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Eigenvalue Problems for Systems of Nonlinear Boundary Value Problems on Time Scales

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Abstract

Values of λ are determined for which there exist positive solutions of the system of nth order dynamic equations on time scales satisfying two-point boundary conditions. A Guo–Krasnosel'skii fixed point theorem is applied.

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1 Introduction

Let $\mathbb{T}$ be a time scale with $a, \sigma^n(b) \in \mathbb{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation

$$J_{\mathbb{T}} = J \cap \mathbb{T}. \tag{1.1}$$

We are concerned with determining values of λ (eigenvalues) for which there exist positive solutions for the system of dynamic equations

$$\begin{aligned} u^{\Delta^{(n)}}(t) + \lambda p(t) f(v(\sigma(t))) = 0, \;\; t \in [a,b]_{\mathbb{T}}, \\ v^{\Delta^{(n)}}(t) + \lambda q(t) g(u(\sigma(t))) = 0, \;\; t \in [a,b]_{\mathbb{T}}, \end{aligned} \tag{1.2}$$

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satisfying the boundary conditions

$$\begin{aligned} &u^{\Delta^{(i)}}(a) = 0, \ \ 0 \le i \le n-2, \ \ u(\sigma^n(b)) = 0, \\ &v^{\Delta^{(j)}}(a) = 0, \ \ 0 \le j \le n-2, \ \ v(\sigma^n(b)) = 0, \end{aligned} \tag{1.3}$$

where

(A1) $f, g \in C([0,\infty),[0,\infty))$,

(A2) $p, q \in C([a,\sigma(b)]_{\mathbb{T}}, [0,\infty))$, and each does not vanish identically on any closed subinterval of $[a,\sigma(b)]_{\mathbb{T}}$,

(A3) All of $f_0 := \lim\limits_{x\to 0^+} (f(x)/x)$, $g_0 := \lim\limits_{x\to 0^+} (g(x)/x)$, $f_\infty := \lim\limits_{x\to\infty} (f(x)/x)$ and $g_\infty := \lim\limits_{x\to\infty} (g(x)/x)$ exist as positive real numbers.

The theory of dynamic equations on time scales (more generally, on measure chains) was introduced in 1988 by Stefan Hilger in his PhD thesis (see [18, 19]). The theory presents a structure where, once a result is established for a general time scale, then special cases can be obtained by taking the particular time scale. If $\mathbb{T} = \mathbb{R}$, then we have the result for differential equations. Choosing $\mathbb{T} = \mathbb{Z}$ we immediately get the result for difference equations. A great deal of work has been done since 1988, unifying and extending the theories of differential and difference equations, and many results are now available in the general setting of time scales and references therein.

On a larger scale, there has been a great deal of study focused on positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from a theoretical sense [11, 12, 15, 20, 31] and as applications for which only positive solutions are meaningful [1, 13, 21, 22]. These considerations are cast primarily for scalar problems, but much attention has been given to boundary value problems for systems of differential equations [16, 17, 27, 30, 32].

The main tool in this paper is an application of the Guo–Krasnosel'skii fixed point theorem for operators leaving a Banach space cone invariant [12]. A Green function plays a fundamental role in defining an appropriate operator on a suitable cone.

2 Preliminaries

By an interval we mean the intersection of the real interval with a given time scale. The jump operators introduced on a time scale $\mathbb{T}$ may be connected or disconnected. To overcome this topological difficulty, the concept of jump operators is introduced in the following way. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$, defined by $\sigma(t)$=Inf $\{s \in \mathbb{T} : s > t\}$ and $\rho(t)$=Sup $\{s \in \mathbb{T} : s < t\}$ are called jump operators. If σ is bounded above and ρ is bounded below, then we define σ(Max $\mathbb{T}$)=Max $\mathbb{T}$ and ρ(Min $\mathbb{T}$)=Min $\mathbb{T}$. These operators allow us to classify the points of time scale $\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, left-dense if $\rho(t) = t$, right-scattered if $\sigma(t) > t$, left-scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$ and dense if $\rho(t) = t = \sigma(t)$. The set $\mathbb{T}^k$ which is derived from the time scale $\mathbb{T}$ as follows

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Finally, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 2.1. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there exists a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t.

If f is delta differentiable for every $t \in \mathbb{T}^\kappa$ we say that $f : \mathbb{T}^\kappa \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$. If f and g are two delta differentiable functions at t, then fg is delta differentiable at t and $(fg)^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t)$.

Definition 2.2. A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.3. Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F which is pre-differentiable with region of differentiation D such that $F^\Delta(t) = f(t)$ holds for all $t \in D$ is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is pre-antiderivative of f.

Definition 2.4. Let β be a real Banach space. A nonempty closed convex set κ is called cone of β if it satisfies the following conditions:
(1). $u \in \beta$, $\sigma \ge 0$, implies $\sigma u \in \kappa$,
(2). $u \in \kappa$, $-u \in \kappa$ implies $u = 0$.

Definition 2.5. Let X and Y be Banach spaces and $T : X \to Y$. T is said to be completely continuous, if T is continuous, and for each bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a convergent subsequence.

3 Green's Function and Bounds

In this section, we state the well-known Guo–Krasnosel'skii fixed-point theorem which we will apply to a completely continuous operator whose kernel, $G(t,s)$, is the Green's function for

$$-y^{\Delta^{(n)}} = 0, \tag{3.1}$$

$$y^{\Delta^{(i)}}(a) = 0, \ \ 0 \le i \le n-2, \quad y(\sigma^n(b)) = 0. \tag{3.2}$$

Theorem 3.1. *The Green's function for the BVP* (3.1), (3.2) *is given by*

$$G(t,s) = \frac{1}{(n-1)!}\begin{cases} \displaystyle\prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))}, & t \le s, \\ \displaystyle\prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))} - \prod_{i=1}^{n-1}(t-\sigma^i(s)), & \sigma(s) \le t. \end{cases}$$

Proof. It is easy to check that the BVP (3.1), (3.2) has only trivial solution. Let $y(t,s)$ be the Cauchy function for $-y^{\Delta^{(n)}} = 0$, and be given by

$$y(t,s) = \frac{-1}{(n-1)!} \underbrace{\int_{\sigma(s)}^t \int_{\sigma^2(s)}^t \cdots \int_{\sigma^{n-1}(s)}^t}_{(n-1) \text{ times}} \Delta\tau\Delta\tau\cdots\Delta\tau = \frac{-1}{(n-1)!}\prod_{i=1}^{n-1}(t-\sigma^i(s)).$$

For each fixed $s \in [a,b]$, let $u(\cdot, s)$ be the unique solution of the BVP

$$-u^{\Delta^{(n)}}(\cdot, s) = 0,$$

$$u^{\Delta^{(i)}}(a,s) = 0, \ \ 0 \le i \le n-2 \ \text{ and } \ u(\sigma^n(b), s) = -y(\sigma^n(b), s).$$

$$y(t,s)\mid_{t=\sigma^n(b)} = \frac{-1}{(n-1)!}\prod_{i=1}^{n-1}(\sigma^n(b)-\sigma^i(s)).$$

Since

$$u_1(t) = 1, u_2(t) = \int_a^t \Delta\tau, \cdots, \ u_n(t) = \underbrace{\int_a^t \int_{\sigma(a)}^t \cdots \int_{\sigma^{n-2}(a)}^t}_{(n-1) \text{ times}} \Delta\tau\Delta\tau\cdots\Delta\tau$$

are the solutions of $-u^{\Delta^{(n)}} = 0$,

$$u(t,s) = \alpha_1(s)\cdot 1 + \alpha_2(s)\cdot \int_a^t \Delta\tau + \cdots + \alpha_n(s)\cdot \underbrace{\int_a^t \int_{\sigma(a)}^t \cdots \int_{\sigma^{n-2}(a)}^t}_{(n-1) \text{ times}} \Delta\tau\Delta\tau\cdots\Delta\tau.$$

By using boundary conditions, $u^{\Delta^{(i)}}(a) = 0,\ \ 0 \le i \le n-2$, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$. Therefore, we have

$$u(t,s) = \alpha_n \underbrace{\int_a^t \int_{\sigma(a)}^t \cdots \int_{\sigma^{n-2}(a)}^t}_{(n-1)\ \text{times}} \Delta\tau\Delta\tau\cdots\Delta\tau = \alpha_n \prod_{i=1}^{n-1}(t - \sigma^{i-1}(a)).$$

Since

$$u(\sigma^n(b), s) = -y(\sigma^n(b), s),$$

it follows that

$$\alpha_n \prod_{i=1}^{n-1}(\sigma^n(b) - \sigma^{i-1}(a)) = \frac{1}{(n-1)!}\prod_{i=1}^{n-1}(\sigma^n(b) - \sigma^i(s)).$$

Thus

$$\alpha_n = \frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))}.$$

Hence $G(t,s)$ has the form for $t \le s$,

$$G(t,s) = \frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(t - \sigma^{i-1}(a))(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))}.$$

For $t \ge \sigma(s)$, $G_n(t,s) = y(t,s) + u(t,s)$. It follows that

$$G(t,s) = \frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(t - \sigma^{i-1}(a))(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))} - \frac{1}{(n-1)!}\prod_{i=1}^{n-1}(t - \sigma^i(s)).$$

This completes the proof. □

Lemma 3.2. *For* $(t,s) \in [a, \sigma^n(b)]_{\mathbb{T}} \times [a, \sigma(b)]_{\mathbb{T}}$*, we have*

$$G(t,s) \le G(\sigma(s), s). \tag{3.3}$$

Proof. For $a \le t \le s \le \sigma^n(b)$, we have

$$\begin{aligned} G(t,s) &= \frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(t - \sigma^{i-1}(a))(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))} \\ &\le \frac{1}{(n-1)!}\prod_{i=1}^{n-1}\frac{(\sigma(s) - \sigma^{i-1}(a))(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))} \\ &= G(\sigma(s), s). \end{aligned}$$

Similarly, for $a \le \sigma(s) \le t \le \sigma^n(b)$, we have $G(t,s) \le G(\sigma(s), s)$. Thus, we have

$$G(t,s) \le G(\sigma(s), s), \quad \text{for all } \ (t,s) \in [a, \sigma^n(b)]_{\mathbb{T}} \times [a, \sigma(b)]_{\mathbb{T}},$$

completing the proof. □

Lemma 3.3. *Let* $I = \left[\frac{\sigma^n(b)+3a}{4}, \frac{3\sigma^n(b)+a}{4}\right]_{\mathbb{T}}$. *For* $(t,s) \in I \times [a, \sigma(b)]_{\mathbb{T}}$, *we have*

$$G(t,s) \geq \frac{1}{16^{n-1}} G(\sigma(s), s). \tag{3.4}$$

Proof. The Green's function for the BVP (3.1), (3.2) as given in the Theorem 3.1, shows that

$$G(t,s) > 0 \text{ on } (a, \sigma^n(b))_{\mathbb{T}} \times (a, \sigma(b))_{\mathbb{T}}. \tag{3.5}$$

For $a \leq t \leq s \leq \sigma^n(b)$ and $t \in I$, we have

$$\begin{aligned}\frac{G(t,s)}{G(\sigma(s),s)} &= \prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma(s)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))} \\ &\geq \prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))}{(\sigma^n(b)-a)} \\ &\geq \frac{1}{4^{n-1}}.\end{aligned}$$

For $a \leq \sigma(s) \leq t \leq \sigma^n(b)$ and $t \in I$, we have

$$\begin{aligned}&\frac{G(t,s)}{G(\sigma(s),s)} \\ &= \frac{\prod_{i=1}^{n-1}(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s)) - \prod_{i=1}^{n-1}(t-\sigma^i(s))(\sigma^n(b)-\sigma^i(a))}{\prod_{i=1}^{n-1}(\sigma(s)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))} \\ &\geq \frac{\prod_{i=1}^{n-1}(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s)) - \prod_{i=1}^{n-1}(t-\sigma^i(s))(\sigma^n(b)-\sigma^i(a))}{\prod_{i=1}^{n-1}(\sigma^n(b)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))} \\ &\geq \frac{[(\sigma(s)-a)(\sigma^2(b)-t)]\prod_{i=2}^{n-1}(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{\prod_{i=1}^{n-1}(\sigma^n(b)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(a))} \\ &\geq \frac{1}{16^{n-1}}.\end{aligned}$$

Therefore

$$\frac{1}{16^{n-1}} G(\sigma(s),s) \leq G(t,s).$$

This completes the proof. □

We note that a pair $(u(t), v(t))$ is a solution of the eigenvalue problem (1.2), (1.3) if and only if

$$u(t) = \lambda \int_a^{\sigma(b)} G(t,s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s, \quad a \leq t \leq \sigma^n(b), \tag{3.6}$$

$$v(t) = \lambda \int_a^{\sigma(b)} G(t,s)q(s)g(u(\sigma(s)))\Delta s, \quad a \leq t \leq \sigma^n(b).$$

Values of λ for which there are positive solutions (positive with respect to a cone) of (1.2), (1.3) will be determined via applications of the following fixed-point theorem [24].

Theorem 3.4 (Krasnosel'skii)**.** *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P} \tag{3.7}$$

be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2,$ *or*

(ii) $\|Tu\| \geq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \leq \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2.$

Then, T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

4 Positive Solutions in a Cone

In this section, we apply Theorem 3.4 to obtain solutions in a cone (i.e., positive solutions) of (1.2), (1.3). Assume throughout that $[a, \sigma^n(b)]_{\mathbb{T}}$ is such that

$$\begin{aligned} \xi &= \min\left\{t \in \mathbb{T} : t \geq \frac{3a + \sigma^n(b)}{4}\right\}, \\ \omega &= \max\left\{t \in \mathbb{T} : t \leq \frac{a + 3\sigma^n(b)}{4}\right\}; \end{aligned} \tag{4.1}$$

both exist and satisfy

$$\frac{3a + \sigma^n(b)}{4} \leq \xi < \omega \leq \frac{a + 3\sigma^n(b)}{4}. \tag{4.2}$$

Next, let $\tau \in [\xi, \omega]_{\mathbb{T}}$ be defied by

$$\int_\xi^\omega G(\tau, s)p(s)\Delta s = \max_{t \in [\xi,\omega]_{\mathbb{T}}} \int_\xi^\omega G(t, s)p(s)\Delta s. \tag{4.3}$$

Finally, we define

$$l = \min_{s \in [a,\sigma(b)]_{\mathbb{T}}} \frac{G(\sigma(\omega), s)}{G(\sigma(s), s)}, \tag{4.4}$$

and let

$$\gamma = \min\left\{\frac{1}{16^{n-1}}, l\right\}. \tag{4.5}$$

For our construction, let $\mathcal{B} = \{x : [a, \sigma^n(b)]_{\mathbb{T}} \to \mathbb{R}\}$ with supremum norm $\|x\| = \sup\{|x(t)| : t \in [a, \sigma^n(b)]_{\mathbb{T}}\}$ and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \Big\{x \in \mathcal{B} \mid x(t) \geq 0, \text{ on } [a, \sigma^n(b)]_{\mathbb{T}}, \text{ and } x(t) \geq \gamma\|x\|, \text{ for } t \in [\xi, \omega]_{\mathbb{T}}\Big\}. \quad (4.6)$$

For our first result, define positive numbers L_1 and L_2 by

$$L_1 := \max\left\{\Big[\gamma \int_\xi^\omega G(\tau, s)p(s)\Delta s f_\infty\Big]^{-1}, \Big[\gamma \int_\xi^\omega G(\tau, s)q(s)\Delta s g_\infty\Big]^{-1}\right\},$$

$$L_2 := \min\left\{\Big[\int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s f_0\Big]^{-1}, \Big[\int_a^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s g_0\Big]^{-1}\right\}.$$

Theorem 4.1. *Assume that conditions* (A1)–(A3) *are satisfied. Then, for each* λ *satisfying*

$$L_1 < \lambda < L_2, \quad (4.7)$$

there exists a pair (u, v) *satisfying* (1.2), (1.3) *such that* $u(x) > 0$ *and* $v(x) > 0$ *on* $(a, \sigma^n(b))_{\mathbb{T}}$.

Proof. Let λ be as in (4.7), and let $\epsilon > 0$ be chosen such that

$$\max\left\{\Big[\gamma \int_\xi^\omega G(\tau, s)p(s)\Delta s(f_\infty - \epsilon)\Big]^{-1}, \Big[\gamma \int_\xi^\omega G(\tau, s)q(s)\Delta s(g_\infty - \epsilon)\Big]^{-1}\right\} \leq \lambda,$$

$$\lambda \leq \min\left\{\Big[\int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s(f_0 + \epsilon)\Big]^{-1}, \Big[\int_a^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s(g_0 + \epsilon)\Big]^{-1}\right\}.$$

Define an integral operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) = \lambda \int_a^{\sigma(b)} G(t, s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s, \; u \in \mathcal{P}. \quad (4.8)$$

By the remarks in Section 3, we seek suitable fixed points of T in the cone $\mathcal{P}$.

Notice from (A1), (A2) and (3.5) that, for $u \in \mathcal{P}$, $Tu(t) \geq 0$ on $[a, \sigma^n(b)]_{\mathbb{T}}$. Also, for $u \in \mathcal{P}$, we have from (3.3) that

$$\begin{aligned} Tu(t) &= \lambda \int_a^{\sigma(b)} G(t, s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s \\ &\leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s \end{aligned}$$

so that

$$\|Tu\| \le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s.$$

Next, if $u \in \mathcal{P}$, we have from (3.4), (4.5), and (4.8) that

$$\begin{aligned}
&\min_{t\in[\xi,\omega]_{\mathbb{T}}} Tu(t)\\
&= \min_{t\in[\xi,\omega]_{\mathbb{T}}} \lambda \int_a^{\sigma(b)} G(t, s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s\\
&\ge \lambda\gamma \int_a^{\sigma(b)} G(\sigma(s), s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s\\
&\ge \gamma\|Tu\|.
\end{aligned}$$

Consequently, $T : \mathcal{P} \to \mathcal{P}$. Moreover, T is completely continuous by a typical application of the Ascoli–Arzela Theorem.

Now, from the definitions of f_0 and g_0, there exists $H_1 > 0$ such that

$$f(x) \le (f_0 + \epsilon)x \text{ and } g(x) \le (g_0 + \epsilon)x, \;\; 0 < x \le H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (3.3) and choice of ϵ, for $a \le s \le \sigma(b)$, that

$$\begin{aligned}
\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r))\Delta r &\le \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)g(u(\sigma(r)))\Delta r\\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)(g_0 + \epsilon)u(r)\Delta r\\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)\Delta r(g_0 + \epsilon)\|u\|\\
&\le \|u\| = H_1.
\end{aligned}$$

As a consequence, we next have from (3.4) and choice of ϵ, for $a \le t \le \sigma^n(b)$, that

$$\begin{aligned}
Tu(t) &= \lambda \int_a^{\sigma(b)} G(t, s)p(s)f\Big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Big)\Delta s\\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_0 + \epsilon)\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Delta s\\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_0 + \epsilon)H_1\Delta s\\
&\le H_1 = \|u\|.
\end{aligned}$$

So, $\|Tu\| \le \|u\|$. If we set

$$\Omega_1 = \{x \in \mathcal{B} : \|x\| < H_1\},$$

then

$$\|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{4.9}$$

Next, from the definitions of f_∞ and g_∞, there exists $\overline{H}_2 > 0$ such that

$$f(x) \geq (f_\infty - \epsilon)x \text{ and } g(x) \geq (g_\infty - \epsilon)x, \quad x \geq \overline{H}_2.$$

Let

$$H_2 = \max\Big\{2H_1, \frac{\overline{H}_2}{\gamma}\Big\}.$$

Let $u \in \mathcal{P}$ and $\|u\| = H_2$. Then,

$$\min_{t\in[\xi,\omega]_{\mathbb{T}}} u(t) \geq \gamma\|u\| \geq \overline{H}_2.$$

Consequently, from (3.4) and choice of ϵ, for $a \leq s \leq \sigma(b)$, we have that

$$\begin{aligned}
\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r &\geq \lambda\int_\xi^{\omega} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r \\
&\geq \lambda\int_\xi^{\omega} G(\tau,r)q(r)g(u(\sigma(r)))\Delta r \\
&\geq \lambda\int_\xi^{\omega} G(\tau,r)q(r)(g_\infty-\epsilon)u(r)\Delta r \\
&\geq \gamma\lambda\int_\xi^{\omega} G(\tau,r)q(r)(g_\infty-\epsilon)\Delta r\|u\| \\
&\geq \|u\| = H_2.
\end{aligned}$$

So, we have from (3.4) and choice of ϵ that

$$\begin{aligned}
Tu(\tau) &= \lambda\int_a^{\sigma(b)} G(\tau,s)p(s)f\big(\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\big)\Delta s \\
&\geq \lambda\int_a^{\sigma(b)} G(\tau,s)p(s)(f_\infty-\epsilon)\lambda\int_a^{\sigma(b)} G(\sigma(s),r)q(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\geq \lambda\int_a^{\sigma(b)} G(\tau,s)p(s)(f_\infty-\epsilon)H_2\Delta s \\
&\geq \gamma H_2 > H_2 = \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \geq \|u\|$. So if we set

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_2\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{4.10}$$

Applying Theorem 3.4 to (4.9) and (4.10), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. As such, and with v being defined by

$$v(t) = \lambda \int_a^{\sigma(b)} G(t,s)q(s)g(u(\sigma(s)))\Delta s,$$

(u, v) is a desired solution of (1.2), (1.3) for the given λ. The proof is complete. □

Prior to our next result, we introduce another hypothesis.

(A4) $g(0) = 0$, and f is an increasing function.

We now define positive numbers L_3 and L_4 by

$$L_3 := \max\Big\{\Big[\gamma \int_\xi^\omega G(\tau,s)p(s)\Delta s f_0\Big]^{-1}, \Big[\gamma \int_\xi^\omega G(\tau,s)q(s)\Delta s g_0\Big]^{-1}\Big\},$$
$$L_4 := \min\Big\{\Big[\int_a^{\sigma(b)} G(\sigma(s),s)p(s)\Delta s f_\infty\Big]^{-1}, \Big[\int_a^{\sigma(b)} G(\sigma(s),s)q(s)\Delta s g_\infty\Big]^{-1}\Big\}.$$

Theorem 4.2. *Assume that conditions* (A1)–(A4) *are satisfied. Then, for each* λ *satisfying*

$$L_3 < \lambda < L_4, \tag{4.11}$$

there exists a pair (u, v) *satisfying* (1.2), (1.3) *such that* $u(x) > 0$ *and* $v(x) > 0$ *on* $(a, \sigma^n(b))_{\mathbb{T}}$.

Proof. Let λ be as in (4.11), and let $\epsilon > 0$ be chosen such that

$$\max\Big\{\Big[\gamma \int_\xi^\omega G(\tau,s)p(s)\Delta s(f_0-\epsilon)\Big]^{-1}, \Big[\gamma \int_\xi^\omega G(\tau,s)q(s)\Delta s(g_0-\epsilon)\Big]^{-1}\Big\} \le \lambda,$$
$$\lambda \le \min\Big\{\Big[\int_a^{\sigma(b)} G(\sigma(s),s)p(s)\Delta s(f_\infty+\epsilon)\Big]^{-1},$$
$$\Big[\int_a^{\sigma(b)} G(\sigma(s),s)q(s)\Delta s(g_\infty+\epsilon)\Big]^{-1}\Big\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (4.8). From the definitions of f_0 and g_0, there exists $H_1 > 0$ such that

$$f(x) \ge (f_0 - \epsilon)x \text{ and } g(x) \ge (g_0 - \epsilon)x, \;\; 0 < x \le H_1.$$

Now, $g(0) = 0$, and so there exists $0 < H_2 < H_1$ such that

$$\lambda g(x) \le \frac{H_1}{\int_a^{\sigma(b)} G(\sigma(s),s)q(s)\Delta s}, \;\; 0 \le x \le H_2.$$

Choose $u \in \mathcal{P}$ with $\|u\| = H_2$. Then, for $a \le s \le \sigma(b)$, we have

$$\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \le \frac{\int_a^{\sigma(b)} G(\sigma(s), r)q(r)H_1\Delta r}{\int_a^{\sigma(b)} G(\sigma(s), s)q(s)\Delta s} \le H_1.$$

Then,

$$\begin{aligned}
Tu(\tau) &= \lambda \int_a^{\sigma(b)} G(\tau, s)p(s)f\big(\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\big)\Delta s \\
&\ge \lambda \int_\xi^{\omega} G(\tau, s)p(s)(f_0 - \epsilon)\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\ge \lambda \int_\xi^{\omega} G(\tau, s)p(s)(f_0 - \epsilon)\lambda \int_\xi^{\omega} G(\tau, r)q(r)g(u(\sigma(r)))\Delta r\Delta s \\
&\ge \lambda \int_\xi^{\omega} G(\tau, s)p(s)(f_0 - \epsilon)\lambda\gamma \int_\xi^{\omega} G(\tau, r)q(r)(g_0 - \epsilon) \parallel u \parallel \Delta r\Delta s \\
&\ge \lambda \int_\xi^{\omega} G(\tau, s)p(s)(f_0 - \epsilon)\|u\|\Delta s \\
&\ge \lambda\gamma \int_\xi^{\omega} G(\tau, s)p(s)(f_0 - \epsilon)\|u\|\Delta s \ge \|u\|.
\end{aligned}$$

So, $\|Tu\| \ge \|u\|$. If we put

$$\Omega_1 = \{x \in \mathcal{B} : \|x\| < H_2\},$$

then

$$\|Tu\| \ge \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{4.12}$$

Next, by definitions of f_∞ and g_∞, there exists $\overline{H}_1$ such that

$$f(x) \le (f_0 - \epsilon)x \text{ and } g(x) \le (g_0 - \epsilon)x, \ x \ge \overline{H}_1$$

There are two cases: (i) g is bounded, and (ii) g is unbounded.

For case (i), suppose $N > 0$ is such that $g(x) \le N$ for all $0 < x < \infty$. Then, for $a \le s \le \sigma(b)$ and $u \in \mathcal{P}$,

$$\lambda \int_a^{\sigma(b)} G(\sigma(s), r)q(r)g(u(\sigma(r)))\Delta r \le N\lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)\Delta r.$$

Let

$$M = \max\Big\{f(x) \mid 0 \le x \le N\lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)\Delta r\Big\},$$

and let

$$H_3 > \max\Big\{2H_2, M\lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s\Big\}.$$

Then, for $u \in \mathcal{P}$ with $\|u\| = H_3$,

$$\begin{aligned} Tu(t) &\le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)M\Delta s \\ &\le H_3 = \|u\| \end{aligned}$$

so that $\|Tu\| \le \|u\|$. If

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_3\},$$

then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{4.13}$$

For case (ii), there exists $H_3 > \max\{2H_2, \overline{H}_1\}$ such that $g(x) \le g(H_3)$, for $0 < x \le H_3$. Similarly, there exists $H_4 > \max\{H_3, \lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)g(H_3)\Delta r\}$ such that $f(x) \le f(H_4)$, for $0 < x \le H_4$. Choosing $u \in \mathcal{P}$ with $\|u\| = H_4$ we have by (A4) that

$$\begin{aligned} Tu(t) &\le \lambda \int_a^{\sigma(b)} G(t, s)p(s)f\big(\lambda \int_a^{\sigma(b)} G(\sigma(r), r)q(r)g(H_3)\Delta r\big)\Delta s \\ &\le \lambda \int_a^{\sigma(b)} G(t, s)p(s)f(H_4)\Delta s \\ &\le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s(f_\infty + \epsilon)H_4 \\ &\le H_4 = \|u\|, \end{aligned}$$

and so $\|Tu\| \le \|u\|$. For this case, if we let

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_4\},$$

then

$$\|Tu\| \le \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{4.14}$$

In either cases, application of part (ii) of Theorem 3.4 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which in turn yields a pair (u, v) satisfying (1.2), (1.3) for the chosen value of λ. The proof is complete. □

5 Example

In this section, we give an example illustrating our result. For the sake of simplicity we take $p(t) = q(t)$ and $f(t) = g(t)$. Let

$$\mathbb{T} = \Big\{\Big(\frac{2}{5}\Big)^n : n \in \mathbb{N}_0\} \cup \{0\Big\} \cup [1, 2].$$

Consider the system of two-point dynamic equations

$$u^{\Delta^2}(t) + \frac{1}{10}\lambda t \frac{kve^{2v}}{c + e^v + e^{2v}} = 0,\ t \in \left[\frac{4}{25}, \frac{2}{5}\right],$$
$$v^{\Delta^2}(t) + \frac{1}{10}\lambda t \frac{kue^{2u}}{c + e^u + e^{2u}} = 0,\ t \in \left[\frac{4}{25}, \frac{2}{5}\right],$$

$$u\left(\frac{4}{25}\right) = 0,\ u\left(\sigma^2\left(\frac{2}{5}\right)\right) = 0,\ v\left(\frac{4}{25}\right) = 0,\ v\left(\sigma^2\left(\frac{2}{5}\right)\right) = 0.$$

Here $p(t) = q(t) = \dfrac{1}{10}t$, $k = 100$, $c = 1000$,

$$f(v) = \frac{kve^{2v}}{c + e^v + e^{2v}},\ g(u) = \frac{kue^{2u}}{c + e^u + e^{2u}}.$$

By simple calculation, we find $\gamma = \dfrac{1}{16}, f_0 = g_0 = \dfrac{k}{c+2} = \dfrac{500}{1002}, f_\infty = g_\infty = k = 500, L_1 = 1.2851, L_2 = 2.13625$. By Theorem 4.1, it follows that for every λ such that $1.2851 < \lambda < 2.13625$, the two-point system of dynamic equation has at least one positive solution.

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