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Using Invariant Manifolds to solve an Anti-Competitive System of Difference Equations

Chris D. Lynd
University of Rhode Island
Department of Mathematics
chris@math.uri.edu

Abstract

In this paper, we analyze the global character of the solutions of an anti-competitive system of rational difference equations. We prove that the solutions of the system can have four types of global behavior, corresponding to different regions of the parameter space. We also show that, in a range of the parameter space, there is a curve of period-2 points. These period-2 points are fixed points of the 2^{nd} iterate of the map corresponding to the system; and each fixed point has a linear global stable manifold. Given an initial point (x_0, y_0), we use the equations of the invariant manifolds to calculate the limit points of the solution of the system of difference equations.

AMS Subject Classifications: 39A10, 39A11, and 39A28.
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1 Introduction

In this paper, we analyze the global character of the solutions of the system of difference equations

$$\left.\begin{aligned} x_{n+1} &= \frac{y_n}{1+x_n} \\ y_{n+1} &= \frac{\beta_2 x_n}{1+B_2 x_n} \end{aligned}\right\} \quad \text{for } n = 0, 1, 2, \ldots \tag{1.1}$$

where the parameters are positive numbers and the initial terms, x_0 and y_0, are nonnegative numbers.

We demonstrate that

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(1) If $\beta_2 \leq 1$, then every solution of the system in (1.1) converges to $(0,0)$.

(2) If $\beta_2 > 1$ and $B_2 < 1$, then almost every solution of the system in (1.1) converges to the unique minimal period-2 solution.

(3) If $\beta_2 > 1$ and $B_2 = 1$, then there is a decreasing curve of minimal period-2 points, and every positive solution of the system in (1.1) contains two subsequences, each of which converges along a linear global stable manifold to a point on the curve.

(4) If $\beta_2 > 1$ and $B_2 > 1$, then every positive solution of the system in (1.1) converges to the unique positive equilibrium point.

The system of difference equations in (1.1) comes from a class of systems whose equations have the form

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\} \text{ for } n = 0, 1, 2, \ldots \tag{1.2}$$

where the parameters and the initial terms, x_0 and y_0, are nonnegative numbers such that the denominators are never equal to zero.

In 2009, Camouzis, Kulenović, Ladas, and Merino [5] introduced a numbering system which divided the class of systems in (1.2) into 2,401 separate systems, where each system of equations has a unique combination of positive parameters. Since then, many papers have been written [1–4, 7–10, 12, 13, 15, 16, 18] analyzing the boundedness, and the global behavior, of the solutions of systems of equations of the form in (1.2).

In 2011, Kalabusic and Kulenović [11], analyzed the two anti-competitive systems of rational difference equations of the form

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \gamma_1 y_n}{A_1 + x_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n}{A_2 + y_n} \end{aligned}\right\} \text{ for } n = 0, 1, 2, \ldots$$

where α_1, α_2, γ_1, and β_2 are positive numbers, A_1 and A_2 are nonnegative numbers such that $A_1 > 0$, if and only if, $A_2 > 0$, and the initial terms are nonnegative numbers such that the denominators are never equal to zero.

In this paper, we analyze the anti-competitive system

$$\left.\begin{aligned} x_{n+1} &= \frac{\gamma_1 y_n}{A_1 + B_1 x_n} \\ y_{n+1} &= \frac{\beta_2 x_n}{A_2 + B_2 x_n} \end{aligned}\right\} \text{ for } n = 0, 1, 2, \ldots \tag{1.3}$$

where the parameters are positive numbers and the initial terms, x_0 and y_0, are nonnegative numbers. We have normalized the equations in the system in (1.3), transforming it into the system in (1.1), which allows the results to be stated in simpler terms.

2 Preliminaries

The following definitions and theorems will be used in this paper.

Definition 2.1. The symbol $\preceq_{SE}$ denotes the *south-east partial ordering* of the nonnegative quadrant. That is, $(x, y) \preceq_{SE} (a, b)$ if $x \leq a$ and $y \geq b$. A strict inequality is defined as $(x, y) \prec_{SE} (a, b)$ if $(x, y) \preceq_{SE} (a, b)$ and $(x, y) \neq (a, b)$. A strong inequality is defined as $(x, y) \prec\prec_{SE} (a, b)$ if $x < a$ and $y > b$.

Definition 2.2. Let R be a subset of $\mathbb{R}^2$ with nonempty interior, and let $F : R \to R$ be a continuous map where $F(x, y) = (f(x, y), g(x, y))$.

(a) The map F is *competitive* if $f(x, y)$ is nondecreasing in x and nonincreasing in y, and $g(x, y)$ is nonincreasing in x and nondecreasing in y.

(b) The map F is *strongly competitive* if $f(x, y)$ is increasing in x and decreasing in y, and $g(x, y)$ is decreasing in x and increasing in y.

(c) The map F is *anti-competitive* if $f(x, y)$ is nonincreasing in x and nondecreasing in y, and $g(x, y)$ is nondecreasing in x and nonincreasing in y.

An important property of competitive maps is that they *preserve the south-east ordering*; that is, if F is a competitive map and $(x, y) \preceq_{SE} (a, b)$, then $F(x, y) \preceq_{SE} F(a, b)$. The map F is *strongly order preserving* when $(x, y) \prec_{SE} (a, b)$ implies that $F(x, y) \prec\prec_{SE} F(a, b)$.

If F is anti-competitive, then F reverses the south-east ordering; that is, if $(x, y) \preceq_{SE} (a, b)$, then $F(a, b) \preceq_{SE} F(x, y)$. So, if the map $F : R \to R$ is anti-competitive, then the map $F^2 : R \to R$, where $F^2(x, y) = F(F(x, y))$, is competitive and preserves the south-east ordering.

Theorem 2.3 (Camouzis and Ladas [6])**.** *Let I be a set of real numbers and let*

$$f : I \times I \to I$$

be a function $f(u, v)$, which decreases in u and increases in v. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation

$$x_{n+1} = f(x_n, x_{n-1}),\ n = 0, 1, 2, \dots ,$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic.

Theorem 2.4 (Kulenović and Merino [14])**.** *Let F be a strongly competitive map on a rectangular region $R \subset \mathbb{R}^2$ that contains a fixed point P in its interior. Suppose that F has a C^1 extension to a neighborhood of P. Suppose also that the Jacobian matrix of F at P has real eigenvalues, λ and μ, such that $0 < |\lambda| < \mu$, $|\lambda| < 1$, and the eigenspace E^λ, corresponding to λ, is not a coordinate axis.*

Then, there exists a curve $C \subset R$ through P that is invariant under F and is a subset of the basin of attraction of P, such that C is tangential to the eigenspace E^λ at P, and C is a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of C in the interior of R are either fixed points, or minimal period-2 points, of the map F.

3 Local Stability Analysis

We will present the results of the local stability analysis of the equilibrium points, and the period-2 points, of the system in (1.1). For an example of how to perform a local stability analysis, as well as how to find the necessary and sufficient conditions for the existence of minimal period-2 points, see [17].

The point $(0,0)$ is an equilibrium point of the system in (1.1), for all positive values of the parameters. If $\beta_2 \leq 1$, there are no other equilibrium points and there are no minimal period-2 points. When $\beta_2 < 1$, the point $(0,0)$ is locally asymptotically stable; and when $\beta_2 = 1$, the point $(0,0)$ is nonhyperbolic.

When $\beta_2 > 1$, the point $(0,0)$ is a repeller. In this range of the parameters, there is also a unique positive equilibrium point and at least two minimal period-2 points.

In order for a point to be a minimal period-2 point of the system in (1.1), it must be on the line segment

$$w_1 = -v_1 + \frac{\beta_2 - 1}{B_2} \text{ with } \beta_2 > 1 \text{ and } 0 \leq v_1 \leq \frac{\beta_2 - 1}{B_2}. \tag{3.1}$$

When $\beta_2 > 1$, the endpoints of the line segment in (3.1), $\left(0, \frac{\beta_2 - 1}{B_2}\right)$ and $\left(\frac{\beta_2 - 1}{B_2}, 0\right)$, are minimal period-2 points of the system in (1.1). The remaining points on the line segment are period-2 points, if and only if, $\beta_2 > 1$ and $B_2 = 1$.

Let F be the map corresponding to the system in (1.1), which is

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{y}{1+x} \\ \dfrac{\beta_2 x}{1 + B_2 x} \end{pmatrix}. \tag{3.2}$$

Then,

$$F^2\begin{pmatrix} x \\ y \end{pmatrix} = F\left(F\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \dfrac{\beta_2 x(1+x)}{(1+B_2x)(1+x+y)} \\ \dfrac{\beta_2 y}{1 + x + B_2 y} \end{pmatrix}. \tag{3.3}$$

If $\beta_2 > 1$ and $B_2 < 1$, then the point $(0,0)$ is a repeller, the positive equilibrium point is a saddle point, and the two minimal period-2 points are locally asymptotically stable fixed points of the map F^2 in (3.3).

If $\beta_2 > 1$ and $B_2 = 1$, then every point on the line segment in (3.1), which includes the positive equilibrium point, is a period-2 point of the map F in (3.2). In this case, the point $(0,0)$ is a repeller and the points on the line segment are nonhyperbolic fixed points of the map F^2. The eigenvalues of the Jacobian matrix of F^2, evaluated at $(x, -x+\beta_2-1)$, are $\mu = 1$ and $0 < \lambda = \dfrac{1}{\beta_2} < 1$.

If $\beta_2 > 1$ and $B_2 > 1$, then the point $(0,0)$ is a repeller, the positive equilibrium point is locally asymptotically stable, and the two minimal period-2 points are saddle points of the map F^2.

4 Global Stability Analysis

We begin by showing that every solution of the system in (1.1) is bounded.

Let $\{x_n, y_n\}$ be a solution of the system in (1.1). Then, $y_{n+1} \leq \dfrac{\beta_2}{B_2}$ for all $n \geq 0$. Thus, $x_{n+1} < y_n \leq \dfrac{\beta_2}{B_2}$ for all $n \geq 1$. So, there is a uniform bound for all solutions of the system in (1.1).

Next, we decouple the system of equations in (1.1) and get the 2$^{\text{nd}}$-order difference equation

$$x_{n+1} = \frac{\beta_2 x_{n-1}}{1 + B_2 x_{n-1} + x_n + B_2 x_n x_{n-1}} \quad \text{for } n = 1, 2, 3, \ldots \tag{4.1}$$

where the parameters are positive numbers and the initial terms, x_0 and x_1, are nonnegative numbers.

Let f be the function corresponding to the difference equation in (4.1), which is

$$f(u,v) = \frac{\beta_2 v}{1 + B_2 v + u + B_2 uv}.$$

Let $\{x_n\}_{n=0}^{\infty}$ be a solution of the difference equation in (4.1). Then for all $n \geq 2$, $x_n < \dfrac{\beta_2}{B_2}$. Also, $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ and f is decreasing in u and is increasing in v. By Theorem 2.3, the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$, of the solution of the difference equation in (4.1), are eventually monotonic. Since every solution is bounded, $\{x_{2n}\}$ and $\{x_{2n+1}\}$ must converge.

Let $\{x_n, y_n\}_{n=0}^{\infty}$ be a solution of the system in (1.1), where x_0 and y_0 are nonnegative numbers. Then, $\{x_n\}_{n=0}^{\infty}$ is the solution of the difference equation in (4.1), where $x_1 = \dfrac{y_0}{1+x_0}$. Since $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=0}^{\infty}$ converge, let

$$\lim_{n\to\infty} x_{2n} = L_1 \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = L_2.$$

Using the first equation in the system in (1.1), we see that for all $n \geq 0$,

$$y_{2n} = x_{2n+1}(1 + x_{2n}) \text{ and } y_{2n+1} = x_{2n+2}(1 + x_{2n+1}).$$

Therefore,

$$\lim_{n\to\infty} y_{2n} = L_2(1 + L_1) \text{ and } \lim_{n\to\infty} y_{2n+1} = L_1(1 + L_2).$$

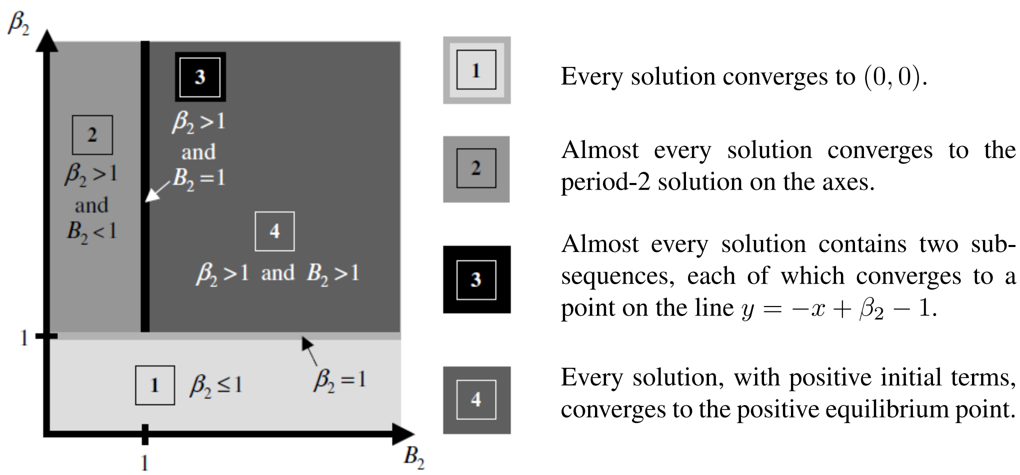
This proves Theorem 4.1 below.

Theorem 4.1. *Let* $\{x_n, y_n\}$ *be a solution of the system of equations in* (1.1)*. Then, there are nonnegative numbers* L_1 *and* L_2*, such that*

$$\{(x_{2n}, y_{2n})\} \to (L_1, L_2(1 + L_1)) \textit{ and } \{(x_{2n+1}, y_{2n+1})\} \to (L_2, L_1(1 + L_2)).$$

We analyze the global behavior of the solutions of the system in (1.1) by examining three regions of the parameter space, including their boundaries. The picture below illustrates the regions and boundaries we will examine and lists the corresponding global behavior of the solutions.

$$\left.\begin{aligned} x_{n+1} &= \frac{y_n}{1+x_n} \\ y_{n+1} &= \frac{\beta_2 x_n}{1+B_2 x_n} \end{aligned}\right\} \text{ for } n = 0, 1, 2, \ldots$$



In the following analysis, $\{(x_n, y_n)\}$ refers to a solution of the system in (1.1) and F refers to the corresponding map, given in (3.2). The subsequences of the solution, $\{(x_{2n}, y_{2n})\}$ and $\{(x_{2n+1}, y_{2n+1})\}$, are iterates of the map F^2, beginning at (x_0, y_0) and (x_1, y_1), respectively.

Region 1: Let $\beta_2 \leq 1$.

Theorem 4.1 states that given any solution of the system in (1.1), the subsequences $\{(x_{2n}, y_{2n})\}$ and $\{(x_{2n+1}, y_{2n+1})\}$ converge. When $\beta_2 \leq 1$, the point $(0, 0)$ is the only fixed point of the map F^2. Therefore, in this region of parameter space, every solution of the system in (1.1) converges to $(0,0)$.

Regions 2-4:

Let $\beta_2 > 1$. Then, there is a positive equilibrium point of the system in (1.1) and at least two minimal period-2 points.

If $(x_0, y_0) = (0,0)$, then the solution of the system in (1.1) is constant. If (x_0, y_0) is a point on the positive x-axis, then every point in the subsequence $\{(x_{2n}, y_{2n})\}$, of the solution of the system in (1.1), is on the x-axis; whereas, every point in $\{(x_{2n+1}, y_{2n+1})\}$ is on the y-axis. Furthermore, $\{(x_{2n}, y_{2n})\}$ converges to $\left(\frac{\beta_2 - 1}{B_2}, 0\right)$ and $\{(x_{2n+1}, y_{2n+1})\}$ converges to $\left(0, \frac{\beta_2 - 1}{B_2}\right)$. The result is similar if (x_0, y_0) is on the positive y-axis.

Let I be the interval $I = (0, \infty)$. For the remainder of the paper, we will consider the case where $(x_0, y_0) \in I \times I$ and $\beta_2 > 1$.

Let E be the positive equilibrium point of the system in (1.1), let $P_1 = \left(0, \frac{\beta_2 - 1}{B_2}\right)$, and let $P_2 = \left(\frac{\beta_2 - 1}{B_2}, 0\right)$. The points P_1 and P_2 are the endpoints of the line segment in (3.1), and they are minimal period-2 points of the system in (1.1).

The map $F : I \times I \to I$, in (3.2), is anti-competitive. The map $F^2 : I \times I \to I$ is strongly competitive and strongly preserves the south-east ordering. By Theorem 4.1, the sequence created by iterating the map F^2, beginning at any $(x_0, y_0) \in I \times I$, converges to a point in $[0, \infty) \times [0, \infty)$.

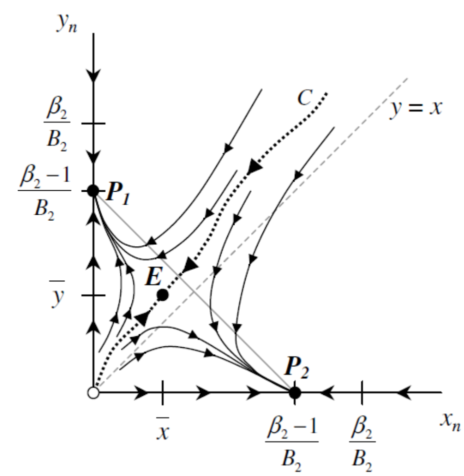
Region 2: Let $\beta_2 > 1$ and $B_2 < 1$.

In this region of parameter space, P_1 and P_2 are locally asymptotically stable fixed points of the map F^2, E is a saddle point of the map F^2, and $(0,0)$ is a repelling point of F^2. Furthermore, $(0,0)$, E, P_1, and P_2 are the only fixed points of the map F^2.

Let $J_F(E)$ denote the Jacobian matrix of F, evaluated at E. If $\vec{v}$ is a non-zero vector that is parallel to an axis, then for any value of λ, $\vec{v} \notin \text{Nul}\,(J_F(E) - \lambda I)$. So, the eigenvectors of $J_F(E)$ are not parallel to either axis. Since the eigenvectors of $J_F(E)$ are the eigenvectors of $(J_F(E))^2 = J_{F^2}(E)$, the eigenvectors of $J_{F^2}(E)$ are not parallel to either axis.

By Theorem 2.4, there is an increasing curve C, through E, that is invariant under the map F^2 and is a subset of the basin of attraction of E. This curve extends indefinitely to the north-east of E and extends from E, in the south-west direction, to the boundary of the region $[0, \infty) \times [0, \infty)$. Since the positive y-axis is in the basin attraction of P_1, and the positive x-axis is in the basin of attraction of P_2, the curve C extends from E to the fixed point (0,0).

When $\beta_2 > 1$ and $B_2 < 1$, almost every solution of the system in (1.1) converges to the period-2 solution on the axes. The picture at the right shows the trajectories of solutions of (1.1), beginning at different values of (x_0, y_0).



Without loss of generality, suppose that (x_0, y_0) is above the curve C. Then, there is a point on curve C, (x_C, y_C), that is to the south-east of (x_0, y_0).

The iterates of the map F^2, beginning at (x_C, y_C), are on curve C, which is an increasing curve. The map F^2 strongly preserves the south-east ordering; thus, all of the points in the subsequence $\{(x_{2n}, y_{2n})\}$, beginning at (x_0, y_0), are above curve C. Since $\{(x_{2n}, y_{2n})\}$ converges and $(0,0)$ is a repeller, it must converge to either E or P_1.

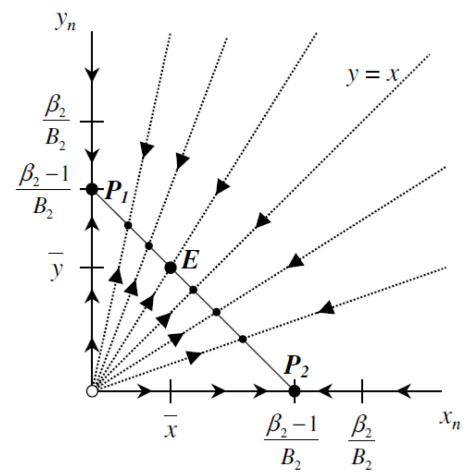
For the sake of contradiction, suppose that $\{(x_{2n}, y_{2n})\}$ converges to E. Since E is a saddle point of the map F^2, and the iterates of F^2, beginning at (x_C, y_C), converge to E, there is some k such that $F^{2k}(x_C, y_C)$ and $F^{2k}(x_0, y_0)$ are both on the local stable manifold through E, which is increasing. This would contradict the fact that the map F^2 strongly preserves the south-east ordering. Thus, $\{(x_{2n}, y_{2n})\}$ must converge to P_1.

The map F reverses the south-east ordering, so $F(x_C, y_C) \preceq F(x_0, y_0) = (x_1, y_1)$. Since E is a fixed point of F and F^2, the iterates of F^2, beginning at $F(x_C, y_C)$, must converge to E. The points in the subsequence $\{(x_{2n+1}, y_{2n+1})\}$ are the iterates of the map F^2, beginning at (x_1, y_1). Using a similar argument as above, one can show that $\{(x_{2n+1}, y_{2n+1})\}$ converges to P_2. Therefore, in Region 2 of the parameter space, whenever (x_0, y_0) is not on curve C, the solution of the system in (1.1) converges to the minimal period-2 solution on the axes.

Region 3: Let $\beta_2 > 1$ and $B_2 = 1$.

In this region of parameter space, every point on the line segment in (3.1), which includes the positive equilibrium point, is a period-2 point of the system in (1.1). The point $(0,0)$ is a repeller and the points on the line segment in (3.1) are nonhyperbolic fixed points of the map F^2. The eigenvalues of the Jacobian matrix of F^2, evaluated at $(x, -x + \beta_2 - 1)$, are $\mu = 1$ and $0 < \lambda = \dfrac{1}{\beta_2} < 1$, and the associated eigenvectors are not parallel to an axis.

When $\beta_2 > 1$ and $B_2 = 1$, almost every solution of the system in (1.1) converges to a period-2 solution on the line $y = -x + \beta_2 - 1$. The picture at the right shows the trajectories of solutions of system (1.1), beginning at different values of (x_0, y_0).



By performing a change of variables, we can show that each positive solution of the system in (1.1) is a set of points on one or two lines.

Let $\{(x_n, y_n)\}$ be a solution of the system in (1.1), where $\beta_2 > 1$, $B_2 = 1$, and $(x_0, y_0) \in I \times I$. Let $z_n = \frac{y_n}{x_n}$, for all n. Then,

$$z_{n+1} = \frac{y_{n+1}}{x_{n+1}} = \frac{\beta_2 x_n}{1 + x_n} \cdot \frac{1 + x_n}{y_n} = \frac{\beta_2}{z_n} \quad \text{for } n \geq 0. \tag{4.2}$$

The solution of the difference equation in (4.2) is

$$z_0 = \frac{y_0}{x_0}, \; z_1 = \frac{\beta_2 x_0}{y_0}, \; z_2 = \frac{y_0}{x_0}, \; z_3 = \frac{\beta_2 x_0}{y_0}, \; \ldots \; .$$

Thus,

$$z_{2n} = \frac{y_{2n}}{x_{2n}} = \frac{y_0}{x_0},$$

which implies that $y_{2n} = \frac{y_0}{x_0} x_{2n}$ for all $n \geq 0$. Also,

$$z_{2n+1} = \frac{y_{2n+1}}{x_{2n+1}} = \frac{\beta_2 x_0}{y_0},$$

which implies that $y_{2n+1} = \frac{\beta_2 x_0}{y_0} x_{2n+1}$ for all $n \geq 0$. Therefore, the points in the subsequence $\{(x_{2n}, y_{2n})\}$ are on the line $y = \frac{y_0}{x_0} x$, and the points in the subsequence $\{(x_{2n+1}, y_{2n+1})\}$ are on the line $y = \frac{\beta_2 x_0}{y_0} x$.

By Theorem 4.1, the subsequences $\{(x_{2n}, y_{2n})\}$ and $\{(x_{2n+1}, y_{2n+1})\}$ converge. The fixed point $(0, 0)$ is a repeller, and every other fixed point of the map F^2 lies on the line segment in (3.1). The lines $y = \frac{y_0}{x_0} x$, $y = \frac{\beta_2 x_0}{y_0} x$, and the line segment in (3.1), are all invariant under the map F^2; and the intersections of these lines are the limit points of the solutions of the system in (1.1).

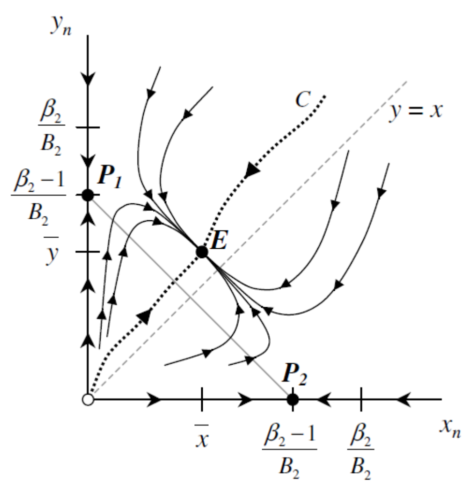
So, in Region 3 of the parameter space, if x_0 and y_0 are positive, then the solution of the system in (1.1) converges to a period-2 solution on the line segment in (3.1), with the subsequence $\{(x_{2n}, y_{2n})\}$ converging to $\left(\frac{x_0(\beta_2 - 1)}{x_0 + y_0}, \frac{y_0(\beta_2 - 1)}{x_0 + y_0}\right)$ along its global stable manifold $y = \frac{y_0}{x_0} x$, and the subsequence $\{(x_{2n+1}, y_{2n+1})\}$ converging to $\left(\frac{y_0(\beta_2 - 1)}{\beta_2 x_0 + y_0}, \frac{\beta_2 x_0(\beta_2 - 1)}{\beta_2 x_0 + y_0}\right)$ along its global stable manifold $y = \frac{\beta_2 x_0}{y_0} x$.

Region 4: Let $\beta_2 > 1$ and $B_2 > 1$.

In this region of parameter space, P_1 and P_2 are saddle points of the map F^2, E is a locally asymptotically stable fixed point of the map F^2, and $(0, 0)$ is a repelling point of F^2. Furthermore, $(0, 0)$, E, P_1, and P_2 are the only fixed points of F^2.

The eigenvectors of $J_{F^2}(E)$ are not parallel to an axis. By Theorem 2.4, there is an increasing curve C, through E, that is invariant under F^2 and is a subset of the basin of attraction of E. As in Region 2 of the parameter space, this curve extends indefinitely to the north-east of E and extends from the south-west of E to the point (0,0).

When $\beta_2 > 1$ and $B_2 > 1$, almost every solution of the system in (1.1) converges to the unique positive equilibrium point. The picture at the right shows the trajectories of solutions of system (1.1), beginning at different values of (x_0, y_0).



Using a similar argument as in Region 2, one can show that if (x_0, y_0) is not on curve C, then the subsequences of points $\{(x_{2n}, y_{2n})\}$ and $\{(x_{2n+1}, y_{2n+1})\}$, of the solution of the system in (1.1), lie on opposite sides of curve C, and each subsequence converges to E.

Therefore, in Region 4 of the parameter space, if x_0 and y_0 are positive, the solution of the system in (1.1) converges to the unique positive equilibrium point.

5 Future Research

When a fixed point of a map has a global stable manifold that is a line, one can easily find its equation, as was done in Region 3 of the parameter space for the system in (1.1). However, in the case of maps corresponding to systems of rational difference equations, the global stable manifold is often a curve whose equation is difficult to determine.

Much research can be done finding the equations of these global stable manifolds. In many cases, this equation will give us an explicit formula for the basin of attraction of a fixed point. Additionally, the equations of the invariant curves can sometimes be used to construct an explicit formula for the solutions of the corresponding system of difference equations.

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