{\alpha}{\epsilon_n^2}, \quad x_{6n+2} > \frac{x_{6n+1}^2}{x_{6n}^2} > \frac{\alpha^2}{\epsilon_n^6}, \quad x_{6n+3} > \frac{x_{6n+1}^2}{x_{6n}^4} > \frac{\alpha^2}{\epsilon_n^8},$$

and

$$x_{6n+4} > \frac{1}{x_{6n}^4} > \frac{1}{\epsilon_n^4}.$$

Also, (5.23) implies that

$$\min\{\frac{\alpha^2}{\epsilon_n^6}, \frac{1}{\epsilon_n^4}, \frac{\alpha^2}{\epsilon_n^8}\} = \frac{1}{\epsilon_n^4} \Rightarrow \min\{x_{6n+2}, x_{6n+3}, x_{6n+4}\} > \frac{1}{\epsilon_n^4}.$$

From (5.20), we have

$$x_{6(n+1)-1} = x_{6n+5} < \left(\alpha\epsilon_n^8 + 1\right)^5 \frac{\epsilon_n^8}{\alpha^4} = \epsilon_{n+1}.$$

Also, directly from the equation and in view of (5.21) and (5.23),

$$x_{6(n+1)} = x_{6n+6} = \frac{\alpha + x_{6n+5}^2}{x_{6n+4}^2} < \alpha\epsilon_n^8 + \left(\alpha\epsilon_n^8 + 1\right)^{10} \frac{\epsilon_n^{24}}{\alpha^8} < \left(\alpha\epsilon_n^8 + 1\right)^5 \frac{\epsilon_n^8}{\alpha^4} = \epsilon_{n+1}.$$

Thus, (5.25) is established and when combined with (5.24), implies that

$$\lim_{n\to\infty} x_{6n} = \lim_{n\to\infty} x_{6n-1} = 0.$$

By taking limits on both sides of

$$x_{6n+1} = \frac{\alpha + x_{6n}^2}{x_{6n-1}^2} \text{ and } x_{6n+2} = \frac{\alpha + x_{6n+1}^2}{x_{6n}^2}, \quad n = 0, 1, \ldots$$

as $n \to \infty$, we find that

$$\lim_{n\to\infty} x_{6n+1} = \lim_{n\to\infty} x_{6n+2} = \infty.$$

From this, and in view of

$$\frac{x_{6n+2}}{x_{6n+1}} = \frac{\alpha + x_{6n+1}^2}{x_{6n+1}} \cdot \frac{1}{x_{6n}^2} \text{ and } \frac{x_{6n+2}}{x_{6n+1}^2} = \frac{\alpha + x_{6n+1}^2}{x_{6n+1}^2} \cdot \frac{1}{x_{6n}^2}, \quad n = 0, 1, \ldots,$$

it follows that

$$\lim_{n\to\infty} \frac{x_{6n+2}}{x_{6n+1}} = \lim_{n\to\infty} \frac{x_{6n+2}}{x_{6n+1}^2} = \infty$$

and so,

$$\lim_{n\to\infty} x_{6n+3} = \lim_{n\to\infty} \frac{\alpha + x_{6n+2}^2}{x_{6n+1}^2} = \infty, \quad \lim_{n\to\infty} \frac{x_{6n+3}}{x_{6n+2}} = \lim_{n\to\infty} \frac{\alpha + x_{6n+2}^2}{x_{6n+2}^2} \cdot \frac{x_{6n+1}}{x_{6n+2}^2} = \infty,$$

and

$$\lim_{n\to\infty} x_{6n+4} = \lim_{n\to\infty} \frac{\alpha + x_{6n+3}^2}{x_{6n+2}^2} = \infty.$$

The proof is complete.

□

6 Future Goals and Research Problems

As we already mentioned in the Introduction of this paper, the boundedness character of solutions of a difference equation is one of the main ingredients in understanding the global behavior of the equation including its global stability.

In fact, now that we have a reasonably good picture of the boundedness character of solutions of (1.1) when $k \in (0, \infty)$, one of our future goals is to study the global character of solutions of (1.1). We are interested in the local and global stability of all equilibrium points of the equation. Also, we wish to investigate the existence of periodic solutions and also conditions under which we have convergence to periodic solutions. Finally, we wish to investigate the existence of solutions with infinitely many accumulation points. In this section we give some directions for these types of investigations.

In Section 2 of this paper, it is established that every solution of each one of the 24 special cases listed in (1.3) is bounded for all $k \in (0, \infty)$. For this group of special cases of (1.1), we pose the following open problem.

Open Problem 6.1. For each one of the 24 special cases listed in (1.3):

1. Investigate the local and global stability of all equilibrium points.
2. Determine ranges of parameters for which every solution converges to a finite limit.
3. Determine whether or not every solution converges to a finite limit when $k \in \left(0, \frac{1}{2}\right)$.
4. Determine whether or not there exist periodic solutions or solutions that converge to periodic solutions.
5. Determine whether or not there exist solutions with an infinite number of accumulation points.

In Section 3 of this paper it is established that in each one of the six special cases:

$$14, \ 27, \ 29, \ 35, \ 42, \ 47, \tag{6.1}$$

every solution is bounded when $k \in (0, 4)$. Also, it is a conjecture, when $k \geq 4$, that there exist unbounded solutions. Now, we pose the following open problem.

Open Problem 6.2. For each one of the six special cases listed in (6.1):

1. Investigate the local and global stability of all equilibrium points when $k \in (0, \infty)$.
2. Determine ranges of parameters and the set of initial conditions for which every solution converges to a finite limit when $k \in (0, 4)$.
3. Determine whether or not every solution converges to a finite limit when $k \in \left(0, \frac{1}{2}\right)$.
4. Determine whether or not there exist periodic solutions or solutions that converge to periodic solutions when $k \in (0, \infty)$.

5. Determine whether or not there exist solutions with an infinite number of accumulation points when $k \in (0, \infty)$.

In Section 4.2 of this paper it is established that, each one of the following eight special cases of (1.1)

$$8, \quad 16, \quad 23, \quad 24, \quad 31, \quad 34, \quad 41, \quad 46, \tag{6.2}$$

possesses unbounded solutions in a certain range of the parameters and for some initial conditions, when $\in [1, \infty)$. Also, except for the two special cases 8 and 23 every solution is bounded when $k \in (0, 1)$ and for all eight special cases every solution is bounded when $k \in \left(0, \frac{1}{2}\right]$. For these eight special cases we pose the following open problem.

Open Problem 6.3. For each one of the eight special cases listed in (6.2):

1. Investigate the local and global stability of all equilibrium points when $k \in (0, \infty)$.

2. Determine the set of initial conditions for which every solution converges to a finite limit when $k \in [1, \infty)$.

3. Determine the set of initial conditions for which there exist periodic solutions or solutions that converge to periodic solutions when $k \in [1, \infty)$.

4. Determine ranges of the parameters and the set of initial conditions for which every solution converges to a finite limit when $k \in (0, 1)$.

5. Determine whether or not every solution converges to a finite limit when $k \in \left(0, \frac{1}{2}\right)$.

6. Determine ranges of parameters for which there exist periodic solutions or solutions that converge to periodic solutions when $k \in (0, 1)$.

7. Determine whether or not there exist solutions with an infinite number of accumulation points when $k \in (0, \infty)$.

APPENDIX

Summary of The Boundedness Character of Solutions of the 49 Special Cases of (1.1):

Number	Equation	$k \in (0,\infty)$	$k \in (0,1]$	$k \in (0,4)$	Comments
1	$x_{n+1} = \frac{\alpha}{A}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. This equation is trivial.
2	$x_{n+1} = \frac{\alpha}{Bx_n^k}$	**U**	**B**	**U**	**U** for each $k \in (1,\infty)$. See Theorem 5.1. **B** when $k \in (0,1]$. See Theorem 5.1. When $k = 1$, every solution is periodic with period two. See [14] and [5].
3	$x_{n+1} = \frac{\alpha}{Cx_{n-1}^k}$	**U**	**B**	**U**	**U** for each $k \in (1,\infty)$. See Theorem 5.1. **B** when $k \in (0,1]$. See Theorem 5.1. When $k = 1$, every solution is periodic with period four. See [14] and [5].
4	$x_{n+1} = \frac{\beta x_n^k}{A}$	**U**	**U**	**U**	**U** for each $k \in [1,\infty)$. See Theorem 4.1. **B** when $k \in (0,1)$. See Theorem 4.2.
5	$x_{n+1} = \frac{\beta x_n^k}{Bx_n^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. This equation is trivial.
6	$x_{n+1} = \frac{\beta x_n^k}{Cx_{n-1}^k}$	**U**	**B**	**U**	**U** for each $k \in (1,\infty)$. See Theorem 5.1. **B** when $k \in (0,1]$.See Theorem 5.1. When $k = 1$, every solution is periodic with period six. See [14] and [5].
7	$x_{n+1} = \frac{\gamma x_{n-1}^k}{A}$	**U**	**U**	**U**	**U** for each $k \in [1,\infty)$. See Theorem 4.1. **B** when $k \in (0,1)$. See Theorem 4.2.
8	$x_{n+1} = \frac{\gamma x_{n-1}^k}{Bx_n^k}$	**U**	**U**	**U**	**U** for each $k \in \left(\frac{1}{2},\infty\right)$. See Theorems 4.3 and 4.5. **B** when $k \in \left(0,\frac{1}{2}\right]$. See Theorem 4.5. When $k = 1$, see [14] and [5].
9	$x_{n+1} = \frac{\gamma x_{n-1}^k}{Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. This equation is trivial.
10	$x_{n+1} = \frac{\alpha}{A + Bx_n^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].

Number	**Equation**	$k \in (0,\infty)$	$k \in (0,1]$	$k \in (0,4)$	Comments
11	$x_{n+1} = \dfrac{\alpha}{A + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
12	$x_{n+1} = \dfrac{\alpha}{Bx_n^k + Cx_{n-1}^k}$	**U**	**B**	**U**	**U** for each $k \in (1,\infty)$. See Thm 5.4. See also [9]. **B** when $k \in (0,1]$. See Theorem 5.3. See also [9]. When $k = 1$, every solution converges to a finite limit. See [9], [14] and [5].
13	$x_{n+1} = \dfrac{\beta x_n^k}{A + Bx_n^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
14	$x_{n+1} = \dfrac{\beta x_n^k}{A + Cx_{n-1}^k}$	**U**	**B**	**B**	**U** for each $k \in [4,\infty)$. See [4]. **B** when $k \in (0,4)$. See [4] and [8]. See also Open Problem 6.2. When $k = 1$, see [14] and [5].
15	$x_{n+1} = \dfrac{\beta x_n^k}{Bx_n^k + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
16	$x_{n+1} = \dfrac{\gamma x_{n-1}^k}{A + Bx_n^k}$	**U**	**U**	**U**	**U** for each $k \in [1,\infty)$. See Theorem 4.3. **B** when $k \in (0,1)$. See Theorem 4.4. See also Open Problem 6.3. When $k = 1$, see [14] and [5].
17	$x_{n+1} = \dfrac{\gamma x_{n-1}^k}{A + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
18	$x_{n+1} = \dfrac{\gamma x_{n-1}^k}{Bx_n^k + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
19	$x_{n+1} = \dfrac{\alpha + \beta x_n^k}{A}$	**U**	**U**	**U**	**U** for each $k \in [1,\infty)$. See Theorem 4.1. **B** when $k \in (0,1)$. See Theorem 4.2.
20	$x_{n+1} = \dfrac{\alpha + \gamma x_{n-1}^k}{A}$	**U**	**U**	**U**	**U** for each $k \in [1,\infty)$. See Theorem 4.1. **B** when $k \in (0,1)$. See Theorem 4.2.
21	$x_{n+1} = \dfrac{\beta x_n^k + \gamma x_{n-1}^k}{A}$	**U**	**U**	**U**	**U** for each $k \in [1,\infty)$. See Theorem 4.1. **B** when $k \in (0,1)$. See Theorem 4.2.

Number	Equation	$k \in (0,\infty)$	$k \in (0,1]$	$k \in (0,4)$	Comments
22	$x_{n+1} = \dfrac{\alpha + \beta x_n^k}{B x_n^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.3. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
23	$x_{n+1} = \dfrac{\alpha + \gamma x_{n-1}^k}{B x_n^k}$	**U**	**U**	**U**	**U** for each $k \in \left[\dfrac{\sqrt{5}-1}{2}, \infty\right)$, see Theorems 4.3 and 4.9. **B** when $k \in \left(0, \dfrac{\sqrt{5}-1}{2}\right)$, see Theorem 4.6. When $k = 1$, see [14] and [5].
24	$x_{n+1} = \dfrac{\beta x_n^k + \gamma x_{n-1}^k}{B x_n^k}$	**U**	**U**	**U**	**U** for each $k \in [1,\infty)$. See Theorem 4.3. **B** when $k \in (0,1)$. See Theorem 4.4. See also Open Problem 6.3. When $k = 1$, see [14] and [5].
25	$x_{n+1} = \dfrac{\alpha + \beta x_n^k}{C x_{n-1}^k}$	**U**	**B***	**U**	**U** for $k = 2$. See Theorem 5.6. **U*** for each $k \neq 2$. See Conjecture 5.5. **B*** when $k \in (0,1)$. See Conjecture 1.7. When $a = \beta = C = k = 1$, this is the Lyness equation in which every solution is periodic with period five. When $k = 1$, see [1–3], [10, 11, 13, 14, 17, 19].
26	$x_{n+1} = \dfrac{\alpha + \gamma x_{n-1}^k}{C x_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.3. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
27	$x_{n+1} = \dfrac{\beta x_n^k + \gamma x_{n-1}^k}{C x_{n-1}^k}$	**U***	**B**	**B**	**U*** for each $k \in [4,\infty)$. See Conjecture 1.3. **B** when $k \in (0,4)$. See Theorem 3.3. See also Open Problem 6.2. When $k = 1$, see [14] and [5].
28	$x_{n+1} = \dfrac{\alpha + \beta x_n^k}{A + B x_n^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
29	$x_{n+1} = \dfrac{\alpha + \beta x_n^k}{A + C x_{n-1}^k}$	**U***	**B**	**B**	**U*** for each $k \in [4,\infty)$. See Conjecture 1.3. **B** when $k \in (0,4)$. See Theorem 3.3 See also Open Problem 6.2. When $k = 1$, see [14] and [5].
30	$x_{n+1} = \dfrac{\alpha + \beta x_n^k}{B x_n^k + C x_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.4. See also Open Problem 6.1. When $k = 1$, see [14] and [5].

Number	**Equation**	$k \in (0, \infty)$	$k \in (0, 1]$	$k \in (0, 4)$	Comments
31	$x_{n+1} = \dfrac{\alpha + \gamma x_{n-1}^k}{A + Bx_n^k}$	**U**	**U**	**U**	**U** for each $k \in [1, \infty)$. See Theorem 4.3. **B** when $k \in (0, 1)$. See Theorem 4.4. See also Open Problem 6.3. When $k = 1$, see [14] and [5].
32	$x_{n+1} = \dfrac{\alpha + \gamma x_{n-1}^k}{A + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0, \infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
33	$x_{n+1} = \dfrac{\alpha + \gamma x_{n-1}^k}{Bx_n^k + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0, \infty)$. See Theorem 2.5. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
34	$x_{n+1} = \dfrac{\beta x_n^k + \gamma x_{n-1}^k}{A + Bx_n^k}$	**U**	**U**	**U**	**U** for all $k \in [1, \infty)$. See Theorem 4.3. **B** when $k \in (0, 1)$. See Theorem 4.4. See also Open Problem 6.3. When $k = 1$, see [14] and [5].
35	$x_{n+1} = \dfrac{\beta x_n^k + \gamma x_{n-1}^k}{A + Cx_{n-1}^k}$	**U***	**B**	**B**	**U*** for each $k \in [4, \infty)$. See Conjecture 1.3. **B** when $k \in (0, 4)$. See Theorem 3.3. See also Open Problem 6.2. When $k = 1$, see [14] and [5].
36	$x_{n+1} = \dfrac{\beta x_n^k + \gamma x_{n-1}^k}{Bx_n^k + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0, \infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
37	$x_{n+1} = \dfrac{\alpha}{A + Bx_n^k + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0, \infty)$. See Theorem 2.1 See also Open Problem 6.1. When $k = 1$, see [14] and [5].
38	$x_{n+1} = \dfrac{\beta x_n^k}{A + Bx_n^k + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0, \infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
39	$x_{n+1} = \dfrac{\gamma x_{n-1}^k}{A + Bx_n^k + Cx_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0, \infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
40	$x_{n+1} = \dfrac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{A}$	**U**	**U**	**U**	**U** for each $k \in [1, \infty)$. See Theorem 4.1. **B** when $k \in (0, 1)$. See Theorem 4.2.

Number	**Equation**	$k \in (0,\infty)$	$k \in (0,1]$	$k \in (0,4)$	Comments
41	$x_{n+1} = \dfrac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{B x_n^k}$	**U**	**U**	**U**	**U** for each $k \in [1,\infty)$. See Theorem 4.3. **B** when $k \in (0,1)$. See Theorem 4.4. See also Open Problem 6.3. When $k = 1$, see [14] and [5].
42	$x_{n+1} = \dfrac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{C x_{n-1}^k}$	**U***	**B**	**B**	**U*** for each $k \in [4,\infty)$. See Conjecture 1.3. **B** when $k \in (0,4)$. See Theorem 3.3. See also Open Problem 6.2. When $k = 1$, see [14] and [5].
43	$x_{n+1} = \dfrac{\alpha + \beta x_n^k}{A + B x_n^k + C x_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
44	$x_{n+1} = \dfrac{\alpha + \gamma x_{n-1}^k}{A + B x_n^k + C x_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
45	$x_{n+1} = \dfrac{\beta x_n^k + \gamma x_{n-1}^k}{A + B x_n^k + C x_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].
46	$x_{n+1} = \dfrac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{A + B x_n^k}$	**U**	**U**	**U**	**U** for all $k \in [1,\infty)$. See Theorem 4.3. **B** when $k \in (0,1)$. See Theorem 4.4. See also Open Problem 6.3. When $k = 1$, see [14] and [5].
47	$x_{n+1} = \dfrac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{A + C x_{n-1}^k}$	**U***	**B**	**B**	**U*** for each $k \in [4,\infty)$. See Conjecture 1.3. **B** when $k \in (0,4)$. See Theorem 3.3. See also Open Problem 6.2. When $k = 1$, see [14] and [5].
48	$x_{n+1} = \dfrac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{B x_n^k + C x_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.3. See also Open Problem 6.1.
49	$x_{n+1} = \dfrac{\alpha + \beta x_n^k + \gamma x_{n-1}^k}{A + B x_n^k + C x_{n-1}^k}$	**B**	**B**	**B**	**B** for all $k \in (0,\infty)$. See Theorem 2.1. See also Open Problem 6.1. When $k = 1$, see [14] and [5].

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