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A Regularization Procedure for Solving some Singular Integral Equations of the Second Kind

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Abstract

We begin by proposing a regularization procedure for solving some singular integral equations of the second kind. After that, we consider a projection method to the regularized equation using a sequence of orthogonal finite rank projections. We prove the convergence of the method and we perform an error analysis. We end with some numerical examples illustrating the obtained theoretical results.

AMS Subject Classifications: 45E05, 45B05.
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1 Introduction and Mathematical Background

During the last two decades, physicists, engineers and mathematicians have shown a strong interest for the theory and numerical modeling of singular integral equations (see [3, 6, 10, 11]). Several problems of engineering physics are described in terms of singular integral equations. In [1, 2, 4, 5], the authors have studied some finite rank approximations using bounded finite rank projections. In [8], we have presented a projection method for solving operator equations with bounded operators in Hilbert spaces, and we have applied the method for solving the Cauchy integral equations for two cases: Galerkin projections and Kulkarni projections, using a sequence of orthogonal finite rank projections. In [7] we have introduced a modified method, which is based on trapezoidal and Simpson's rules, for solving Volterra integral equations of the second kind. In [9], we have studied projection approximations for solving Cauchy integro-differential equations using airfoil polynomials of the first kind. Projection methods play an important role in numerical analysis. In particular, they are an effective means of numerically solving integral and integro-differential equations. The goal of the present

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paper is to introduce a regularization procedure for solving some singular integral equations of the second kind.

Denote by $\mathcal{H} := L^2([-1,1],\mathbb{C})$ the space of complex-valued Lebesgue square integrable (classes of) functions on $[-1,1]$. Let the universe under consideration be the space $\mathcal{H}$ and consider a generalized integral equation with Cauchy kernel,

$$\fint_{-1}^{1} \frac{k(s,\tau)x(\tau)}{\tau - s} d\tau = z\varphi(s) + f(s), \quad -1 \le s \le 1, \tag{1.1}$$

where z is real and nonzero. We assume that k is continuous and

$$\overline{k(s,\tau)} = k(\tau,s).$$

Let

$$Kx(s) \ := \ \fint_{-1}^{1} \frac{k(s,\tau)x(\tau)}{\tau - s} d\tau, \quad x \in \mathcal{H}, \quad -1 \le s \le 1.$$

We recall that $K \in BL(\mathcal{H})$ and $K^* = -K$. Equation (1.1) can be rewritten in operator form as follows:

$$(K - zI)\varphi = f.$$

Theorem 1.1. *For each right-hand side $f \in \mathcal{H}$, the Cauchy integral equation* (1.1) *has a unique solution $\varphi \in \mathcal{H}$.*

Proof. Since

$$\overline{k(s,\tau)} = k(\tau,s),$$

it is clear that K is skew-Hermitian, hence $sp\ (K) \subseteq \mathrm{i}\mathbb{R}$. This shows that $z \notin sp\ (K)$, and consequently the operator $K - zI$ is invertible. □

Theorem 1.2. *The following estimate holds:*

$$\|(K - zI)^{-1}\| \le \frac{1}{|z|}.$$

Proof. For all $x \in \mathcal{H}$,

$$\begin{aligned} Re\,\langle (K - zI)\,x, x\rangle &= \frac{1}{2}\left[\langle (K - zI)\,x, x\rangle + \overline{\langle (K - zI)\,x, x\rangle}\right] \\ &= \frac{1}{2}\left[-2z\,\langle x, x\rangle + \langle Kx, x\rangle + \langle x, Kx\rangle\right] \\ &= -z\,\langle x, x\rangle\,, \end{aligned}$$

so that

$$|z|\,\|x\|^2 \le |\mathrm{Re}\ \langle (K - zI)x, x\rangle| \le |\langle (K - zI)x, x\rangle| \le \|(K - zI)x\|\,\|x\|\,,$$

which yields

$$\|(K - zI)^{-1}\| \le \frac{1}{|z|}.$$

This concludes the proof. □

We define the regularized operator K_ϵ for $\epsilon > 0$ by

$$K_\epsilon \varphi(s) := \int_{-1}^{1} \frac{(\tau - s)k(s,\tau)\varphi(\tau)}{(\tau - s)^2 + \epsilon^2} d\tau, \quad -1 < s < 1,$$

which is compact and skew-Hermitian from $\mathcal{H}$ into itself. We denote by φ_ϵ the solution of the regularized integral equation

$$(K_\epsilon - zI)\varphi_\epsilon = f.$$

2 Numerical Approximation

Let $(\ell_n)_{n\geq 0}$ denote a sequence of Legendre polynomials and

$$e_j := \sqrt{\frac{2j+1}{2}} \ell_j$$

be the corresponding normalized sequence. Let us consider $(\Pi_n)_{n\geq 1}$ a sequence of bounded projections, each one of finite rank, such that

$$\Pi_n x := \sum_{j=0}^{n-1} \langle x, e_j \rangle e_j.$$

We recall that

$$\lim_{n\to\infty} \|\Pi_n \psi - \psi\| = 0$$

for all $\psi \in \mathcal{H}$. Let $\mathcal{H}_n$ denote the space spanned by the first n of Legendre polynomials. Consider the approximate operator

$$K_{\epsilon,n} := \Pi_n K_\epsilon \Pi_n.$$

Hence, the equation

$$K_{\epsilon,n} \varphi_{\epsilon,n} - z\varphi_{\epsilon,n} = \Pi_n f \tag{2.1}$$

has a unique solution $\varphi_{\epsilon,n}$ given by

$$\varphi_{\epsilon,n} = \sum_{j=1}^{n} x_{n,j} e_{n,j}$$

for some scalars $x_{n,j}$. Equation (2.1) reads as

$$\sum_{j=1}^{n} x_{n,j} \left[\Pi_n K_\epsilon e_{n,j} - z e_{n,j}\right] = \Pi_n f,$$

so that

$$\sum_{j=1}^{n} x_{n,j} \left[\sum_{i=1}^{n} \langle K_\epsilon e_{n,j}, e_{n,i} \rangle e_{n,i} - z e_{n,j} \right] = \sum_{i=1}^{n} \langle f, e_{n,i} \rangle e_{n,i},$$

that is, the coefficients $x_{n,j}$ are obtained by solving the linear system

$$(A_{n,\epsilon} - zI) x_n = b_n,$$

where

$$\begin{aligned} A_{n,\epsilon}(k,j) &:= \sqrt{\frac{2i+1}{2}} \sqrt{\frac{2j+1}{2}} \int_{-1}^{1} \oint_{-1}^{1} \frac{\ell_j(\tau) k(s,\tau)}{(\tau - s)^2 + \epsilon^2} d\tau ds, \\ b_n(k) &:= \sqrt{\frac{2k+1}{2}} \int_{-1}^{1} \ell_k(s) f(s) ds. \end{aligned}$$

Theorem 2.1. *The following estimate holds:*

$$\|(K_{\epsilon,n} - zI)^{-1}\| \leq \frac{1}{|z|}.$$

Proof. It is clear that $K_{\epsilon,n}$ is skew-Hermitian, so that the proof is rather similar to the proof of Theorem 1.1. This shows that for n large enough, the operator $K_{\epsilon,n} - zI$ is invertible and the constant $\sup_n \|(K_{\epsilon,n} - zI)^{-1}\|$ is finite. □

Theorem 2.2. *The following estimate holds:*

$$\|\varphi_{\epsilon,n} - \varphi_\epsilon\|_2 \leq \frac{1}{|z|} \left[\|(\Pi_n - I) f\|_2 + \|(\Pi_n - I) K_\epsilon \varphi_\epsilon\|_2 + \|K_\epsilon\| \, \|(\Pi_n - I)\varphi_\epsilon\|_2 \right].$$

Proof. One has

$$\begin{aligned} \varphi_{\epsilon,n} - \varphi_\epsilon &= (K_{\epsilon,n} - zI)^{-1} \Pi_n f - (K_\epsilon - zI)^{-1} f \\ &= (K_{\epsilon,n} - zI)^{-1} \Pi_n f - (K_{\epsilon,n} - zI)^{-1} f \\ &+ (K_{\epsilon,n} - zI)^{-1} f - (K_\epsilon - zI)^{-1} f \\ &= (K_{\epsilon,n} - zI)^{-1} (\Pi_n - I) f \\ &+ (K_{\epsilon,n} - zI)^{-1} \left[(K_\epsilon - zI) - (K_{\epsilon,n} - zI) \right] (K_\epsilon - zI)^{-1} f \\ &= (K_{\epsilon,n} - zI)^{-1} \left[(\Pi_n - I) f + (K_\epsilon - K_{\epsilon,n}) \varphi_\epsilon \right]. \end{aligned}$$

On the other hand, due to Theorem 2.1,

$$(K_{\epsilon,n} - K_\epsilon)\varphi_\epsilon = (\Pi_n - I) K_\epsilon \varphi_\epsilon + \Pi_n K_\epsilon (\Pi_n - I) \varphi_\epsilon.$$

We get the desired result because $\|\Pi_n\| = 1$. □

n	$\|\varphi - \varphi_n\|_2$
4	5.17e-2
5	3.78e-2
6	5.02e-3
7	1.91e-3
8	7.36e-4
9	3.67e-4
10	2.13e-4
11	8.19e-5
12	4.25e-5

Table 1: *Numerical results*

Theorem 2.3. *The solution $\varphi_{\epsilon,n}$ of the equation* (2.1) *converges to the solution φ of equation* (1.1) *if, first, $n \to \infty$, and then $\epsilon \to 0$.*

Proof. We have

$$\varphi_\epsilon - \varphi = (K_\epsilon - zI)^{-1}[K - K_\epsilon]\varphi.$$

Because

$$\left\|(K_\epsilon - zI)^{-1}\right\| \leq \frac{1}{|z|},$$

it follows that

$$\|\varphi_\epsilon - \varphi\|_2 \leq \frac{1}{|z|}\|(K - K_\epsilon)\varphi\|_2 \to 0 \text{ as } \epsilon \to 0.$$

Hence,

$$\|\varphi_{\epsilon,n} - \varphi\|_2 \leq \|\varphi_\epsilon - \varphi\|_2 + \|\varphi_{\epsilon,n} - \varphi_\epsilon\|_2 \to 0,$$

if, first, $n \to \infty$, and then $\epsilon \to 0$. □

3 Numerical Results and Discussion

Consider the generalized integral equation with Cauchy kernel (1.1) with f such that

$$\varphi(s) = s^2 + 1$$

and

$$k(s,t) = \cos(s - t), \quad z = \frac{1}{\pi}.$$

For the regularization process, take $\epsilon = 10^{-63}$. We present in Table 1 the corresponding absolute errors for this example.

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