International Journal of Difference Equations
ISSN 0973-6069, Volume 8, Number 1, pp. 3–14 (2013)
http://campus.mst.edu/ijde

Results on Shared Values of Entire Functions and their Homogeneous Differential Polynomials

Rabab Bouabdelli and **Benharrat Belaïdi**
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University Abdelhamid Ibn Badis, Mostaganem (UMAB), Algeria
bouabdelli.rabab@gmail.com, belaidi@univ-mosta.dz

Abstract

The present paper introduces various results for studying the uniqueness of entire functions and the linear combination of its derivatives $L[f]$ sharing one value and two small entire functions. The results improve some previous results known as the conjecture of Brück.

AMS Subject Classifications: 30D30, 30D35.
Keywords: Entire functions, order of growth, shared values, uniqueness theorems, small functions.

1 Introduction and Main Results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. We shall use the standard notations in Nevanlinna value distribution of meromorphic functions [6, 8, 13] such as $T(r, f)$, $N(r, f)$ and $m(r, f)$. For any nonconstant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying

$$\lim_{r\to+\infty} \frac{S(r,f)}{T(r,f)} = 0,$$

possibly outside a set of finite linear measure in $\mathbb{R}_+$. A meromorphic function a is said to be a small function of f, provided $T(r, a) = S(r, f)$. We say that two meromorphic functions f and g share a IM (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

In this paper we also need the following definitions.

Received June 29, 2012; Accepted January 17, 2013
Communicated by Delfim F. M. Torres

Definition 1.1 (See [6]). Let f be a nonconstant entire function. The order of f is defined as

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r},$$

where, and in the following, $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 1.2 (See [13]). Let f be a nonconstant entire function. The lower order of f is defined as

$$\mu(f) = \liminf_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}.$$

Definition 1.3 (See [13]). Let f be a nonconstant entire function. The hyper-order of f is defined as

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

Definition 1.4 (See [13]). Let f be a nonconstant entire function. The lower hyper-order of f is defined as

$$\mu_2(f) = \liminf_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \liminf_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

The uniqueness theory of entire and meromorphic functions has grown up to an extensive subfield of the value distribution theory, see monograph [13] by Yang and Yi. A widely studied subtopic of the uniqueness theory has been to consider shared value problems relative to a meromorphic function f and its derivatives $f^{(k)}$. Some of the basic papers in this direction are due to Rubel and Yang [10] (they proved that if an entire function f shared two distinct finite complex numbers CM with f', then $f \equiv f'$), Gundersen [4], Jank, Mues and Volkmann [7] and Yang [12]. A much investigated problem in this direction is the following conjecture proposed by Brück [2]:

Conjecture. Let f be a nonconstant entire function. Suppose that

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f' - a}{f - a} = c$$

for some nonzero constant c. The conjecture has been verified, if one of the following three assumptions is satisfied:

(i) f is of finite order, see [5];

(ii) $a = 0$, see [2];

(iii) $N\left(r, \dfrac{1}{f'}\right) = S(r, f)$, see [2].

However, the corresponding conjecture for meromorphic functions fails in general, as shown by Gundersen and Yang [5], while it remains true in the case of $N\left(r, \dfrac{1}{f'}\right) = S(r, f)$. Recently, Li and Yi posed the following question.

Question 1.1 What can be said when a nonconstant entire function f share one finite value a with a linear differential polynomial related to f?

Regarding Question 1.1, they proved the following theorem. Consider the following linear differential polynomial related to f:

$$L[f] = f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_1 f' + a_0 f, \tag{1.1}$$

where k is a positive integer, $a_{k-1}, \ldots, a_1$ and a_0 are k finite complex numbers.

Theorem 1.5 (See [9]). *Let $Q(z)$ be a nonconstant polynomial, and let $a\,(\neq 0)$ be a finite complex number. If f is a nonconstant solution of the differential equation*

$$L[f] - a = (f - a)\, e^{Q(z)},$$

where $L[f]$ is defined by (1.1)*, then one of the following cases will occur:*

(i) *If $\mu(f) > 1$, then $\mu(f) = \infty$ and $\mu_2(f) = \sigma_2(f) = \gamma_Q$, where γ_Q is the degree of $Q(z)$;*

(ii) *If $\mu(f) \leq 1$, then $\mu(f) = 1$ and $Q(z) = p_1 z + p_0$, where $p_1\,(\neq 0)$ and p_0 are two finite complex numbers, $a_0, a_1, \ldots, a_{k-1}$ are not all equal to zero.*

In this paper, we will prove the following theorem which improves Theorem 1.5.

Theorem 1.6. *Let $Q(z)$ be a nonconstant polynomial and let $a\,(\neq 0)$ be a finite complex number. If f is a nonconstant solution of the differential equation*

$$L^l[f] - a = \left(f^l - a\right) e^{Q(z)}, \tag{1.2}$$

where $L[f]$ is defined by (1.1) *and $l \geq 1$ is an integer, then one of the following cases will occur:*

(i) *If $\mu(f) > 1$, then $\mu(f) = \infty$ and we have also $\mu_2(f) = \sigma_2(f) = \gamma_Q$, where and in the following, γ_Q is the degree of $Q(z)$;*

(ii) *If $\mu(f) \leq 1$, then $\mu(f) = 1$ and $Q(z) = p_1 z + p_0$, where $p_1 (\neq 0)$ and p_0 are two finite complex numbers, $a_0, a_1, \ldots, a_{k-1}$ are not all equal to zero.*

Remark 1.7. If we take $l = 1$ in Theorem 1.6, then we obtain Theorem 1.5.

From Theorem 1.6, we get the following two corollaries.

Corollary 1.8. *Let $Q(z)$ be a polynomial, and let a be a nonzero complex number. If f is a nonconstant solution of the differential equation* (1.2) *such that $\mu_2(f)$ is not a positive integer, where $L[f]$ is defined by* (1.1), *then f and $L[f]$ assume one of the following two relations:*

(i) $L^l[f] - a = c\left(f^l - a\right)$, *where $c (\neq 0)$ is some finite complex number;*

(ii) $L^l[f] - a = \left(f^l - a\right) e^{b_1 z + b_0}$, *where $\mu(f) = 1$ and $b_1 (\neq 0)$, b_0 are two finite complex numbers, and $a_0, a_1, \ldots, a_{k-1}$ are not all equal to zero.*

Corollary 1.9. *Let f be a nonconstant entire function such that $\mu(f) < \infty$, and let a be a nonzero complex number. If $f^l - a$ and $L^l[f] - a$ share 0 CM, where $L[f]$ is defined by* (1.1), *then f and $L[f]$ assume one of the two relations (i) and (ii) of Corollary 1.8.*

Question 1.2 What can be said when a nonconstant entire function f share a small entire function a with their derivative $f^{(k)}$?

Regarding Question 1.2, Xiao and Li proved the following theorem.

Theorem 1.10 (See [11]). *Let $Q(z)$ be a nonconstant polynomial, and let $a(z)$ be a small entire function of f, such that $\sigma(a) < \gamma_Q$. If f is a nonconstant solution of the differential equation*

$$f^{(k)} - a = (f - a)\, e^{Q(z)},$$

then $\sigma_2(f) = \gamma_Q$, and f is an entire function of infinite order.

In this paper, we will prove the following theorem which improves Theorem 1.10.

Theorem 1.11. *Let $Q(z)$ be a nonconstant polynomial, and let $b_i(z)\,(i = 1, 2)$ be small entire functions of f, such that $\sigma(b_i) < 1$. If f is a nonconstant solution of the differential equation*

$$L^l[f] - b_1 = \left(f^l - b_2\right) e^{Q(z)}, \tag{1.3}$$

where $L[f]$ is defined by (1.1), *then one of the following cases will occur:*

(i) *If $\mu(f) > 1$, then $\mu(f) = \infty$ and we have also $\mu_2(f) = \sigma_2(f) = \gamma_Q$ where γ_Q is the degree of $Q(z)$;*

(ii) *If $\mu(f) \leq 1$, then $\mu(f) = 1$ and $Q(z) = p_1 z + p_0$, where $p_1 (\neq 0)$ and p_0 are two finite complex numbers, $a_0, a_1, \ldots, a_{k-1}$ are not all equal to zero.*

Remark 1.12. If we take $a_{k-1} = a_{k-2} = \cdots = a_0 = 0$, $b_1 = b_2 = a$ and $l = 1$ in Theorem 1.11, then we obtain Theorem 1.10.

2 Auxiliary Lemmas

In order to prove our theorems, we need the following lemmas.

Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be an entire function. We define by

$$\mu(r) = \max\{|a_n| r^n : n = 0, 1, 2, \ldots\}$$

the maximum term of f, and define by $\nu(r, f) = \max\{m : \mu(r) = |a_m| r^m\}$ the central index of f.

Lemma 2.1 (See [9]). *Let $f(z)$ be an entire function of infinite order, with the lower order $\mu(f)$ and the lower hyper-order $\mu_2(f)$. Then*

(i) $\mu(f) = \liminf_{r\to+\infty} \frac{\log \nu(r, f)}{\log r}$*;*

(ii) $\mu_2(f) = \liminf_{r\to+\infty} \frac{\log\log \nu(r, f)}{\log r}$*, where here, and in the following, $\nu(r, f)$ denotes the central index of $f(z)$.*

Lemma 2.2 (See [8]). *Let $g : (0, \infty) \to \mathbb{R}$, $h : (0, \infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \le h(r)$ outside of an exceptional set E of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(r^\alpha)$ for all $r > r_0$.*

Lemma 2.3 (See [8]). *If f is an entire function of order $\sigma(f)$, then*

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log \nu(r, f)}{\log r}.$$

Lemma 2.4 (See [3]). *If f is a transcendental entire function of hyper-order $\sigma_2(f)$, then*

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log \nu(r, f)}{\log r}.$$

Lemma 2.5 (See [6]). *Let f be a transcendental meromorphic function and $k \ge 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log(rT(r, f)))$$

outside of a possible exceptional set E of finite linear measure, and if f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 2.6 (See [1]). *Let $g : (0, \infty) \to \mathbb{R}$, $h : (0, \infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \le h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

3 Proofs of the Main Results

Proof of Theorem 1.6. Suppose that f is polynomial. Then, from (1.1) and (1.2), we see that there exists a nonzero constant c such that $e^{Q(z)} \equiv c$. We obtain a contradiction with Q a nonconstant polynomial. Next we suppose that f is a transcendental entire function. We discuss the following two cases.

Case 1. Suppose that

$$\liminf_{r\to+\infty}\frac{\log\nu(r,f)}{\log r}>1. \tag{3.1}$$

Then from (3.1) and (i) of Lemma 2.1 we get

$$\mu(f)=\liminf_{r\to+\infty}\frac{\log\nu(r,f)}{\log r}>1. \tag{3.2}$$

Because $Q(z)$ is a nonconstant polynomial, we have

$$Q(z)=b_nz^n+\cdots+b_1z+b_0, \tag{3.3}$$

where $b_n\,(\neq 0),\ldots,b_1$ and b_0 are complex numbers. It follows from (3.3) that

$$\lim_{|z|\to+\infty}\frac{|Q(z)|}{|b_nz^n|}=1.$$

From this, we see that there exists a sufficiently large positive number r_0, such that

$$\frac{|Q(z)|}{|b_nz^n|}>\frac{1}{e},\ \ |z|>r_0.$$

From this and (1.2), we deduce that

$$\begin{aligned}\log|b_n|+n\log|z|-1<\log|Q(z)|&=\log\left|\log e^{Q(z)}\right|\\ &\le\left|\log\log e^{Q(z)}\right|=\left|\log\log\frac{L^l[f]-a}{f^l-a}\right|\ \ (|z|>r_0).\end{aligned} \tag{3.4}$$

Knowing that f is a nonconstant entire function, we have

$$M(r,f)\to\infty,\ as\ r\to+\infty. \tag{3.5}$$

Let

$$M(r,f)=|f(z_r)|, \tag{3.6}$$

where $z_r=re^{i\theta(r)}$ and $\theta(r)\in[0,2\pi)$. From (3.6) and the Wiman–Valiron theory, we see that there exists a subset $F_j\subset(1,\infty)\,(1\le j\le k)$ with finite logarithmic measure, i.e., $\int\limits_{F_j} \frac{dt}{t} < \infty$, such that for some point $z_r = re^{i\theta(r)}$ ($\theta(r) \in [0, 2\pi)$) satisfying $|z_r| = r \notin F_j$ and $M(r, f) = |f(z_r)|$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = \left(\frac{\nu(r,f)}{z_r}\right)^j (1 + o(1)) \quad (1 \le j \le k, r \notin F_j, r \to +\infty). \tag{3.7}$$

Set

$$\frac{L^l[f] - a}{f^l - a} = \frac{\frac{L^l[f]}{f^l} - \frac{a}{f^l}}{1 - \frac{a}{f^l}}. \tag{3.8}$$

Since f is a transcendental entire function, a is a nonzero constant. From (3.5) and (3.6), we deduce that

$$\lim_{r\to+\infty} \frac{|a|}{|f^l(z_r)|} = \lim_{r\to+\infty} \frac{|a|}{M^l(r,f)} = 0. \tag{3.9}$$

From this and using (1.1), (3.5)–(3.8), we obtain

$$\left|\frac{L^l[f(z_r)] - a}{f(z_r) - a}\right| = \left(\frac{\nu(r,f)}{r}\right)^{lk} |1 + o(1)|^l, \quad \left(r \notin \bigcup_{j=1}^{k} F_j, r \to +\infty\right). \tag{3.10}$$

From (3.10), we have

$$\log\left|\frac{L^l[f(z_r)] - a}{f^l(z_r) - a}\right| = kl(\log\nu(r,f) - \log r) + O(1) \quad \left(r \notin \bigcup_{j=1}^{k} F_j, r \to +\infty\right); \tag{3.11}$$

from (3.4) we deduce that

$$\begin{aligned} \log|b_n| + n\log|z_r| - 1 &\le \left|\log\log\frac{L^l[f(z_r)] - a}{f^l(z_r) - a}\right| \\ &= \left|\log\left|\log\frac{L^l[f(z_r)] - a}{f^l(z_r) - a}\right| + i\arg\left(\log\frac{L^l[f(z_r)] - a}{f^l(z_r) - a}\right)\right| \\ &\le \left|\log\left|\log\frac{L^l[f(z_r)] - a}{f^l(z_r) - a}\right|\right| + 2\pi. \end{aligned} \tag{3.12}$$

From (3.11), (3.12), Lemma 2.2 and $|z_r| = r$, we see that for any $\beta > 1$, there exists a sufficiently large positive number r_0, such that

$$\log|b_n| + n\log r - 1 \le \log\log\nu\left(r^\beta, f\right) + \log\log r^\beta + O(1), \; r > r_0. \tag{3.13}$$

From (3.13) and Lemma 2.4, we deduce

$$\frac{n}{\beta} \le \limsup_{r^\beta\to+\infty} \frac{\log\log\nu\left(r^\beta, f\right)}{\log r^\beta} = \limsup_{r\to+\infty} \frac{\log\log\nu(r,f)}{\log r} = \sigma_2(f).$$

By letting $\beta \to 1^{+}$, we have
$$n \leq \sigma_2 (f) . \tag{3.14}$$
In the same manner as above and by (ii) of Lemma 2.1, we get
$$n \leq \mu_2 (f) . \tag{3.15}$$
It is well-known that
$$\sigma \left(e^{Q}\right) = \gamma_Q = n. \tag{3.16}$$
From (3.14) and (3.16), we get
$$\sigma \left(e^{Q}\right) \leq \sigma_2 (f) . \tag{3.17}$$
On the other hand, from (1.2), (3.7) and (3.8) we have
$$\left(\frac{\nu (r, f)}{z_r}\right)^{kl} (1 + o (1))^{l} = e^{Q(z_r)},\ r \notin \bigcup_{j=1}^{k} F_j,\ r \to +\infty.$$
From this, we obtain
$$\nu^{lk} (r, f) \leq C r^{kl} M \left(r, e^{Q}\right),\ r \notin \bigcup_{j=1}^{k} F_j,\ r \to +\infty, \tag{3.18}$$
where $C > 0$ is a constant. From (3.18) and Lemma 2.2, we see that for any $\beta > 1$, there exists a sufficiently large positive number r_0, such that
$$\nu^{lk} (r, f) \leq C r^{lk\beta} M \left(r^{\beta}, e^{Q}\right), r > r_0. \tag{3.19}$$
From (3.19), Lemma 2.4 and Definition 1.3, we get
$$\begin{aligned}\sigma_2 (f) &= \limsup_{r\to+\infty}\frac{\log\log \nu (r, f)}{\log r} = \limsup_{r\to+\infty}\frac{\log\log \nu^{kl} (r, f)}{\log r}\\ &\leq \beta\limsup_{r\to+\infty}\frac{\log\log \left(C r^{kl\beta} M \left(r^{\beta}, e^{Q}\right)\right)}{\log r^{\beta}}\\ &= \beta\limsup_{r\to+\infty}\frac{\log\log M \left(r, e^{Q}\right)}{\log r} = \beta\sigma \left(e^{Q}\right),\end{aligned}$$
namely
$$\sigma_2 (f) \leq \beta\sigma \left(e^{Q}\right) . \tag{3.20}$$
By letting $\beta \to 1^{+}$ on both sides of (3.20) we have
$$\sigma_2 (f) \leq \sigma \left(e^{Q}\right) . \tag{3.21}$$
From (3.16), (3.17) and (3.21), we deduce
$$\sigma_2 (f) = \sigma \left(e^{Q}\right) = n. \tag{3.22}$$

Noting that $\mu_2(f) \leq \sigma_2(f)$, from (3.2), (3.15), (3.16) and (3.22), then we get

$$\sigma_2(f) = \mu_2(f) = n = \gamma_Q. \tag{3.23}$$

If $\mu(f) < +\infty$, then it follows from (3.23) that $\mu_2(f) = \gamma_Q = 0$. Thus Q is a constant: a contradiction since Q is a nonconstant polynomial. Hence, $\mu(f) = +\infty$.

Case 2. Suppose that

$$\liminf_{r\to+\infty} \frac{\log \nu(r,f)}{\log r} \leq 1. \tag{3.24}$$

From (3.24) and (i) of Lemma 2.1, we have

$$\mu(f) = \liminf_{r\to+\infty} \frac{\log \nu(r,f)}{\log r} \leq 1. \tag{3.25}$$

From (1.1), (1.2) and Lemma 2.5, we deduce

$$T\left(r, e^Q\right) \leq O\left(T(r,f)\right) + O\left(\log rT(r,f)\right),\ r \notin E.$$

From this and Lemma 2.6, we see that for r a sufficiently large positive number

$$T\left(r, e^Q\right) \leq O\left(T(2r,f)\right) + O\left(\log 2r + \log T(2r,f)\right),\ r > r_0.$$

From this and (3.3), we deduce

$$1 \leq n = \gamma_Q = \sigma\left(e^Q\right) = \mu\left(e^Q\right) \leq \mu(f).$$

It follows from (3.25) that $n = \gamma_Q = \mu(f) = 1$ and $Q(z) = p_1 z + p_0$. If $a_j = 0$ $(0 \leq j \leq k-1)$, then (1.2) can be rewritten as $\left(f^{(k)}\right)^l - a = \left(f^l - a\right) e^{p_1 z + p_0}$. Using the same reasoning as in the Case 1, we get (3.15) and so $\mu_2(f) \geq 1$, which contradicts $\mu(f) = 1$. Thus, $a_0, a_1, \ldots, a_{k-1}$ are not all equal to zero. Theorem 1.6 is thus completely proved. □

Proof of Theorem 1.11. Noting that b_1 and b_2 are small entire functions of f, from (1.3) and Lemma 2.5 we deduce

$$T\left(r, e^Q\right) \leq O\left(T(r,f)\right) + O\left(\log rT(r,f)\right), r \notin E. \tag{3.26}$$

From (3.26) and Lemma 2.6, we see that there exists a sufficiently large positive number r_0 such that

$$T\left(r, e^Q\right) \leq O\left(T(2r,f)\right) + O\left(\log 2r + \log T(2r,f)\right), r > r_0. \tag{3.27}$$

From (3.27), we deduce $\mu\left(e^Q\right) \leq \mu(f)$. Combining $\mu\left(e^Q\right) = \sigma\left(e^Q\right) = \gamma_Q \geq 1$ and $\sigma(b_i) < 1\,(i = 1,2)$, we get

$$\mu(f) > \sigma(b_i), \quad (i = 1,2). \tag{3.28}$$

Combining (3.28) and the fact that f is a transcendental entire function, it follows from Definition 1.1 and Definition 1.2 that

$$\frac{b_i\left(z_r\right)}{f\left(z_r\right)} \to 0, \quad (i=1,2)$$

as $\left|z_r\right| \to \infty$, and where $z_r = re^{i\theta(r)}$ for $\theta\left(r\right) \in [0, 2\pi)$. Proceeding as in the proof of Theorem 1.6, we obtain Theorem 1.11. □

Proof of Corollary 1.8. If $Q\left(z\right)$ is a constant, then the conclusion (i) of Corollary 1.8 is valid. If we suppose that $Q\left(z\right)$ is not a constant, then it follows from (1.2) that f is a transcendental entire function, and so conclusions (i) and (ii) of Theorem 1.6 hold. If (ii) of Theorem 1.6 holds, we get (ii) of Corollary 1.8. If (i) of Theorem 1.6 holds, then $\mu\left(f\right) = \infty$ and $\mu_2\left(f\right) = \sigma_2\left(f\right) = \gamma_Q$. Combining (1.2) and the condition that $\mu_2\left(f\right)$ is not a positive integer, we have $\gamma_Q = 0$ and so there exists some finite nonzero complex number c such that $L^l\left[f\right] - a = c\left(f^l - a\right)$. From (3.10), we have

$$\left(\frac{\nu\left(r,f\right)}{r}\right)^{kl} \left|1+o\left(1\right)\right|^l = \left|c\right|, \quad \left(r \notin \bigcup_{j=1}^{k} F_j, r \to +\infty\right)$$

and so

$$\frac{\nu\left(r,f\right)}{r} = O\left(1\right), \quad \left(r \notin \bigcup_{j=1}^{k} F_j, r \to +\infty\right). \tag{3.29}$$

From (3.29) and Lemma 2.1, we deduce $\mu\left(f\right) = 1$ which is impossible. Corollary 1.8 is thus completely proved. □

Proof of Corollary 1.9. From the condition that $f^l - a$ and $L^l\left[f\right] - a$ share 0 CM, we have (1.2). From (1.1), (1.2) and Lemma 2.5, we deduce

$$T\left(r, e^Q\right) \leq O\left(T\left(r,f\right)\right) + O\left(\log rT\left(r,f\right)\right), r \notin E.$$

From this and Lemma 2.6 and the condition $\mu\left(f\right) < \infty$, we deduce

$$\sigma\left(e^Q\right) = \mu\left(e^Q\right) \leq \mu\left(f\right) < \infty.$$

From this, we see that $Q\left(z\right)$ is a polynomial. If $Q\left(z\right)$ is a constant, then the conclusion (i) of Corollary 1.8 is valid. If we suppose that $Q\left(z\right)$ is not a constant, then it follows from (1.2) that f is a transcendental entire function, and so conclusions (i) and (ii) of Theorem 1.6 hold. If (ii) of Theorem 1.6 holds, then we get (ii) of Corollary 1.8. If (i) of Theorem 1.6 holds, then we have $\mu\left(f\right) = \infty$. This contradicts the supposition $\mu\left(f\right) < \infty$. Corollary 1.9 is thus completely proved. □

Remark 3.1. In Corollary 1.9 the result remains true if we take $\mu_2\left(f\right) < \frac{1}{2}$ with $\mu\left(f\right) = \infty$. Indeed, from condition (i) of Theorem 1.6 we have $\mu_2\left(f\right) = \gamma_Q$. Since $\mu_2\left(f\right) < \frac{1}{2}$ and γ_Q is a positive integer, we obtain that $\gamma_Q = 0$, and so Q is a constant.

Acknowledgements

The authors are very grateful to the anonymous referee for his/her helpful remarks and suggestions to improve the paper.

References

[1] S. Bank, *A general theorem concerning the growth of solutions of first-order algebraic differential equations*, Compositio Math. 25 (1972), 61–70.

[2] R. Brück, *On entire functions which share one value CM with their first derivative*, Results Math. 30 (1996), no. 1-2, 21–24.

[3] Z. X. Chen and C. C. Yang, *Some further results on the zeros and growths of entire solutions of second order linear differential equations*, Kodai Math. J. 22 (1999), no. 2, 273–285.

[4] G. Gundersen, *Meromorphic functions that share two finite values with their derivative*, Pacific J. Math. 105 (1983), no. 2, 299–309.

[5] G. Gundersen and L. Z. Yang, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl. 223 (1998), no. 1, 88–95.

[6] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs Clarendon Press, Oxford 1964.

[7] G. Jank, E. Mues and L.Volkmann, *Meromorphe Funktionen, die mit ihrer ersten und zweiten Ableitung einen endlichen Wert teilen*, (German) [Meromorphic functions which share a finite value with their first and second derivative] Complex Variables Theory Appl. 6 (1986), no. 1, 51–71.

[8] I. Laine, *Nevanlinna theory and complex differential equations*, de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.

[9] X. M. Li and H. X. Yi, *Some results on the regular solutions of a linear differential equation*, Comput. Math. Appl. 56 (2008), no. 9, 2210–2221.

[10] L. A. Rubel and C. C. Yang, *Values shared by an entire function and its derivative*, Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), pp. 101–103. Lecture Notes in Math., Vol. 599, Springer, Berlin, 1977.

[11] Y. H. Xiao and X. M. Li, *An entire function sharing one small function with its derivative*, Applied Mathematics E-Notes, 8 (2008), 238–245.

[12] L. Z. Yang, *Solution of a differential equation and its applications*, Kodai Math. J. 22 (1999), no. 3, 458–464.

[13] C. C. Yang and X. X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.